Curves of Constant Curvatures in Möbius Geometry

Dedicated to the memory of Alfred Gray

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Introduction

In this notebook we apply *Mathematica* to describe the curves of constant curvatures in the three-dimensional Möbius geometry. The calculations and examples illustrate and complete the presentation of the subject in my not yet finished book [MDG]. In our book [OS3] one finds the foundations of elementary Möbius geometry in terms of linear algebra; also hyperbolic and elliptic geometry as used in this notebook are contained there.

The classical work treating Möbius differential geometry is W. Blaschke & G. Thomsens book [BT]. A contemporary presentation of this field is given by Udo Hertrich-Jeromin in his book [H-J]. There one can find more literature and some aspects of the history of this field. This interesting book accumulates a lot of material of the subject, but it contains nothing about the Möbius differential geometry of curves. In my paper [MGII] I treated this matter by E. Cartan's method of moving frames. The curves of constant curvatures are classified there solving a differential equation. In the presentation [MDG] and in chapter 2 below I applied the classification of 1-parameter subgroups of the Möbius group to describe all the curves of constant curvatures in the 3-dimensional Möbius space. In the present notebook the theory as described in [MDG] is applied and illustrated. Some calculations not contained in [MDG] are performed only here. I recommend to look at chapter 3 of [MDG]; this notebook can be considered as a part of that chapter. A special role play the circles and lines in Möbius geometry. These curves are locally Möbius equivalent; they form a Cartan class of immersions for which local invariants like curvatures do not exist. Of course they are also orbits of 1-parametric subgroups. Under a curve of constant curvatures in the Möbius space we always understand a generally curved curve with constant curvatures.

Any 1-parametric subgroup whose orbit is a curve of constant curvatures can be completed to a uniquely defined connected 2-dimensional maximal abelian subgroup of the Möbius group, see chapter 3 below. The orbits of these subgroups are the Dupin cyclides. The Dupin cyclides can be characterised as surfaces being envelopes of 1-parametric sphere families in a twofold way: There exist two distinct such families having the same envelope. They appear as the main part of homogeneous surfaces in the 3-dimensional Möbius space, see [MGV]. The characteristics of the generating sphere families form two families of circles (or lines) each generating the cyclide. These families are transformed into themselves by the 1-parametric subgroup generating the curve of constant curvature defining the cyclide; since the transformations are conformal the curves of constant curvatures are isogonal trajectories of the circle families. A well known example are the helices on the circle cylinder in Euclidean geometry. In dimension 3 the classification in chapter two shows that any curve of constant Möbius curvatures is Möbius equivalent to a curve of constant Riemannian curvatures in an Euclidean, hyperbolic or elliptic space, whose Riemannian geometries are subgeometries of the Möbius space: their isometry groups are subgroups of the Möbius group.

For the application of *Mathematica* to Euclidean differential geometry we mention the pioneering work of Alfred Gray, who introduced me into *Mathematica*, see [G06]. A short presentation of n-dimensional Euclidean curve theory is given in my paper [ECG] which together with a corresponding notebook EuCurves.nb may be downloaded from my homepage. From there one may download also other notebooks treating Möbius elementary geometry, pseudo-Euclidean geometry and Lie algebras.

I hope that the present notebook may serve as a good example of applying *Mathematica* to differential geometry. Of special interest are the many symbolical calculations carried out here. This and the graphical and numerical tools of *Mathematica* have been very useful in exploring the subject.

Keywords

Möbius group, curves of constant curvatures, space forms (conformal models), Dupin cyclides, cone, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, spherical reflections, 2D-spirals, spiral cylinder, stereographic projection, pseudo-Euclidean space

Initialization

Before starting to work interactively with this notebook first time, read this section carefully and make the necessary preparation. Later it suffices to call the menu item "Evaluation. Evaluate Initialization Cells".

The needed packages

For working with the notebook you need the packages
euvec.m, vectorcalc.m, neuvec.m, eudiffgeo.m, mdg.m.
You may download these and other packages from my Homepage . From there you find all the packages mentioned above in the file mdgpack.zip, download.
Before initializing the notebook ensure that these packages are laying in a directory of your choice, e.g. ~/mathpack, in your

```
In[1]:= $Path
```

If necessary, insert the addresses of yor package- and your working directory into the next cells which correspond to your operating system:

For Windows:

```
In[2]:= PrependTo[$Path, "E:\\mymath\\mathpack"];
In[3]:= SetDirectory["E:\\mymath\\diffgeo\\mdg"];
```

For Linux:

```
PrependTo[$Path, "~/mymath/mathpack"];
SetDirectory["~/mymath/diffgeo/mdg"];
Directory[]
```

Now give the cells corresponding to your operating system the properties "Cell Evaluatable" and "Initialization cell" (Menu Cell/Cell properties), and inactivate these properties for the cells corresponding to the other operating system. If this is done, save the notebook. Next time you may start the notebook directly with the evaluation of the initialization, as follows:

The Initialization

Before starting to work interactively with the notebook

Activite Evaluation/Evaluate Initialization from the menu.

The first command loads the packages mentioned above:

```
In[4]:= Needs["mdg`"]
```

In this notebook we shall consider the 3-dimensional Möbius space as the elliptical quadrik, the 3-dimens-sional sphere, in the 4-dimensional real projective space. Therefore we set the dimension dim and the index ind of the vector space as follows:

```
In[5]:= dim::usage = "dim is the dimension of the vector space
            under consideration; it must be set within the Global Context.";

        ? dim

In[6]:= dim = 5;

In[7]:= ind ::usage =
          "ind is the index of the pseudo-Euclidean vector space under consideration;
            it must be set within the Global Context.";

In[8]:= ind = 1;
```

- **Examples:**
- **$Assumptions**

1. List of Symbols and their Usages

In this section one finds tables of the symbols introduced in the imported packages and in the Global Con-text.
To get the usages click on the name! If this does not work,enable Dynamic Updating in the Evaluation Menu.

1.1. Symbols in the Package vectorcalc.m

```
In[10]:= ? vectorcalc`*
```

▼ vectorcalc`

basis	matrix	outzero	randomv	stb
blf	noprops	projpt	rank	trp
dotnorme	null	projpt3D	renorm	vec
dv	nullmatrix	randommatrix	smoothing	wedge

■ 1.2. Symbols in the Package euvec.m

In[11]:= ? euvec`*

▼ euvec`

center3pts	normed	sphericalreflection
circle3D	orthocomplement	spherpt3D
circle3pts	plotcircle3D	sterproj
cross	plotcircle3pts	subsphere
esorthonorm	plotgencircle	subspheremf
gencircle	posvec3pts	tg4frame
hyperplane	print	tube
innerprod	radius3pts	unit
invsterproj	sph3D	unitvec
neglect	sphere	
norm	sphereplot3D	

■ 1.3. Symbols in the Package eudiffgeo.m

In[12]:= ? eudiffgeo`*

▼ eudiffgeo`

arc	circ2D	curvatures	spiral	torsion
arclength	circle2D	frenet	spirgr	
assu	curvature	graph	tangent	

■ 1.4. Symbols in the Package neuvec.m

In[13]:= ? neuvec`*

▼ neuvec`

ch	ide	orthonorm	pscross	psgram
chsort	indexorder	orthopair	psCross	pssp
dual	normalize	pr	psfilter	

■ Examples

■ 1.5. Symbols in the Package mdg.m

In[14]:= ? mdg`*

▼ mdg`

dimm	io	mfre3D	transio
indd	iob	sphmap	transoi

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```
mdg`dimm
```

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```
dimm = dim
```

■ Examples

■ 1.6. Symbols in the Global Context

Evaluate the next cell to get the actual list of the symbols in the Global context.

In[138]:=
```
?Global`*
```

▼ Global`

a	euclideaneval	gdcc	nn$
abelian2D	euclideanevec	gr	n$
abeliangr2D	euclidpt	grell	pscurvecchyp
b	eucurvecc	grhyp	psogram
b$	eucurvecc0	gsp	psomatrix
c	euklid	h	ptell
cc	eval	hyp2D	pthyp2D
cca	evec	hyp3D	pthyp3D
ccell	evprods	hypccc	r
cchyp	felln	hyperboliccc	rot
chB	ff2	i	rr
confell	ff3	ind	rr$
confeuklid	fn	iomatrix	s
confhyp	fr3	i$	sphhyp2D
conus	fr3ell	j	sphhyp3D
curvecc	frell	k	t
curvecchyp	frell2	lambda	tor
curveellcc	frell2n	larot	tr
cy	frell3	ldio	u
cy1	frell3n	mcurvell1	v
d	frmat	mcurvell2	v0
dim	fy	mk1	w
dupin	fy2	mk2	x
ellambda	fy3	n	y
ellcc	gd	nfr3ell	z
ellmgcc	gda	nn	zerof

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```
Global`dupin
```

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```
dupin[k_, h_][u_, v_] :=
 sterproj[sphmap[transoi[].Transpose[abeliangr2D[k, h][u, v]][[1]]]]
```

2. Space Curves of Constant Curvatures

In this chapter we describe the curves of constant curvatures as orbits of 1-parameter subgroups of the Möbius group, see section 3.2 of [MDG]. There is shown that any curve of constant Möbius curvatures k, h is Möbius equivalent to an orbit of the 1-parametric subgroup g(t) = exp[cc[k,h]t] of the Möbius group. In section 2.1 we calculate these subgroups, what leads to a very large complex formula, hardly to work with manually. Nevertheless, by the help of *Mathematica* we may classify and visualize the curves of constant curvatures as shown in the subsequent sections of this chapter. Since equivalent curves (under the action of the Möbius group) correspond to conjugated subgroups we obtain at the same time the classification of the 1-

parametric subgroups of the Möbius group of S^3 in conjugacy classes.

- **2.1. The 1-Parametric Subgroups of the Möbius Group Generate the Curves of Constant Curvatures**
- **2.2. Classification and Properties of the 1-Parameter Subgroups**
- **2.3. The Eigensystem of cc[k,h]**
- **2.4. Case 3, the Euclidean Case chB=0**
- **2.5. Case 2, the Hyperbolic Case chB>0**
- **2.6. Complete Systems of Representatives in the Hyperbolic Case**
- **2.7. The Elliptic Case chB<0**
- **2.8. Transformation between the Euclidean and the Möbius Space**
- **2.9. Lines. The Conformal Representation of the Translation Group**
- **2.10. Circles. The Conformal Representation of the Rotation Group.**

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3. Dupin Cyclides

The Dupin cyclides are the orbits of the two-parametric Abelian subgroups of the Möbius space, see [MGV]. In section 3.1 we start with the element cc[k,h] of the Lie algebra and complete it to a maximal Abelian Lie subalgebra of the Lie algebra of the Möbius group. In section 3.2 we consider the corresponding two-dimensional Abelian subgroups and apply the classification obtained in chapter 2. In each case we plot a corresponding two-dimensional orbit and a curve of constant curvatures k, h lying in it.

- **3.1. Two-Parametric Abelian Subgroups of the Möbius Group**
- **3.2 The Dupin Cyclides as Orbits of 2-Dimensional Commutative Subgroups**
- **3.3. The Euclidean Realization of the Hyperbolic Curves of Constant Curvatures**

References

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http://www-irm.mathematik.hu-berlin.de/~sulanke