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Differential Geometry of the Möbius Space I

Curve Theory

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Preface

The manuscript presented here is the first part of a planned book about the differential geometry of submanifolds of the Möbius space. It is devoted to curve theory, developed systematically using E. Cartan's method of moving frames [4]. In our papers R. Sulanke, A. Švec [29], see also [25], we described an interpretation of E. Cartan's method, and we wanted to test our work in a homogeneous space in which invariant linear connections don't exist and the methods of Riemannian geometry can not be applied directly. Appropriate such spaces are the n-dimensional Möbius spaces, whose geometry coincides with the conformal geometry of the n-spheres S^n. The advantage of Möbius geometry is not only that it is very near to Euclidean geometry and admits clear graphic illustrations of the geometric objects in low dimensions, but also that it contains the classical Riemannian geometries of constant curvature as subgeometries, what allows metric interpretations of the conformal invariants.

There exist important very interesting monographs containing Möbius differential geometry. The earliest I know is the monograph [3] of W. Blaschke and G. Thomsen which appeared in 1929. It is a very rich collection of matters about the geometry of circles and spheres presenting much more than Möbius geometry; it contains also the Laguerre and Lie geometries. A multilateral and very attractive presentation of the recent developement of the subject is contained in the monograph [10] of U. Hertrich-Jeromin. A very general monograph about conformal structures is the book [1] of M. A. Akivis and V. V. Goldberg; in this book the method of G. F. Laptev is applied. In spite of the rich material contained in these monographs they may not be considered as systematic presentations of Möbius differentiual geometry; for example the differential geometry of curves in the 3-dimensional Möbius space is treated very incompletely in [3] as a serie of exercises, and not mentioned in the monographs [10] and [1]. The more modest aim of our book is to give a detailed description of the geometry of submanifolds of the n-dimensional Möbius space emphasizing the manifolds of points and considering the families of subspheres as tools only. The presentation of the subject is as elementary as possible, applying the elements of the theory of Lie groups and homogeneous spaces.

The first chapter contains basic concepts of Möbius geometry, the Möbius group, the simply connected space forms: the spherical, hyperbolical and Euclidean spaces embedded into the Möbius space, their isometry groups as subgroups of the Möbius group which is the conformal group of he standard n-sphere. The deciding object for the differential geometry of a homogeneous space G/H is the structure form of the group G and the isotropy action of H on it. We elaborate this concept for linear homogeneous spaces and consider the special cases needed in our presentation of the Möbius geometry.

The second chapter starts with a description of E. Cartan's method of moving frames. Section 2.1 may be read as a heuristic approach to this method; for a deeper understanding the study of the relevant literatur is necessary. For a reader who is mainly interested in a special geometry it can be said that a general understanding of the method suffices to apply it in a given situation; any Lie group requires a special investigation guided by the general method leading to interesting results which sometimes may be considered as corollaries of general theorems but often need special proofs. Section 2.2 based on a joined paper [23] with Ch. Schiemangk contains such a special application to immersions into the Möbius space. The there defined Möbius structure of an m-dimensional immersion contains an invariant Riemannian metric on the immersed manifold and an invariant linear connection of its normal bundle. The Möbius structure becomes singular at umbilical points. A criterion for an immersion to be a part of a subsphere is proved.

Chapter 3 is devoted to the Möbius geometry of curves, it is an elaboration of my paper [26]. For generally curved curves a natural parameter and a complete system of invariants, its conformal curvatures, are found. A so-called Fundamental Theorem, stating the existence and uniqueness (up to Möbius transformations) of a curve with given curvatures is proved. Curves contained in a subsphere are characterized by the vanishing of some higher curvatures. In section 3.2 the curves of constant curvatures are treated; they are identified as orbits of 1-parametric subgroups of the Möbius group. In the last section some results about curves in linear Lie groups and linear homogeneous spaces are collected, forming the theoretical background of curve theory in homogeneous spaces.

Connected with this book some Mathematica notebooks about Möbius Geometry are published on my homepage. There one find tools for the calculation of the moving frame and the invariants and the visualization of the curves and surfaces treated in chapter 3. Moreover, a framework for working in the geometry of the Möbius space and the spaces of constant curvatures is developed.

In the forthcoming not yet written chapters the geometry of hypersurfases, particularly surfaces, and special problems of the Möbius differential geometry will be considered as far as I am able to complete my program.

Grabow, November 16, 2010, Rolf Sulanke.

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Möbius Space. Space Forms

1.1 The Projective Model of the Möbius Space S^n

The *Möbius space* is defined to be the n-dimensional sphere S^n considered as the non-degenerate quadric of index 1 in the $(n+1)$-dimensional real projective space P^{n+1}. As usual we consider the *real* $n+1$*-dimensional projective geometry* as the geometry in the lattice of subspaces of an $(n+2)$-dimensional real vector space V^{n+2}. The *points* of the projective space P^{n+1} are the 1-dimensional subspaces of V^{n+2}, the k*-dimensional projective subspaces*, shortly named k*-planes*, are the $(k+1)$-dimensional vector subspaces of V^{n+2}. In the following we apply concepts, notations, and results of projective and Möbius geometry as described in [19]. Sometimes the fact is confusing that the same object, e. g. a vector subspace, is considered from the vectorial or the projective point of view leading to other dimensions: Denoting the usual dimension of a vector subspace by dim and the projective dimension by Dim, one has

$$\operatorname{Dim} U = k \Longleftrightarrow \dim U = k+1.$$

This is valid also for the null space $U = \{\mathfrak{o}\}$; considered from the projective point of view it is called the *nopoint* and denoted by o. If the k-planes are interpreted as point sets, i. e. as the sets of its 1-dimensional subspaces of the vector space, the nopoint is the empty set. It has the projective dimension $\operatorname{Dim} o = -1$.

Any point $x \in P^{n+1}$ can be represented by a basis vector $\mathfrak{x} \neq \mathfrak{o}$ of the corresponding 1-dimensional subspace of V^{n+2}; we write $x = [\mathfrak{x}]$, and generally $[A]$ for the span of a set $A \subset V$. The vector coordinates x^i of $\mathfrak{x}$ with respect to a frame $(\mathfrak{b}_i)$ are the corresponding *homogeneous coordinates* of the point x with respect to this frame; they are defined up to a common factor $\lambda \neq 0$.

All the non-degenerate n-dimensional quadrics of index 1 are projectively equivalent. We may fix an $(n+2)$-frame in the vector space V such that the equation of the quadric S^n in the corresponding homogeneous coordinates of P^{n+1} has the normal form

$$\langle \mathfrak{x}, \mathfrak{x} \rangle = -(x^0)^2 + \sum_{i=1}^{n+1} (x^i)^2 = 0. \tag{1}$$

Here $\langle , \rangle$ denotes the *pseudo-orthogonal scalar product of index 1* in the real $(n+2)$-dimensional vector space $V = V^{n+2}$, which defines the projective space P^{n+1}, the vectors $\mathfrak{x} \in V, \mathfrak{x} \neq \mathfrak{o}$, solving (1) describe the *isotropic cone* $J \subset V^{n+2}$, whose generators represent the points of S^n, and

$$\mathfrak{x} = \mathfrak{e}_j x^j := \sum_{j=0}^{n+1} \mathfrak{e}_j x^j \tag{2}$$

is the basis representation of $\mathfrak{x}$ with respect to the fixed *pseudo-orthonormal frame* $(\mathfrak{e}_j)$ of V, fulfilling

$$\epsilon_{ij} := \langle \mathfrak{e}_i, \mathfrak{e}_j \rangle = \epsilon_i \delta_{ij}, \text{ with } \epsilon_0 = -1, \epsilon_i = 1 \text{ for } i = 1, \ldots, n+1, \tag{3}$$

δ_{ij} the Kronecker symbol. As in (2) we will often use the *sum convention* that over the full range of indices appearing as subscripts and superscripts of an expression is to summarize.

The *m-dimensional subspheres* $\Sigma^m \subset S^n$, $m = 0, \ldots, n$, are defined as intersections

$$\Sigma^m = S^n \cap A^{m+1} \subset S^n$$

of S^n with projective $(m+1)$-planes A, containing at least two points; such $(m+1)$-planes are called *planes intersecting* S^n. We speak about *point pairs* $\Sigma^0 = \{x, y\} \subset S^n \times S^n, x \neq y$, as 0-spheres, *circles* are 1-spheres, and *hyperspheres* $\Sigma^{n-1} \subset S^n$ are subspheres of dimension $n-1$. Since the subspheres Σ^m contain at least two points of S^n, the vector subspaces W^{m+2} defining the intersecting projective $m+1$-planes $A^{m+1} = W^{m+2} \subset V$ contain at least two linearely independent isotropic vectors; therefore they correspond bijectively to the pseudo-Euclidean vector subspaces W^{m+2} of index 1 and dimension $m+2$. They are m-dimensional Möbius spaces themselves.

The *Möbius group* $M(n)$ is defined to be the group of all projective transformations of P^{n+1} preserving S^n. In [19] the Möbius group is identified with the *pseudo-orthogonal group* $O(1, n+1)^+$ of all linear transformations preserving the scalar product in the $(n+2)$-dimensional pseudo-Euclidean vector space V^{n+2} of index 1 and a time orientation of V. Indeed, we prove

Proposition 1. *The Möbius group $M(n)$, $n \geq 1$, is isomorphic to the pseudo-orthogonal group $O(1, n+1)^+$.*

P r o o f. Clearly, each projective transformation f generated by a linear transformation $g \in O(1, n+1)^+$ satisfies $f(S^n) = S^n$, see (1). On the other hand, let the projective transformation f preserving S^n be generated by the linear transformation g, and define

$$b(\mathfrak{x}, \mathfrak{y}) := \langle g\mathfrak{x}, g\mathfrak{y} \rangle.$$

Since f preserves S^n it follows

$$b(\mathfrak{x},\mathfrak{x}) = 0 \Longleftrightarrow \langle\mathfrak{x},\mathfrak{x}\rangle = 0.$$

We consider a line $[\mathfrak{x} + t\mathfrak{y}]$ starting at a point $x = [\mathfrak{x}] \in S^n$ in a non-isotropic direction $\mathfrak{y}, \langle\mathfrak{y},\mathfrak{y}\rangle \neq 0$. A point of the line belongs to the sphere S^n if and only if $t = 0$ or $t = -2\langle\mathfrak{x},\mathfrak{y}\rangle/\langle\mathfrak{y},\mathfrak{y}\rangle$. The same argument is valid for the scalar product replaced by the bilinear form b, and it follows

$$b(\mathfrak{x},\mathfrak{y})\langle\mathfrak{y},\mathfrak{y}\rangle = \langle\mathfrak{x},\mathfrak{y}\rangle b(\mathfrak{y},\mathfrak{y}) \text{ if } \langle\mathfrak{x},\mathfrak{x}\rangle = 0,\ \langle\mathfrak{y},\mathfrak{y}\rangle \neq 0.$$

We consider the pseudo-orthonormal basis $(\mathfrak{e}_i)$ of V^{n+2} satisfying (3) and evaluate the last equation for $\mathfrak{x} = \mathfrak{e}_0 + \mathfrak{e}_i, i > 0$, and $\mathfrak{y} = \mathfrak{e}_0$. It follows

$$b(\mathfrak{e}_0,\mathfrak{e}_i) = 0 \text{ for } i > 0.$$

Similarely, evaluating the equation for the same $\mathfrak{x}$ and $\mathfrak{y} = \mathfrak{e}_j$, $j > 0$, we conclude

$$b(\mathfrak{e}_i,\mathfrak{e}_j) = \delta_{ij}\lambda_j \text{ with } \lambda_j := b(\mathfrak{e}_j,\mathfrak{e}_j).$$

Since $\mathfrak{e}_0 + \mathfrak{e}_i$ for $i > 0$ is an isotropic vector we obtain

$$b(\mathfrak{e}_0 + \mathfrak{e}_i, \mathfrak{e}_0 + \mathfrak{e}_i) = \lambda_0 + \lambda_i = 0 \text{ for } i > 0.$$

Setting $\lambda := \lambda_1$ and applying Sylvester's theorem of inertia and $n + 2 \geq 3$ we conclude $b(\mathfrak{x},\mathfrak{y}) = \lambda\langle\mathfrak{x},\mathfrak{y}\rangle$ with $\lambda > 0$. Since the projective transformation f can be generated also by $\pm g\lambda^{-1/2}$ we may restrict to the linear transformations $g \in O(1, n+1)^+$. □

Another proof of Proposition 1, valid for $n \geq 2$ only, can be given by the argument: Any projective transformation f preserving the hyperquadric S^n maps hyperspheres, as intersections of S^n with hyperplanes, into hyperspheres. By Proposition 2.7.3 in [19] the map f can be generated by a linear transformation $g \in O(1, n+1)^+$.

Möbius geometry is the geometry of the transitive transformation group $[M(n), S^n]$, i.e. the geometry of the hypersphere under the action of the Möbius group. Möbius differential geometry is the theory of submanifolds of S^n under the action of $M(n)$. It is well known that $M(n)$, $n > 1$, coincides with the group of all conformal transformations of S^n provided with the Riemannian metric of constant curvature (induced by the standard embedding of S^n as the hypersphere of radius 1 in the Euclidean space E^{n+1}), see e.g. U. Hertrich-Jeromin [10], § 1.5. Therefore, instead of Möbius geometry one often speaks about the *conformal geometry* of the hypersphere.

The scalar product defines a *polarity* F in the lattice of projective subspaces, corresponding to the orthogonality of the vector subspaces:

$$F : A \longmapsto F(A) := A^\perp.$$

The map F is an involutive bijection of the lattice inverting the order. For the dimensions it follows

$$\mathrm{Dim}\, A = k \Longleftrightarrow \mathrm{Dim}\, A^\perp = n - k,\ (A \subset P^{n+1}). \tag{4}$$

To each point $x \in P^{n+1}$ corresponds its *polar* $x^\perp$, a hyperplane of P^{n+1}. The polars of points $x \in S^n$ can be characterized as the hyperplanes intersecting S^n in a single point, namely $x \in x^\perp$; this hyperplanes are called *tangent spaces of* S^n and denoted by $T_xS^n := x^\perp$ for $x \in S^n$. More generally, a projective subspace $A \subset P^{n+1}$ is said to be *tangent to* S^n *at the point* $x \in S^n$, if $x \in A \subset T_xS^n$ takes place. On the tangent subspace A, considered as vector subspaces of V, the scalar product degenerates; *the projective tangent subspaces correspond to the isotropic vector subspaces.* Projective k-planes A, $k > 0$, with $A \cap S^n = \emptyset$ are named *outside* k-planes; *the projective outside subspaces correspond to the Euclidean vector subspaces, on which the induced scalar product is positively definite.* The *outside points* $x = [\mathfrak{x}]$ are represented by *spacelike* vectors $\mathfrak{x}$ fulfilling $\langle \mathfrak{x}, \mathfrak{x} \rangle > 0$; the *inside points* are represented by *timelike vectors* characterized by $\langle \mathfrak{x}, \mathfrak{x} \rangle < 0$. The points of S^n are the autopolar points represented by the *isotropic vectors* fulfilling (1) and $\mathfrak{x} \neq \mathfrak{o}$. Finally, the *pseudo-Euclidean vector subspaces for which the induced scalar product is pseudo-Euclidean of index* 1*, correspond to the intersecting projective subspaces, which intersect* S^n *at least in two points*, defining subspheres (see above). On the other hand, any outside $(h-1)$-plane B corresponds to an Euclidean vector subspace $U^h \subset V^{n+2}$ defining uniquely its orthogonal complement, a pseudo-Euclidean vector $(n+2-h)$-subspace of V, representing a uniquely defined $(n-h)$-subsphere $\Sigma^{n-h} = S^n \cap B^\perp$. All these correspondences are bijections and equivariant under the action of the Möbius group $M(n)$. Especially, the outside points correspond bijectively to the hyperspheres of S^n. As a consequence of Proposition 1 one proves adapting pseudo-orthonormal frames to the subspheres:

Corollary 2. *The Möbius group $M(n)$ acts transitively on the manifold $S_{n,m}$ of the m-dimensional subspheres $\Sigma^m \subset S^n$. Let the m-sphere Σ^m be defined by the $(m+2)$-dimensional pseudo-Euclidean subspace $W \subset V$. The stationary subgroup of Σ^m under the action of $M(n)$ is isomorphic to $M(m) \times O(n-m)$, where $M(m)$ is the Möbius group of Σ^m and $O(n-m)$ denotes the orthogonal group of the Euclidean vector subspace $W^\perp$; thus it results the isomorphism of transformation groups*

$$S_{n,m} \cong O^+(1, n+1)/(O^+(1, m+1) \times O(n-m)).$$

The m-spheres Σ^m are m-dimensional Möbius spaces in a canonical way. □

Here and in the following we identify the Möbius group $M(n)$ with the pseudo-orthogonal group $O(1, n+1)^+$ of the vector space V. Any element $g \in O(1, n+1)$ is represented by its matrix (γ^i_j) with respect to the pseudo-orthonormal frame $(\mathfrak{e}_i)$:

$$g \in O(1, n+1) \mapsto (\gamma^i_j) \in GL(n+2, \mathbf{R}),\ \text{if}\ g(\mathfrak{e}_j) = \mathfrak{e}_i \gamma^i_j. \tag{5}$$

Evaluating the pseudo-orthogonality relations for the images $g(\mathfrak{e}_i)$ one obtains: A linear transformation $g \in GL(n+2, \mathbf{R})$ belongs to $O(1, n+1)$ iff its representing matrix fulfils the *pseudo-orthonormality relations*:

$$(\gamma_i^k)'(\epsilon_{kl})(\gamma_j^l) = (\epsilon_{ij}), \tag{6}$$

$(\gamma_i^k)'$ denotes the matrix transposed to (γ_i^k). Left multiplication of (6) with the matrix (ϵ_{hi}) shows that (6) is equivalent with

$$(\gamma_h^l)^{-1} = (\epsilon_{hi})(\gamma_i^k)'(\epsilon_{kl}).$$

Formula (6) is a system of quadratic equations for the coefficients γ_j^i of the pseudo-orthogonal matrices representing $g \in O(1, n+1)$. By symmetry we may restrict on the equations with $i \leq j$. One may prove that the remaining $(n+2)(n+3)/2$ equations are independent and define a regular algebraic submanifold of the space $\mathrm{End}(V) \cong \mathbf{R}^{(n+2)^2}$ with

$$\dim O(1, n+1) = (n+2)(n+1)/2.$$

(Apply the implicit function theorem.) The group $O(1, n+1)$ provided with this submanifold structure becomes a linear Lie group with four connected components characterized by

$$\det(\gamma_j^i) = \pm 1, \ \mathrm{sign}\, \gamma_0^0 = \pm 1.$$

Equation (6) implies $(det(\gamma_j^i))^2 = 1$ and the equation

$$\langle g\mathfrak{e}_0, g\mathfrak{e}_0 \rangle = -(\gamma_0^0)^2 + (\gamma_1^1)^2 + \ldots + (\gamma_{n+1}^{n+1})^2 = -1$$

yields $(\gamma_0^0)^2 \geq 1$. In the theory of special relativity ($\dim V = 4$) the four components correspond to the sets of transformations preserving or inversing the time and the space orientations, cf. P. K. Raschewski [21], Kap. 4. The Möbius group $M(n) = O(1, n+1)^+$ consists of two components:

$$\begin{aligned} &M(n) = SM(n) \cup M(n)^-, \\ &SM(n) := \{g \in O(1, n+1) | \gamma_0^0 > 0, \det(\gamma_j^i) = 1\}, \\ &M(n)^- := \{g \in O(1, n+1) | \gamma_0^0 > 0, \det(\gamma_j^i) = -1\}; \end{aligned} \tag{7}$$

the transformations $g \in SM(n)$ preserve the orientation of S^n, and those of $M(n)^-$ invert it. The group $M(n)$ acts simply transitively on the manifold of pseudo-orthonormal frames with a fixed time orientation, which can be considered as the group space of $M(n)$ as a submanifold of the space of square matrices $M_{n+2}(\mathbf{R}) \cong \mathbf{R}^{(n+2)^2}$, inheriting the differentiable (analytic) structure and topology induced by this embedding:

$$\iota : g \in M(n) \longmapsto (g\mathfrak{e}_i)_{i=0,\ldots,n+1} \longmapsto (\gamma_j^i) \in M_{n+2}(\mathbf{R}), \tag{8}$$

where $(\mathfrak{e}_i)$ is a fixed pseudo-orthonormal frame, and (γ_j^i) the matrix of g defined by (5).

1.1.1 The Isotropy Group of the Möbius Space

In the following we need the representation of the Möbius space as a factor space $S^n \cong M(n)/H_a$, where H_a denotes the isotropy group of the point

$$a := [\mathfrak{e}_0 - \mathfrak{e}_{n+1}] \in S^n.$$

To this aim we introduce another kind of frames, the *isotropic-orthonormal frames* $(\mathfrak{c}_i), i = 0, \ldots, n+1$, with scalar products fulfilling

$$\begin{aligned} &\langle \mathfrak{c}_0, \mathfrak{c}_0 \rangle = \langle \mathfrak{c}_{n+1}, \mathfrak{c}_{n+1} \rangle = 0, \\ &\langle \mathfrak{c}_0, \mathfrak{c}_{n+1} \rangle = \langle \mathfrak{c}_{n+1}, \mathfrak{c}_0 \rangle = -1, \\ &\langle \mathfrak{c}_i, \mathfrak{c}_j \rangle = \delta_{ij} \text{ for } i, j = 1, \ldots, n, \\ &\langle \mathfrak{c}_0, \mathfrak{c}_i \rangle = \langle \mathfrak{c}_{n+1}, \mathfrak{c}_i \rangle = 0 \text{ for } i, = 1, \ldots, n. \end{aligned} \tag{9}$$

If $(\mathfrak{b}_i)$ runs over the manifold of pseudo-orthonormal frames, the frame $(\mathfrak{c}_i)$, defined by

$$\mathfrak{c}_0 := (\mathfrak{b}_0 - \mathfrak{b}_{n+1})/\sqrt{2}, \ \mathfrak{c}_{n+1} := (\mathfrak{b}_0 + \mathfrak{b}_{n+1})/\sqrt{2}, \ \mathfrak{c}_i := \mathfrak{b}_i \text{ for } i = 1, \ldots, n, \tag{10}$$

runs over the manifold of isotropic-orthonormal frames, thus giving another representation of the group space $M(n)$ as a manifold of frames. Let us denote by $(\mathfrak{a}_i)$ the isotropic-orthonormal frame corresponding to the fixed pseudo-orthonormal frame $(\mathfrak{e}_i)$. One can prove (see Lemma 2.6.19 and proposition 2.7.1 in [19])

Proposition 3. *Given an isotropic-orthonormal frame $(\mathfrak{a}_i)$ the isotropy group H_a of the point $a = [\mathfrak{a}_0]$ consists of all transformations $h \in M(n)$ which have a matrix of the following form in this basis:*

$$h(A, \mathfrak{a}, \lambda) := \begin{pmatrix} \lambda^{-1} & \lambda^{-1}\mathfrak{a}'A & \langle \mathfrak{a}, \mathfrak{a} \rangle / 2\lambda \\ \mathfrak{o} & A & \mathfrak{a} \\ 0 & \mathfrak{o}' & \lambda \end{pmatrix} \ \textit{with } A \in O(n), \mathfrak{a} \in \mathbf{R}^n, \lambda > 0. \tag{11}$$

The Lie group H_a is isomorphic to the group $CE(n)$ of all similarities of the n-dimenional Euclidean space (also named the conformal Euclidean group*). The Möbius space under the transitive action of the Möbius group is a homogeneous space isomorphic to the factor space*

$$S^n \cong M(n)/H_a.$$

□

The isomorphism of H_a with $CE(n)$ is explicitely given in subsection 2.1 below, see (2.8).

1.1.2 A Vector Model of the Möbius Space

For many calculations it is useful to have a fixed vector model of the Möbius space S^n as a submanifold of the Euclidean hyperplane E^{n+1} defined by $\langle \mathfrak{e}_0, \mathfrak{z}\rangle = -1$. It intersects the isotropic cone J defined by (1) in the standard unit hypersphere $S^n_1 = J \cap E^{n+1}$:

$$\mathfrak{z} \in S^n_1 \Longleftrightarrow \mathfrak{z} = \mathfrak{e}_0 + \mathfrak{y},\ \langle \mathfrak{e}_0, \mathfrak{y}\rangle = 0 \text{ and } \langle \mathfrak{y}, \mathfrak{y}\rangle = 1. \tag{12}$$

With other words, we represent the point $x = [\mathfrak{z}] \in S^n$ by its *normed representative* $\mathfrak{z}$ satisfying $\langle \mathfrak{e}_0, \mathfrak{z}\rangle = -1$. The n-sphere S^n_1 with the embedding into V defined in (12) we take as the *vector model* of the Möbius space S^n. The norming of the arbitrary representative $\mathfrak{x} \in J$ is the projection

$$q : \mathfrak{x} \in J \longmapsto q(\mathfrak{x}) := \mathfrak{x}/x^0 = \mathfrak{e}_0 + \mathfrak{y} \in S^n_1,\ x^0 = -\langle \mathfrak{e}_0, \mathfrak{x}\rangle. \tag{13}$$

We prove

Lemma 4. *The differential of the map $q : \mathfrak{x} \mapsto \mathfrak{z}$ has the kernel $[\mathfrak{x}] \subset T_{\mathfrak{x}}J$. Restricted to the arbitrary Euclidean subspace $W^n \subset T_{\mathfrak{x}}J$ it is conformal; it fulfils*

$$\langle d\mathfrak{z}, d\mathfrak{z}\rangle = \langle d\mathfrak{y}, d\mathfrak{y}\rangle = \langle d\mathfrak{x}, d\mathfrak{x}\rangle/(x^0)^2. \tag{14}$$

P r o o f. The first statement follows from the surjectivity of q and the property $q(\mathfrak{x}t) = q(\mathfrak{x})$ for all $t \in \mathbf{R}$. By (13) we obtain

$$d\mathfrak{z} = d\mathfrak{y} = \frac{d\mathfrak{x}}{x^0} - \frac{\mathfrak{x}}{(x^0)^2} dx^0. \tag{15}$$

Calculating the scalar products we get (14), since from (1) it follows $\langle \mathfrak{x}, d\mathfrak{x}\rangle = 0$. Any Euclidean subspace W^n is complementary to the defect subspace $[\mathfrak{x}]$ of $T_{\mathfrak{x}}J$. Thus the statement follows from (14). □

Corollary 5. *Let $(\mathfrak{c}_i)$ be an isotropic-orthonormal frame at the point $x = [\mathfrak{c}_0] \in S^n$. Then the vectors*

$$\mathfrak{t}_i := dq_{\mathfrak{x}}(\mathfrak{c}_i),\ i = 1, \ldots, n,\ \mathfrak{x} := \mathfrak{c}_0, \tag{16}$$

form an orthogonal basis of the tangential space $T_{\mathfrak{z}}S^n_1$, $\mathfrak{z} = q(\mathfrak{x})$.

P r o o f. The vectors $\mathfrak{c}_i$ fulfil $\langle \mathfrak{c}_i, \mathfrak{x}\rangle = 0$; therefore they span an Euclidean subspace $W^n \subset T_{\mathfrak{x}}J$. By Lemma 4 their images are orthogonal, but not necessarily normed vectors. □

1.2 Space Forms and their Isometry Groups.

In this section we consider the *simply connected n-dimensional Riemannian spaces of constant curvature*, also called *space forms,* see S. Kobayashi, K. Nomizu [14]: The *Euclidean spaces* E^n, the *n-spheres* $S^n(r), n > 1,$ *of radius* $r > 0$ with a standard Riemannian metric of constant curvature $c = r^{-2}$, and the *hyperbolic spaces* $H^n(r)$ *of curvature* $c = -r^{-2}$. It is well known that their isometry groups are subgroups of the Möbius group $M(n)$. We will describe embeddings of these spaces into the Möbius space S^n and embeddings of their isometry groups into $M(n)$, in this way finding models of the Riemannian geometries of constant curvatures within the Möbius geometry. Later we will apply these embeddings to get metric expressions of the conformal invariants of immersions appropriate for studying the relations between metric and conformal properties.

1.2.1 Simply Connected Space Forms E_c^n embedded in S^n

First we construct models of these space forms as submanifolds $E_c^n \subset V^{n+2}, c \in \mathbf{R}$, of the pseudo-Euclidean vector space V. The span

$$E^{n+1} := [\{\mathfrak{e}_1, \ldots, \mathfrak{e}_{n+1}\}]$$

is an Euclidean vector space; it serves as the vector space of all the Euclidean hyperplanes defined by $x^0 = -\langle \mathfrak{x}, \mathfrak{e}_0 \rangle = r$. Here and in the following $(\mathfrak{e}_i), i = 0, \ldots, n+1$, denotes a fixed pseudo-orthonormal frame of V, and $(\mathfrak{a}_i)$ the corresponding fixed isotropic-orthonormal frame, see (1.9), (1.10). Denote by J^+ the upper half of the *isotropic cone*:

$$J^+ := \{\mathfrak{x} \in V | \langle \mathfrak{x}, \mathfrak{x} \rangle = 0, \langle \mathfrak{x}, \mathfrak{e}_0 \rangle < 0\}. \tag{1}$$

Intersecting J^+ with the hyperplane $x^0 = r > 0$, we obtain the standard n-sphere of radius r:

$$E_c^n := \{\mathfrak{x} \in V | \langle \mathfrak{x}, \mathfrak{e}_0 \rangle = -r, \langle \mathfrak{x}, \mathfrak{x} \rangle = 0\}, \text{if } c = r^{-2} > 0 \tag{2}$$

as submanifold of J^+. The solution of (2) is

$$\mathfrak{x} = \mathfrak{e}_0 r + \mathfrak{u}, \mathfrak{u} \in E^{n+1} \text{ with } \langle \mathfrak{u}, \mathfrak{u} \rangle = r^2, \tag{3}$$

and this is the standard equation of the hypersphere of radius r in the $(n+1)$-dimensional Euclidean space. For $r = 1$ we obtain the unit hypersphere as the vector model S_1^n of the Möbius space, see (1.12).

Analogously, intersecting J^+ with the pseudo-Euclidean hyperplane

$$x^{n+1} = \langle \mathfrak{x}, \mathfrak{e}_{n+1} \rangle = -r < 0,$$

we obtain

$$E_c^n := \{\mathfrak{x} \in V | \langle \mathfrak{x}, \mathfrak{e}_{n+1} \rangle = -r, \langle \mathfrak{x}, \mathfrak{e}_0 \rangle < 0, \langle \mathfrak{x}, \mathfrak{x} \rangle = 0\}, \text{if } c = -r^{-2} < 0 \qquad (4)$$

as submanifold of J^+. The solution of (4) is

$$\mathfrak{x} = -\mathfrak{e}_{n+1} r + \mathfrak{u}, \mathfrak{u} \in [e_{n+1}]^\perp \text{ with } \langle \mathfrak{u}, \mathfrak{u} \rangle = -r^2, \langle \mathfrak{u}, \mathfrak{e}_0 \rangle < 0, \qquad (5)$$

and this is the standard equation of the upper part of the hypersphere of imaginary radius $\mathfrak{i}\, r$ in the $(n+1)$-dimensional pseudo-Euclidean space $[e_{n+1}]^\perp$, i.e. the pseudo-Euclidean standard model of the hyperbolic n-space $E_c^n, c = -r^2$, see e.g. P. K. Raschewski [21]. In affine geometry it consists of one sheet of the n-dimensional two-sheeted hyperboloid in the pseudo-Euclidean subspace $[\mathfrak{e}_{n+1}]^\perp = [\{\mathfrak{e}_0, \ldots, \mathfrak{e}_n\}]$.

The spherical resp. hyperbolical space is obtained as an intersection of the isotropic cone J^+ with an Euclidean resp. pseudo-Euclidean hyperplane, and vice versa: each intersection with any such hyperplane not containing the origin is pseudo-orthogonally equivalent to a spherical resp. hyperbolic space. Now we consider the intersection of J^+ with the isotropic hyperplane, characterised by the equation $\langle \mathfrak{a}_0, \mathfrak{x} \rangle = -1$. This hyperplane has the parameter representation with respect to the isotropic-orthonormal frame $(\mathfrak{a}_i)$

$$\mathfrak{x} = \mathfrak{a}_0 u^0 + \mathfrak{u} + \mathfrak{a}_{n+1} \text{ with } \mathfrak{u} \in E^n = [\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}], u^0 \in \mathbf{R}. \qquad (6)$$

The intersection with J^+ obtained inserting (6) into the equation $\langle \mathfrak{x}, \mathfrak{x} \rangle = 0$ yields the model for the Euclidean space form

$$E_0^n := \{\mathfrak{x} \in V | \mathfrak{x} = \mathfrak{a}_0 \langle \mathfrak{u}, \mathfrak{u} \rangle / 2 + \mathfrak{u} + \mathfrak{a}_{n+1}, \mathfrak{u} \in E^n\}. \qquad (7)$$

This is the elliptical hyperparaboloid in the isotropic intersecting hyperplane characterised by the equation

$$2u^0 = \langle \mathfrak{u}, \mathfrak{u} \rangle = \sum_1^n (u^i)^2.$$

The induced Riemannian metric of this submanifold of V is the Euclidean metric; indeed:

$$ds^2 = \langle d\mathfrak{x}, d\mathfrak{x} \rangle = \langle d\mathfrak{u}, d\mathfrak{u} \rangle = \sum_1^n (du^i)^2.$$

Now it is easy to prove the statement $H_a \cong CE(n)$ of Proposition 1.3. Multiplying the matrix (1.11) with the coordinate vector of $\mathfrak{x} \in E_0^n$, one gets

$$\mathfrak{y} = h(A, \mathfrak{a}, \lambda)\mathfrak{x} = \begin{pmatrix} \lambda^{-1} & \lambda^{-1}\mathfrak{a}' A & \langle \mathfrak{a}, \mathfrak{a} \rangle / 2\lambda \\ \mathfrak{o} & A & \mathfrak{a} \\ 0 & \mathfrak{o}' & \lambda \end{pmatrix} \begin{pmatrix} \langle \mathfrak{u}, \mathfrak{u} \rangle / 2 \\ \mathfrak{u} \\ 1 \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{v}, \mathfrak{v} \rangle / 2 \\ \mathfrak{v} \\ 1 \end{pmatrix} \lambda,$$

where

$$\mathfrak{v} = (A\mathfrak{u} + \mathfrak{a}) / \lambda \qquad (8)$$

defines the coordinates of the image $y = [\mathfrak{y}]$ in E_0^n. Here we used $[\mathfrak{y}] = [\mathfrak{y}/\lambda]$. Since $A \in O(n)$, the transformation $[\mathfrak{x}] \mapsto [\mathfrak{y}]$ of $F_0(E_0^n)$ corresponds to a similarity of E_0^n; here F_0 denotes the special case $c = 0$ of the *embeddings*;

$$F_c : \mathfrak{x} \in E_c^n \longmapsto F_c(\mathfrak{x}) := [\mathfrak{x}] \in S^n. \tag{9}$$

By (2) it follows that $F_c(E_c^n) = S^n$ for $c > 0$; for any point $x = [\mathfrak{x}] \in S^n$ exists a representative $\mathfrak{x}$ with $x^0 = -\langle \mathfrak{e}_0, \mathfrak{x} \rangle = r$. In the case $c < 0$ the equation $x^{n+1} = \langle \mathfrak{e}_{n+1}, \mathfrak{x} \rangle = 0$ defines a hypersphere of the Möbius space; representatives of $x = [\mathfrak{x}] \in S^n$ satisfying (4) only exist for the points of the halfspace S_-^n defined by $x^{n+1} < 0$; it coincides with the spherical model of the hyperbolic space, the half-hypersphere, see example 2.5.3 in [19]. For the Euclidean case, $c = 0$, using (1.13) one easily shows that $F_0(E_0^n)$ is the complement of the point $a = [\mathfrak{e}_0 - \mathfrak{e}_{n+1}]$.

1.2.2 Frame Bundles

The Möbius group acts simply transitively on the manifold of all time-oriented isotropic-orthonormal frames of the pseudo-Euclidean vector space V^{n+2} of index 1, see (1.9), (1.10). This allows to identify the group space $M(n)$ with this frame manifold. Fixing an isotropic-orthonormal frame $(\mathfrak{a}_i)$ we identify

$$c \in M(n) \longmapsto (\mathfrak{c}_i) := (c(\mathfrak{a}_i)). \tag{10}$$

The matrix of c with respect to $(\mathfrak{a}_i)$:

$$\mathfrak{c}_i = c(\mathfrak{a}_i) = \mathfrak{a}_j c_i^j, (c_i^j) \in GL(n+2, \mathbf{R}) \subset M_{n+2}(\mathbf{R}) \tag{11}$$

yields a representation of $M(n)$ as a linear group and as a submanifold of the real vector space $M_{n+2}(\mathbf{R}) \cong \mathbf{R}^{(n+2)^2}$. Since we consider time-oriented pseudo-orthonormal frames only, and since the $M(n)$-isomorphism (1.10) preserves the time orientation, the frame manifold corresponding to $M(n)$ consists of those isotropic-orthonormal frames fulfilling aside of (1.9) the condition

$$c_0^0 = -\langle \mathfrak{c}_0, \mathfrak{a}_{n+1} \rangle > 0. \tag{12}$$

It has two connected components characterised by $\det(c_i^j) = \pm 1$, containg the transformations $c \in M(n)$ preserving resp. inverting the orientation of S^n. Fixing the frame $(\mathfrak{a}_i)$ implies fixing the *origin* $a := [\mathfrak{a}_0]$ of S^n. Under the identification (10) the isotropy group H_a is the set of all frames $(\mathfrak{c}_i)$ at the origin: $a = [\mathfrak{c}_0]$. Generally, the canonical map $p : M(n) \to M(n)/H_a$ of this coset space appears as the projection of the *principal fibre bundle $M(n)(S^n, p, H_a)$ with basis space S^n, projection p, and structure group H_a*:

$$p : c = (\mathfrak{c}_i) \in M(n) \mapsto p(c) = [\mathfrak{c}_0] \in S^n. \tag{13}$$

The *fibres* $p^{-1}(x), x \in S^n$, are the cosets; the fibre $p^{-1}(x)$ consists of all frames $(\mathfrak{c}_i) \in M(n)$ with $x = [\mathfrak{c}_0]$. The map p is a $M(n)$-map, one has

$$p \circ L_g = l_g \circ p, \; g \in M(n), \tag{14}$$

where L_g denotes the *left action* of $M(n)$ on itself: $L_g(c) = gc$, and l_g the action of $M(n)$ on S^n. Furthermore, the structure group H_a acts from the right on $M(n)$ preserving the fibres:

$$\begin{gathered}(g,h) = ((\mathfrak{c}_i),(h^i_j)) \in M(n) \times H_a \longmapsto R_h(g) := (\mathfrak{c}_i)(h^i_l) \in M(n), \\ p \circ R_h = p, \; h \in H_a.\end{gathered} \tag{15}$$

Now following the paper R. Sulanke [28] we describe the group manifolds of the isometry groups of the space forms E^n_c as the bundles of their orthonormal frames and embed them into $M(n)$. The embeddings $E^n_c \to V$ defined by (3), (5), (7) are isometries on spacelike submanifolds. In the case $c > 0$ (3) implies $d\mathfrak{x} = d\mathfrak{u} \in E^{n+1}$ and $\langle \mathfrak{u}, d\mathfrak{u}\rangle = 0$; for $c < 0$ formula (5) gives again $d\mathfrak{x} = d\mathfrak{u}$ and $\langle \mathfrak{u}, d\mathfrak{u}\rangle = 0$; $\mathfrak{u}$ is timelike, therefore the image of $d\mathfrak{x}$ must be an Eucldean subspace for any $\mathfrak{x} \in E^n_c$. In the Euclidean case $c = 0$ derivation of (7) yields

$$d\mathfrak{x} = \mathfrak{a}_0 \langle \mathfrak{u}, d\mathfrak{u}\rangle + d\mathfrak{u}. \tag{16}$$

Now $\mathfrak{u} \in E^n$ leads to the tangential vectors at the point $\mathfrak{x}(\mathfrak{u})$:

$$\mathfrak{t}_i := d\mathfrak{x}(\mathfrak{a}_i) = \mathfrak{a}_0 u^i + \mathfrak{a}_i, \; i = 1, \ldots, n, \tag{17}$$

and this is an orthonormal basis of the tangential space $T_{\mathfrak{x}} E^n_0$. We denote by $[O_c(n), E^n_c, q, O(n)]$ the *bundle of orthonormal frames* of E^n_c. The bundle space $O_c(n)$ is the disjoint union of the fibres

$$q^{-1}(\mathfrak{x}) := \{(\mathfrak{t}_i) | (\mathfrak{t}_i) \text{ orthonormal basis of } T_{\mathfrak{x}} E^n_c\}, \; \mathfrak{x} \in E^n_c.$$

The structure group is the orthogonal group $O(n)$ with the right action preserving the fibres

$$z = (\mathfrak{t}_i) \in O_c(n), \, h = (h^j_i) \in O(n) \longmapsto R_h(z) := (\mathfrak{t}_j h^j_i) \in O_c(n), \tag{18}$$

$$q \circ R_h = q. \tag{19}$$

The isometry group $\mathrm{Iso}(E^n_c)$ acts simply transitively on the frame bundle $O_c(n)$. Indeed, it is well known that any distance preserving map of a Riemannian manifold into itself preserves the Riemannian structure, see e.g. S: Helgason [9], theorem 1.11.1. Therefore the isometry group acts on the bundle of orthonormal frames permuting the fibres:

$$L_g : g \in \mathrm{Iso}(E^n_c), z = (\mathfrak{t}_i) \in O_c(n) \longmapsto g(z) := (dg_{q(z)}(\mathfrak{t}_i)) \in O_c(n), \tag{20}$$

$$q \circ L_g = g \circ q. \tag{21}$$

If for $f, g \in \mathrm{Iso}(E^n_c)$ a frame $z \in O_c(n)$ exists with $f(z) = g(z)$ then for $\mathfrak{x} = q(z)$ the conditions

$$f(\mathfrak{x}) = g(\mathfrak{x}), \; df_{\mathfrak{x}} = dg_{\mathfrak{x}}$$

of Lemma 1.11.2 in [9] are fulfilled, and it follows $f = g$. By elementary considerations one can prove that for any two frames $z_1, z_2 \in O_c(n)$ there exists an isometry g with $g(z_1) = z_2$, see e.g. exercise 6.2.13 in [18] or exercise 1.5.7 in [19] for the Euclidean group, exercise 2.5.17 in [19] for the spheres , and Corollary 2.6.10 in [19] for the hyperbolic space. We shall give another proof below, embedding the frame bundles $O_c(n)$ into the Möbius group $M(n)$ and applying the Möbius transformations preserving E^n_c. To obtain these embeddings we remark that for any frame $z = (\mathfrak{t}_i) \in O_c(n)$ the projection $q(z) = \mathfrak{x} \in E^n_c$ is an isotropic vector orthogonal to the orthonormal tangent vectors $\mathfrak{t}_i$. One easily proves

Lemma 1. *Let $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_n$ be $n+1$ vectors in the pseudo-Euclidean vector space V^{n+2} of index 1 satisfying the relations (1.9) for $i, j = 0, 1, \ldots, n$. Then there exists a uniquely defined vector $\mathfrak{c}_{n+1}$ such that the relations (1.9) are fulfilled for the sequence $(\mathfrak{c}_i), i = 0, \ldots, n+1$.* □

With other words, any element $z \in O_c(n)$,

$$\mathfrak{c}_0 := q(z) = \mathfrak{x}, \mathfrak{c}_i := \mathfrak{t}_i,\ i = 1, \ldots, n, \tag{22}$$

defines uniquely an isotropic-orthonormal basis $w = (\mathfrak{c}_i) \in M(n)$, where $\mathfrak{c}_{n+1}$ is well defined by Lemma 1. The next proposition describes the embeddings obtained applying the lemma:

Proposition 2. *Let $F_c : \mathfrak{x} \in E^n_c \mapsto [\mathfrak{x}] \in S^n$ denote the embeddings, where the space Forms E^n_c are defined by (3), (5), and (7). For $z = (\mathfrak{t}_i) \in O_c(n)$, $\mathfrak{x} = q(z)$ the vector*

$$\begin{aligned} \mathfrak{c}_{n+1} &:= (\mathfrak{e}_0 r - \mathfrak{u})/(2r^2) \ \text{if } c = r^{-2}, \\ \mathfrak{c}_{n+1} &:= (\mathfrak{e}_{n+1} r + \mathfrak{u})/(2r^2) \ \text{if } c = -r^{-2}, \\ \mathfrak{c}_{n+1} &:= \mathfrak{a}_0 \ \text{if } c = 0, \end{aligned} \tag{23}$$

completes the vectors (22) to an isotropic-orthonormal frame $\bar{F}_c(z) := (\mathfrak{c}_i) \in M(n)$. The map $\bar{F}_c : O_c(n) \to M(n)$ is an injective bundle morphism of principal fibre bundles where the corresponding homomorphism of the structure groups is the embedding

$$\iota : A \in O(n) \longmapsto \iota(A) = \begin{pmatrix} 1 & \mathfrak{o}' & 0 \\ \mathfrak{o} & A & \mathfrak{o} \\ 0 & \mathfrak{o}' & 1 \end{pmatrix} \in H_a. \tag{24}$$

Proof. One checks by a direct calculation that the vectors defined by (22) together with (23) fulfil the isotropic-orthonormality relations (1.9). By the definition it is clear that $\bar{F}_c$ is a bundle morphism fulfilling

$$F_c \circ q = p \circ \bar{F}_c,\ \bar{F}_c \circ R_A = R_{\iota(A)} \circ \bar{F}_c,\ A \in O(n). \tag{25}$$

□

Proposition 3. *The isometry group* $\mathrm{Iso}(E^n_c)$ *acts simply transitively on the bundle of orthonormal frames* $O_c(n)$*. Fixing a frame the frame bundles can be identified with the group spaces:*

$$O_c(n) \cong O(n+1) \text{ if } c > 0,\ O_c(n) \cong O^+(1,n) \text{ if } c < 0,\ O_0(n) \cong E(n), \quad (26)$$

where $E(n)$ *denotes the Euclidean group. The embeddings of these groups into* $M(n)$ *constructed using Lemma 1 are the subgroups of* $M(n)$ *preserving* E^n_c*.*

Proof. Let again $z_0 := (\mathfrak{a}_0, \ldots, \mathfrak{a}_{n+1})$ denote the fixed isotropic-orthonormal frame obtained applying the transformation (1.10) on the fixed standard pseudo-orthonormal basis $(\mathfrak{e}_i)$. We shall carry out the proof in case $c = 0$ only; the other cases can be shown analogously. As the origin of E^n_0 we choose $\mathfrak{d}_0 := \mathfrak{x}(\mathfrak{o}) = \mathfrak{a}_{n+1}$. Let

$$z_0 := (\mathfrak{d}_i) := (\mathfrak{a}_i) \in q^{-1}(\mathfrak{d}_0),\ i = 1, \ldots, n,$$

be the fixed orthonormal frame. From (22) for $\mathfrak{u} = \mathfrak{o}$ it follows that z_0 is a tangential orthonormal frame of E^n_0 at $\mathfrak{d}_0$. The vector $\mathfrak{d}_{n+1} := \mathfrak{a}_o$ completes the orthonormal frame to the corresponding isotropic-orthonormal frame $w_o = \bar{F}_0(z_0) \in p^{-1}([\mathfrak{a}_{n+1}])$. Formula (22) gives for $\mathfrak{u} \in E^n = [\mathfrak{a}_1, \ldots, \mathfrak{a}_n]$: The map

$$\mathfrak{s} \in E^n \longmapsto \mathfrak{t} = d\mathfrak{x}(\mathfrak{s}) = \mathfrak{a}_0\langle \mathfrak{x}, \mathfrak{s}\rangle + \mathfrak{s} \in T_{\mathfrak{x}} E^n_0,\ \mathfrak{x} = \mathfrak{x}(\mathfrak{u}), \quad (27)$$

is an isometry of Euclidean vector spaces with respect to the scalar product induced by that of V. Here we used $d\mathfrak{u}(\mathfrak{s}) = \mathfrak{s}$ and $\langle \mathfrak{u}, \mathfrak{s}\rangle = \langle \mathfrak{x}, \mathfrak{s}\rangle$. Now let $z = (\mathfrak{t}_i) \in q^{-1}(\mathfrak{x})$ be an arbitrary orthonormal frame. By (22) and (23) the corresponding isotropic orthonormal frame is $w = \bar{F}_0(z) = (\mathfrak{x}, \mathfrak{t}_1, \ldots, \mathfrak{t}_n, \mathfrak{a}_0) \in M(n)$. Denote by g the transformation defined by $gw_0 = w$, in detail

$$g(\mathfrak{a}_{n+1}) = \mathfrak{x}, g(\mathfrak{a}_i) = \mathfrak{t}_i = \mathfrak{a}_0\langle \mathfrak{x}, \mathfrak{s}_i\rangle + \mathfrak{s}_i, g(\mathfrak{a}_0) = \mathfrak{a}_0.$$

Evaluating these conditions regarding (7) and (27) one sees that g has the matrix $a(A, \mathfrak{u}, 1)$ for $\mathfrak{x} = \mathfrak{x}(\mathfrak{u})$, see (1.11). The subgroup $E(n)$ of $M(n)$ defined by all these matrices with $A \in O(n)$ acts on E^n_0 as the Euclidean group, and any $h \in M(n)$ preserving E^n_c belongs to $E(n)$, what proves the statement. □

1.3 Structure Forms

In this subsection we introduce the structure form of linear Lie groups and apply it to the Möbius space and the space forms. We start with the case of a linear Lie group admitting an elementary approach with the method of moving frames. This special case suffices for the applications to most of the differential geometries of classical groups. The general theory goes back to E. Cartan [4]. In the following we refer to its description in R. Sulanke, A. Švec [29] and [25].

1.3.1 Structure Forms of Linear Lie Groups

Definition 1. A *linear Lie group* is defined to be a subgroup and a submanifold of the general linear group $GL(N, \mathbf{R})$ contained as an open manifold in the matrix space $M_N(\mathbf{R})$. Fixing the frame $(\mathfrak{a}_i), i = 1, \dots, N$, we obtain a *coordinate representation* of the linear Lie group $G \subset GL(N, \mathbf{R})$ by

$$g \in G \longmapsto (\gamma^i_j) \in M_N(\mathbf{R}) \text{ if } \mathfrak{c}_j := g(\mathfrak{a}_j) = \mathfrak{a}_i \gamma^i_j. \tag{1}$$

The analytic functions $\gamma^i_j = \gamma^i_j(g)$ are called the *coordinates* of g with respect to $(\mathfrak{a}_i)$. Since $\det((\gamma^i_j)) \neq 0$, the coordinate representation (1) is also an embedding of G into the linear group $GL(N, \mathbf{R})$. The differential of the identity id_G has the coordinate representation

$$dg: \; d\mathfrak{c}_j = \mathfrak{c}_k \omega^k_j, \tag{2}$$

where

$$\omega := g^{-1}dg; \; (\omega^k_j) := (\gamma^k_i)^{-1}(d\gamma^i_j) \tag{3}$$

denotes the *structure form* of the linear Lie group G. The matrix (ω^k_j) is the coordinate matrix of the structure form ω. If the frame $\mathfrak{a}_i$ remains fixed, one often identifies $\omega = (\omega^k_j)$. □

Formula (3) follows immediately deriving $\mathfrak{c}_j$ and expressing the fixed frame $(\mathfrak{a}_i)$ by the moving frame $(\mathfrak{c}_i)$:

$$d\mathfrak{c}_j = \mathfrak{a}_i d\gamma^i_j = \mathfrak{c}_k \omega^k_j.$$

In many applications to differential geometry equations (2) are named the *derivation equations*. The main properties of the structure form are the left invariance and its behaviour under right translations on G. Let $a \in G$ be a fixed element with matrix (α^i_j). Then the *left translation* L_a and *right translation* R_a of G are defined by

$$L_a g := ag; \text{ in matrix form: } L_a(\gamma^i_j) = (\alpha^i_k)(\gamma^k_j), \; g \in G, \tag{4}$$

$$R_a g := ga; \text{ in matrix form: } R_a(\gamma^i_j) = (\gamma^i_k)(\alpha^k_j), \; g \in G. \tag{5}$$

Proposition 1. *The structure form ω is left invariant, i. e. $L_a^*\omega = \omega$ for all $a \in G$. Under a right translation it transforms as*

$$R_a^*\omega = a^{-1} \cdot \omega \cdot a. \tag{6}$$

P r o o f. Since a is constant, we have $d(ag) = adg$. It follows

$$L_a^*\omega = (ag)^{-1}d(ag) = g^{-1}a^{-1}adg = g^{-1}dg = \omega.$$

Analogously,

$$R_a^*\omega = (ga)^{-1}d(ga) = a^{-1}g^{-1}dga = a^{-1}\omega a.$$

Replacing the group elements by their matrices one gets the corresponding formulas for the coordinate forms. □

Corollary 2. *The Pfaffian forms ω^i_j are left invariant, i. e. $L_a^*\omega^i_j = \omega^i_j$ for any $a \in G$. Under the right translation R_a with $a = (\alpha^i_j)$ they transform as*

$$R_a^*\omega^i_j = \overset{-1}{\alpha}{}^i_k\, \omega^k_l \alpha^l_j, \tag{7}$$

where

$$(\overset{-1}{\alpha}{}^i_k) := (\alpha^i_k)^{-1} \tag{8}$$

and the sum convention are used. □

The left invariant Pfaffian forms on G are called *Maurer-Cartan forms.* Clearly, they form a vector space of dimension $\dim G$, since every left invariant 1-form θ is defined by its value at the unit element $e \in G$:

$$\theta(g, \mathfrak{t}) = \theta(e, dL_g^{-1}(\mathfrak{t})),\ (\mathfrak{t} \in T_gG).$$

The N^2 Maurer-Cartan forms ω^i_j are not linearely independent, if $\dim G < N^2$; for the full linear group $G = GL(N, \mathbf{R})$ they are a basis of the Maurer-Cartan forms. In the general case on may adapt the forms to the subgroup G by an appropriate choice of the fixed basis $(\mathfrak{a}_i)$, as we shall see in some concrete examples below.

Now we return to the derivation equations (2). The differentials $d\mathfrak{c}_i$ are 1-forms with values in the vector space V^N. Exterior differentiation [1] of the derivation equations yields

$$\begin{aligned} 0 = dd\mathfrak{c}_j &= d\mathfrak{c}_k \wedge \omega^k_j + \mathfrak{c}_k d\omega^k_j, \\ 0 &= \mathfrak{c}_i\omega^i_k \wedge \omega^k_j + \mathfrak{c}_i d\omega^i_j. \end{aligned}$$

and it follows

Proposition 3. *The structure form ω of the linear Lie group G satisfies the* structure equation *of G:*

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_i,\ i, j = 1, \dots, N, \tag{9}$$

or, in coordinatefree terms:

$$d\omega = -\omega \wedge \omega. \tag{10}$$

□

[1] The exterior calculus for differential forms with values in a vector space is described in many textbooks on differential geometry, see e. g. W. Blaschke, H. Reichardt [2], § 84, R. W. Sharpe [24], R. Sulanke, P. Wintgen [30].

In the theory of Lie groups is proved that the structure equation defines the Lie group up to local isomorphy: Lie groups with the same structure equation have isomorphic neighbourhoods of the unit elements. In the differential geometries of homogeneous spaces with transformation group G its structure equation plays a fundamental role, see the remarks following Theorem 6 below.

Example 1. Let $N = n + 1$, and number the basis vectors with indices $j = 0, 1, \dots, n$. We consider the n-dimensional real affine space A^n embedded in the $n+1$-dimensional vector space V^{n+1} as the hyperplane defined by $x^0 = 1$:

$$A^n := \{x \in V | x = \mathfrak{a}_0 + \sum_{j=1}^{n} \mathfrak{a}_j x^j\}.$$

The *affine group* $A(n)$ is the group of all linear transformations preserving A^n:

$$A(n) := \{g \in GL(n+1, \mathbf{R}) | gA^n = A^n\}.$$

The axioms of affine geometry, see e. g. [18], § 4.3, are fulfilled if one considers A^n as the set of points and the subspace $W^n := [\mathfrak{a}_1, \dots, \mathfrak{a}_n]$ as the vector space acting on A^n as the group of translations

$$t_{\mathfrak{b}}(x) := x + \mathfrak{b} \in A^n,\ \mathfrak{x} \in A^n, \mathfrak{b} \in W^n.$$

An element $g \in GL(n+1, \mathbf{R})$ belongs to the affine group $A(n)$ iff its matrix (γ^i_j) with respect to the fixed frame $(\mathfrak{a}_j)$ has the shape (see the projective model of affine geometry, formula (I.5.10) in [19])

$$\mathfrak{c}_j := g(\mathfrak{a}_j) = \mathfrak{a}_i \gamma^i_j,\ (\gamma^i_j) = \begin{pmatrix} 1 & \mathfrak{o}' \\ \mathfrak{x} & C \end{pmatrix},\ \mathfrak{x} \in \mathbf{R}^n, C \in GL(n, \mathbf{R}), \tag{11}$$

$\mathfrak{o}'$ denotes the zero row vector. We identify the affine group with the set of affine frames

$$g \in A(n) \longmapsto (\mathfrak{c}_j) \in GL(V^{n+1}), \tag{12}$$

with $\mathfrak{c}_j$ defined by (11). The points $x \in A^n$ are represented as $x = \mathfrak{a}_0 + \mathfrak{x}$ with $\mathfrak{x} \in W^n$. We take the point $a_0 = \mathfrak{a}_0$ as origin of A^n. The isotropy group of a_0 is the linear group $GL(W^n)$, and we obtain the factor space representation of the affine space with the canonical projection p:

$$p : g \in A(n) \longmapsto x = g\mathfrak{a}_0 = \mathfrak{a}_0 + \mathfrak{x} \in A^n \cong A(n)/GL(W^n). \tag{13}$$

The fibres $p^{-1}(x) \subset A(n)$ consist of all tangential bases $(x, \mathfrak{c}_1, \dots, \mathfrak{c}_n)$, $\mathfrak{c}_i \in T_x A^n$. Specialising (2) to the affine case we obtain for the structure form of the affine group

$$d\mathfrak{c}_0 = d\mathfrak{x} = \sum_{i=1}^{n} \mathfrak{a}_i dx^i = \sum_{i=1}^{n} \mathfrak{c}_i \omega^i, \tag{14}$$

$$d\mathfrak{c}_j = \sum_{i=1}^{n} \mathfrak{a}_i d\gamma^i_j = \sum_{i=1}^{n} \mathfrak{c}_i \omega^i_j, \tag{15}$$

with

$$\omega^i := \omega_0^i = \sum_{k=1}^{n} \overset{-1}{\gamma}{}^i_k \, dx^k, \omega_j^i = \sum_{k=1}^{n} \overset{-1}{\gamma}{}^i_k \, d\gamma_j^k, (\overset{-1}{\gamma}{}^k_j) = C^{-1}. \tag{16}$$

Formulas (11) and (16) show that the forms ω^i, ω_j^i are the components of the structure form of the affine group:

$$\omega = g^{-1}dg = (\gamma_l^h)^{-1}(d\gamma_k^l) = \begin{pmatrix} 0 & \mathfrak{o}' \\ (\omega^i) & (\omega_j^i) \end{pmatrix}. \tag{17}$$

Spezializing (9) we get the structure equation of the affine group $A(n)$ by exterior differentiation of (14), (15) and inserting (15) in the result:

$$d\omega^k = -\omega_i^k \wedge \omega^i, \tag{18}$$

$$d\omega_j^k = -\omega_i^k \wedge \omega_j^i, \tag{19}$$

here all the indices run over $1, \ldots, n$, and the sum convention is applied. The $n(n+1)$ forms ω^k, ω_j^k are linearely independent and thus a basis of the Maurer-Cartan forms of the affine group $A(n)$. The isotropy group is the Lie subgroup $GL(W^n) \subset A(n)$. Its tangential vectors are the solution of $d\mathfrak{x} = 0$. By (14) this leads to the Pfaffian system

$$\omega^1 = \omega^2 = \ldots = \omega^n = 0.$$

The integrability conditions of this system are just the equations (18); thus they are fulfilled. The solutions are the connected components of the fibres, i. e. the cosets $gGL(W^n)$, $g \in A(n)$.□

Example 2. Now we specialize the affine geometry fixing a scalar product. Consider the foregoing example and assume that in the real vector space W^n a scalar product $\langle , \rangle$ is given, that is a symmetric or skew symmetric, non-degenerate bilinear form on W. For the fixed frame $(\mathfrak{a}_i), i = 1, \ldots, n$, denote by

$$gr := (\epsilon_{ij}) := (\langle \mathfrak{a}_i, \mathfrak{a}_j \rangle), \; i, j, = 1, \ldots n, \tag{20}$$

the *Gram matrix* of the scalar product with respect to the basis $(\mathfrak{a}_i)$. The Gram matrix is a symmetric respectively skew symmetric square matrix with determinant $\det(gr) \neq 0$. We denote the isotropy group of the scalar product under the action of $GL(W)$ by $GO(n)$, explicitely:

$$GO(n) := \{g \in GL(W) | \langle g(\mathfrak{x}), g(\mathfrak{y}) \rangle = \langle \mathfrak{x}, \mathfrak{y} \rangle \text{ for all } \mathfrak{x}, \mathfrak{y} \in W\}. \tag{21}$$

The group generated by $GO(n)$ and the group of translations of W^n is the *affine group preserving* $\langle , \rangle$; it is denoted by $GA(n) \subset A(n)$. The transformations $g \in GA(n)$ are represented by matrices of the shape (11) with $C \in GO(n)$. The embedding (12) restricted to the group $GA(n)$ gives the group space as a frame manifold, for which the derivation equations (14), (15) are valid. The invariance of the scalar product leads to the relations

$$\langle \mathfrak{c}_i, \mathfrak{c}_j \rangle = \epsilon_{ij}. \tag{22}$$

The ϵ_{ij} are constant on this frame manifold. Deriving (22) and inserting the derivation equation (15) yields the equations

$$\epsilon_{kj}\omega_i^k + \epsilon_{ik}\omega_j^k = 0,\ i, j, k = 1 \ldots, n, \tag{23}$$

which show that the Maurer-Cartan forms restricted to the frame manifold of $GO(n)$ are linearely dependent. At any rate, in applying or evaluating the structure equations (18), (19) we have to complete them by the relations (23). Introducing the forms

$$\beta_{ij} := \epsilon_{ik}\omega_j^k, \tag{24}$$

formula (23) can be written as a matrix equation

$$(\beta_{ij})' = -(\beta_{ij}) \text{ if } \langle , \rangle \text{ is symmetric}, \tag{25}$$

$$(\beta_{ij})' = (\beta_{ij}) \text{ if } \langle , \rangle \text{ is skew symmetric}. \tag{26}$$

Leaving the symplectic geometry: $\langle , \rangle$ skew symmetric, here aside, we consider now the symmetric case only. The vector spaces and the corresponding affine spaces are pseudo-Euclidean of index k, if the bilinear from defining the scalar product has index k. The most important case is the Euclidean defined by $k = 0$; in this case the motion group is the *Euclidean group* $E(n)$ and the structure group of the fibre bundle $p : E(n) \to E^n$ is the *orthogonal group* $O(n)$. Usually one considers orthonormal frames, for which $\epsilon_{ij} = \delta_{ij}$ equals the Kronecker symbol. Then the structure form ω has a skew symmetric matrix $(\omega_j^i) = (\beta_{ij})$. In the *pseudo-Euclidean spaces* one usually considers pseudo-orthonormal frames with scalar products

$$\epsilon_{ij} = \epsilon_i\delta_{ij}, \epsilon_i = -1 \text{ for } i \leq k,\ \epsilon_i = 1 \text{ for } i > k. \tag{27}$$

For example, in the case $k = 1, n = 4$, the matrix of the structure form of the pseudo-orthogonal group with respect to pseudo-orthonormal frames has the shape

$$\omega \doteq \begin{pmatrix} 0 & \omega_2^1 & \omega_3^1 & \omega_4^1 \\ \omega_2^1 & 0 & \omega_3^2 & \omega_4^2 \\ \omega_3^1 & -\omega_3^2 & 0 & \omega_4^3 \\ \omega_4^1 & -\omega_4^2 & -\omega_4^3 & 0 \end{pmatrix}. \tag{28}$$

The structure form of the pseudo-orthogonal group $O(k, n-k)$ of index k with respect to a pseudo-orthonormal frame can be written as a block matrix

$$\omega \doteq \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$$
$$\text{with } A' = -A \in M_k, C' = -C \in M_{n-k}, B \in M_{k,n-k}. \tag{29}$$

Of course, the coefficients of the matrices are real 1-forms on $O(k, n-k)$, and not real numbers. The set $\{\omega_k^i | 1 \leq i < k \leq n\}$ is a basis of the Maurer-Cartan forms, and it results

$$\dim O(k, n-k) = n(n-1)/2 \tag{30}$$

for all k, $0 \leq k \leq n$. For the pseudo-Euclidean point space E_k^n, that is the affine space A^n under the action of the pseudo-Euclidean group $E(n,k)$ being generated by $O(k, n-k) \subset GL(W^n)$ and the translations, we must enlarge the structure form according to (11), where now $C \in O(k, n-k)$ is valid. As in the affine geometry in the foregoing example we get a frame bundle as a principal fibre bundle

$$p : E(n,k) \to E_k^n = E(n,k)/O(k, n-k) \tag{31}$$

analogously to (13), the connected components of the fibres, the cosets

$$p^{-1}(x) = gO(k, n-k), \mathfrak{c}_0 = g\mathfrak{a}_0 = \mathfrak{a}_0 + \mathfrak{x},$$

are the solutions of the completely integrable Pfaffian system

$$\omega^1 = \omega^2 = \ldots = \omega^n = 0. \tag{32}$$

□

1.3.2 The Structure Form of the Möbius Group

As pointed out in section 1, the Möbius group $M(n)$ consists of two connected components of the pseudo-orthogonal group $O(1, n+1)$, see Propostion 1.1 and (1.7). For certain purposes, e.g. investigating the geometry of circles or subspheres, it may be convenient to work with pseudo-orthonormal frames. Then the formulas of example 2 in the special case k=1 may be applied, in particular formula (29) for the structure form. For the geometry of the Möbius space S^n itself, as a manifold of points, it is useful to work with isotropic-orthonormal frames defined by (1.9), see also Proposition 1.3 and formula (1.11). Thus we consider the group space $M(n)$ as the manifold af all time-oriented isotropic -orthonormal frames; formulas (1.9) and (2.12) are valid. The Gram matrix of any isotropic-orthonormal frame is

$$(\epsilon_{ij}) = \begin{pmatrix} 0 & \mathfrak{o}' & -1 \\ \mathfrak{o} & I_n & \mathfrak{o} \\ -1 & \mathfrak{o}' & 0 \end{pmatrix}, \, i,j = 0, \ldots, n+1. \tag{33}$$

Here I_n denotes the unit matrix of order n. Evaluating the relations (23), where the indices now run from 0 to $n+1$, one obtains

$$\omega_0^{n+1} = \omega_{n+1}^0 = 0, \, \omega_0^0 = -\omega_{n+1}^{n+1}, \tag{34}$$

$$\omega_0^i = \omega_i^{n+1}, \, \omega_i^0 = \omega_{n+1}^i, \tag{35}$$

$$\omega_j^i + \omega_i^j = 0 \text{ for } i,j = 1, \ldots, n. \tag{36}$$

Thus the matrix of the structure form has the block structure

$$(\omega^i_j) = \begin{pmatrix} \omega^0 & \tau' & 0 \\ \theta & C & \tau \\ 0 & \theta' & -\omega^0 \end{pmatrix} \tag{37}$$

with $\omega^0 := \omega^0_0$; $\theta := (\omega^i_0)$ and $\tau := (\omega^i_{n+1})$ are 1-forms on $M(n)$ with values in $\mathbf{R}^n$, and $C := (\omega^i_j)$ is a 1-form on $M(n)$ with values in the space of skew symmetric $n \times n$-matrices, isomorphic to the orthogonal Lie algebra $\mathfrak{o}(n)$ (see the next subsection). The structure equations are obtained evaluating (19) with indices running from 0 to $n+1$, since $M(n)$ is a subgroup of $GL(n+2, \mathbf{R})$. Regarding (34) - (36) we obtain *the structure equations of the Möbius group:*

$$d\omega^0_0 = -\omega^0_i \wedge \omega^i_0, \tag{38}$$

$$d\omega^i_0 = -\omega^i_0 \wedge \omega^0 - \omega^i_j \wedge \omega^j_0, \tag{39}$$

$$d\omega^i_j = -\omega^i_0 \wedge \omega^0_j - \omega^i_k \wedge \omega^k_j - \omega^0_i \wedge \omega^j_0, \tag{40}$$

$$d\omega^0_i = -\omega^0 \wedge \omega^0_i - \omega^0_j \wedge \omega^j_i \text{ for } i, j, k = 1, \ldots, n. \tag{41}$$

Now we return to formulas (2.11), (2.13) describing the principal fibre bundle of the Möbius space. The geometrical interpretation of an isotropic-orthonormal frame $c = (\mathfrak{c}_i) \in M(n)$ is the following: The vector $\mathfrak{x} = \mathfrak{c}_0$ represents the point $x = p((\mathfrak{c}_i)) = [\mathfrak{c}_o] \in S^n$, see equation (1.1). Defining $\omega^i := \omega^i_0$ we have by (34)

$$d\mathfrak{x} = d\mathfrak{c}_0 = \mathfrak{x}\omega^0 + \mathfrak{c}_i\omega^i. \tag{42}$$

The sequence $(\mathfrak{x}, \mathfrak{c}_1, \ldots, \mathfrak{c}_n)$ is a basis of the tangential space of the isotropic cone J at the point $\mathfrak{x}$. The first term $\mathfrak{x}\omega^0$ describes the variation along the generator $[\mathfrak{x}] \subset J$ of the isotropic cone $J \subset V$, defined by $\langle \mathfrak{x}, \mathfrak{x} \rangle = 0$, and the second term the variation transversal to $[\mathfrak{x}]$. On the other hand, the vectors $(\mathfrak{c}_i)$ correspond to an orthogonal basis of the tangential space $T_x S^n$, see Corollary 1.5. By Lemma 2.1 the vector $\mathfrak{c}_{n+1}$ is uniquely defined by the other vectors of the frame; conversely, it defines a hyperspace of $T_{\mathfrak{x}} J$ complementary to $[\mathfrak{x}]$. The forms $\omega^i, i > 0$, define the fibres of the fibre bundle as solutions of the completely integrable system

$$\omega^1 = 0, \ldots, \omega^n = 0. \tag{43}$$

These solution are the orbits of the right action of the isotropy group H_a on $M(n)$, namely the cosets $cH_a \subset M(n)$.

1.3.3 Basis Forms

Let us for a moment return to the situation of subsection 1. We define:

Definition 2. Let $G \subset GL(N, \mathbf{R})$ be a linear Lie group and $M^n \cong G/H$ be a homogeneous space, i.e. a manifold with a transitive action of G on M^n, where H denotes the isotropy group of the arbitrarily fixed point $a \in M^n$. A basis $(\omega^i), i = 1, \ldots, N := \dim G$ of the Maurer-Cartan forms of G is named a *basis adapted to the homogeneous space*, if the cosets $gH \subset G$ are solutions of the completely integrable Pfaffian system (43). The forms $\omega^i, i = 1, \ldots, n$, are called *basis forms of the homogeneous space* $M^n \cong G/H$. □

Formula (43) contains the basis forms of the Möbius space. Example 1 shows that the basis considered there is adapted to the affine space; the forms (ω^i) are basis forms, and the same considerations show that they are basis forms for the Euclidean and pseudo-Euclidean spaces too. We prove

Lemma 4. *Let* $p : g \in G \mapsto gH \in M^n := G/H$ *be the principal fibre bundle associated to the homogeneous space* M^n. *Then there exist adapted bases of the Maurer-Cartan forms of* G.

P r o o f. In the theory of Lie groups is proved the basic fact that the isotropy groups of a Lie transformation group $[G, M^n]$ are closed Lie subgroups of G. Thus the fiber bundle $p : G \to G/H$ is analytic. Consider the unit element $e \in H \subset G$. If $\dim G = r$ we have $\dim H = r - n$. For the tangential spaces at e we have $T_eH \subset T_eG$, and there exist a basis (ω_e^i), $i = 1, \ldots, n$, of the annihilator $T_eH^\perp \subset T_e^*G$ of T_eH. Then the Pfaffian forms on G defined by

$$\omega^i(g, \mathfrak{t}) := \omega_e^i(dL_g^{-1}(\mathfrak{t})),\ i = 1, \ldots, n,\ (\mathfrak{t} \in T_gG), \tag{44}$$

are left invariant. We complete them to a basis (ω^ρ), $\rho = 1, \ldots r$, of the Maurer-Cartan forms of G. For the tangential space of the cosets we have

$$T_g gH = (dL_g)_e T_eH,$$

and from (44) one obtains $\omega^i(\mathfrak{t}) = 0$ for all $\mathfrak{t} \in T_gH$, showing that the cosets are solutions of the Pfaffian system (43). □

To find invariant geometric properties of the homogeneous space M^n by E. Cartan's method of moving frames one has to consider *local sections* of the associated principal fibre bundle. Generally, if $p : F^r \to M^n$ is a smooth surjective map of differentiable manifolds and $U \subset M^n$ an open subset, a smooth map $s : U \to F^r$ is said to be a local section of p, if $p \circ s = \mathrm{id}_U$. In case that p is the projection of a fibre bundle one also names s a local section of the fibre bundle. The condition for s to be a local section implies

$$\operatorname{rank} ds_x = \operatorname{rank} dp_g = \dim M^n = n \tag{45}$$

and it follows

Lemma 5. *Let* G *be a linear Lie group. For any local section* $s : U \to G$ *of the principal fibre bundle* $p : G \to M^n = G/H$ *the* local structure form $s^*\omega = \omega \circ ds = (s^*\omega_j^i)$ *is a matrix of* 1*-forms defined on* U. *If* (ω^i), $i = 1, \ldots, r = \dim G$, *is a basis of the Maurer-Cartan forms adapted to* G/H, *the forms* $\sigma^i(x) := \omega^i \circ ds_x$, $i = 1, \ldots, n$, *are a basis of the dual tangential space* $T_x^*M^n$. □

1.3.4 Lie Algebras

For a deeper understanding of the structural background of E. Cartan's method of moving frames the knowledge of some elementary facts about the structure of Lie groups and Lie algebras is useful. [2] The *Lie algebra* $\mathfrak{g}$ *of the Lie group* G is the set of all left invariant vector fields on the Lie group G. Let $e \in G$ denote the unit element. Then for any tangential vector $X_e \in T_eG$ there exists a uniquely defined left invariant vector field, namely

$$X(g) := (dL_g)_e X_e \text{ with } X(e) = X_e.$$

Argumentwise addition and multiplication with scalars $\lambda \in \mathbf{R}$ give $\mathfrak{g}$ the structure of an r-dimensional real vector space, $r = \dim G$, which is the dual of the space of the Maurer-Cartan forms. Clearly $\theta(X) =$ *constant* for any Maurer-Cartan form θ and any $X \in \mathfrak{g}$. Since the map $X \in \mathfrak{g} \mapsto X(e) \in T_eG$ is a linear isomorphism one often identifies $\mathfrak{g}$ with T_eG.

Considering the vector fields as differential operators, the commutator of two vector fields is defined by

$$[X,Y](f) := X(Y(f)) - Y(X(f)), \ f \in C_\infty(G); \tag{46}$$

it is a vector field again, and it is left invariant if X, Y have this property. The *commutator* is a bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ being *anti-commutative*

$$[X,Y] + [Y,X] = 0 \tag{47}$$

and satisfying the *Jacobi identity*:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0. \tag{48}$$

Generally, a vector space $\mathfrak{g}$ over a commutative field K, on which a commutator operation $[,]$ is defined, is called a *Lie algebra*. A trivial example one obtains setting $[X,Y] := 0$ for all $X, Y \in \mathfrak{g}$; such a Lie algebra is named an *Abelian Lie algebra*. Another interesting example is the infinite dimensional Lie algebra of all vector fields of class C_∞ on a differentiable manifold M^n with the commutator defined by (46). Later on we will describe the Lie algebras of the pseudo-orthogonal groups, in particular, the Möbius group..

The following fundamental theorem summarizes the main relations between Lie groups and Lie algebras (see C. Chevalley [6], S. Helgason [9]):

Theorem 6. *The map* $G \mapsto \mathfrak{g}$, $\phi \in \mathrm{Hom}(G,H) \mapsto d\phi \in \mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ *is a covariant functor of the category of Lie groups with the continuous homomorphisms as morphisms into the category of the finite dimensional real Lie*

[2] The matter necessary for our purposes is contained in [30]. For a profound treatmaent see S. Helgason [9], where numerous hints to the literatur about Lie theory can be found.

algebras. This functor maps Lie subgroups to Lie subalgebras of the same dimension, and normal Lie subgroups to ideals of the corresponding Lie algebras. Lie groups with isomorphic Lie algebras are locally isomorphic. To any finite dimensional real Lie algebra there exists up to isomorphism exactly one connected, simply connected Lie group whose Lie algebra is isomorphic to the given one. □

We remark that a continuous homomorphism of Lie groups is always analytic, thus its differential is defined; it maps left invariant fields to left invariant fields. To characterize $d\phi$ it suffices to know the differential at the unity $d\phi_e$.

Now we can define the *structure form* ω of the Lie group G as the one-Form on G with values in the Lie algebra $\mathfrak{g}$ of G generalizing (3):

$$\omega : \mathfrak{t} \in T_g G \longmapsto \omega(\mathfrak{t}) := dL_g^{-1}\mathfrak{t} \in T_e G = \mathfrak{g},$$

or shortly $\omega = g^{-1}dg$. Again ω is left invariant. The rule (6) generalizes to

$$R_a^* \omega = \mathrm{Ad}(a^{-1}) \circ \omega,$$

where $\mathrm{Ad}(a) := dL_a \circ dR_{a^{-1}}$ denotes the *adjoint representation* of the Lie group (see also (51) below).

Let(X_i), $i = 1, \ldots, r = \dim \mathfrak{g}$, be a basis of the Lie algebra $\mathfrak{g}$. Decomposing the commutator

$$[X_i, X_j] = X_k C_{ij}^k, \ i, j, k = 1, \ldots, r,$$

we obtain the *structure constants* of the Lie algebra $\mathfrak{g}$, being the coordinates of a twofold covariant, onefold contravariant tensor defining the Lie algebra structure and, by Theorem 6, locally also the Lie group. Let (ω^i) be the dual basis for the Maurer-Cartan forms:

$$\omega^i(X_j) = \delta_j^i.$$

For any Maurer-Cartan form θ and any left invariant vector field $X \in \mathfrak{g}$ the value $\theta(X)$ is constant. Thus the general formula for the value of the exterior differential of a Pfaffian form on arbitrary vector fields X, Y (see [30], (II.4.51))

$$d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])$$

yields

$$d\omega^i(X_j, X_k) = -\omega^i(X_h C_{jk}^h) = -C_{jk}^i.$$

On the other hand, decomposing the exterior differentials $d\omega^i$ with respect to the natural basis $(\omega^j \wedge \omega^k)$, $j < k$, in the space of biforms, one obtains with the same constants

$$d\omega^i = -\frac{1}{2} C_{jk}^i \omega^j \wedge \omega^k = -\sum_{j<k} C_{jk}^i \omega^j \wedge \omega^k.$$

Comparing this with the structure equations (9) of the general linear group $GL(n,\mathbf{R})$ we recognize the deep meaning of these equations: they define the structure of the linear Lie groups. In the case that $G \subset GL(n,\mathbf{R})$ is not the full linear group, the forms ω^i_j restricted to G are Maurer-Cartan forms on G. In case $\dim G < n^2$ they are linearely dependent. Since exterior differentiation commutes with restriction of the forms, the structure equations (9) are valid for the forms restricted to G. Regarding the linear depedicies as we will do in the examples considered later one one obtaines the structure equations for G evaluating (9).

The main tool for relating the properties of a Lie group with those of the corresponding Lie algebra is the exponential, which we are going to define now. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G and $X \in \mathfrak{g}$ a left invariant vector field. Then the solution of the differential equation

$$\frac{dg}{dt} = X(g) \text{ with } g(0) = e$$

is uniquely defined and exists for all $t \in \mathbf{R}$; we denote it by $\exp tX$. By the left invariance of X the trajectory of the field $X(g)$ through the element $g \in G$ is given by $g(t) = g \exp tX$, and it follows

$$\exp(s+t)X = \exp sX \exp tX.$$

Corollary 7. *For any $X \in \mathfrak{g}$ the map $t \in \mathbf{R} \mapsto \exp tX \in G$ is an analytic homomorphism of the additive group $[\mathbf{R}, +]$ into the Lie group G, named a* one-parameter subgroup. *Every connected one-dimensional Lie subgroup of the Lie group G is such a one-parameter subgroup; thus it is Abelian.*□

The Lie subalgebra corresponding to the one-parameter subgroup $\exp tX$ is the vector subspace of $\mathfrak{g}$ spanned by X; it is an Abelian subalgebra. Now we define the *exponential* exp *of the Lie group G* as the map

$$\exp : X \in \mathfrak{g} \longmapsto \exp X := \exp tX|_{t=1} \in G. \tag{49}$$

We prove

Lemma 8. *The map $\exp : \mathfrak{g} \to G$ is analytic. There exists a neighbourhood U of the zero element $0 \in \mathfrak{g}$ such that $\exp|U$ is an analytic diffeomorphism of U onto the neighbourhood $\exp(U) \subset G$ of the unity $e \in G$.*

P r o o f. The solution of the differential equation depends analytically on the parameter X. For the differential of exp at the zero element 0 one obtains

$$(d\exp)_0(X) = \frac{d \exp tX}{dt}|_{t=0} = X,$$

and it follows $d\exp_0 = \mathrm{id}_{\mathfrak{g}}$. By the inverse function theorem the map exp is invertible in a certain neighbourhood U of 0.□

For any $a \in G$ the *inner automorphism* σ_a is defined by

$$\sigma_a(g) := a \cdot g \cdot a^{-1},\ g \in G. \tag{50}$$

Obviously, one has $\sigma_a = L_a \circ R_{a^{-1}}$. By Theorem 6 the differential $d\sigma_a$ is an automorphism of the Lie algebra $\mathfrak{g}$. One easily proves

Corollary 9. *For any Lie group G with Lie algebra $\mathfrak{g}$ the map*

$$\mathrm{Ad} : a \in G \longmapsto \mathrm{Ad}(a) := d\sigma_a \in \mathrm{Aut}(\mathfrak{g}) \tag{51}$$

is a representation of G as a group of automorphisms of the Lie algebra $\mathfrak{g}$, called the adjoint representation *of G.* □

Clearly the relation

$$\mathrm{Ad}(a \cdot b) = \mathrm{Ad}(a) \circ \mathrm{Ad}(b)$$

is fulfilled for all $a, b \in G$.

Example 3. Now we specialize to the case of linear Lie groups and Lie algebras, see e. g. H. Freudenthal, H. De Vries [8]. The general linear group consists of all real non-degenerate matrices

$$g \in GL(n, \mathbf{R}) \Longleftrightarrow g = (\gamma^i_j) \in M_n(\mathbf{R}) \text{ with } \det(\gamma^i_j) \neq 0;$$

thus it is an open set in the real vector space $M_n(\mathbf{R})$ of real square matrices of order n with coordinates γ^i_j; its dimension is n^2. The group operation is the usual matrix product. The corresponding Lie algebra is the vector space

$$\mathfrak{g}l(n, \mathbf{R}) = T_e GL(n, \mathbf{R}) \cong M_n(\mathbf{R}),$$

with the matrix commutator

$$[X, Y] := X \cdot Y - Y \cdot X. \tag{52}$$

For any $X \in \mathfrak{g}l(n, \mathbf{R})$ the infinite series of matrices

$$\mathrm{e}^{tX} := \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} \tag{53}$$

converges for all values of t. The well known formula

$$\det(\mathrm{e}^X) = \mathrm{e}^{\mathrm{tr}\, X}, \tag{54}$$

where $\mathrm{tr}\, X$ denotes the trace of the matrix X, implies $\det(\mathrm{e}^X) \neq 0$, and deriving (53) one gets

$$\frac{d(\mathrm{e}^{tX})}{dt} = \mathrm{e}^{tX} \cdot X = X \cdot \mathrm{e}^{tX} \,. \tag{55}$$

Therefore e^{tX} is the trajectory of the left invariant vector field $X(g) = g \cdot X$ through the unity $\mathrm{e}^{0X} = e = (\delta^i_j)$ and coincides with the exponential defined for general Lie groups. The differentials $d\gamma^i_j$ form a cobasis at the point $e = (\delta^i_j)$; one has

$$d\gamma^i_j(X) = \xi^i_j \text{ for } X = (\xi^k_l) \in M_n(\mathbf{R}) = T_e GL(n, \mathbf{R}),$$

thus the corresponding basis of $T_e GL(n, \mathbf{R})$ consists of all matrices

$$X^k_l = (\delta^i_l \delta^k_j)_{i,j},$$

which have the coordinates $\xi^k_l = 1$ and $\xi^i_j = 0$ if $i \neq k$ or $j \neq l$. The left translation of these vectors resp. forms give the basis of the Lie algebra resp. the Maurer-Cartan forms as fields on $GL(n, \mathbf{R})$. Since the group multiplication is generated by the bilinear matrix multiplication, the adjoint representation of $GL(n, \mathbf{R})$ can be expressed as a product of matrices

$$\mathrm{Ad}(a)(X) = a \cdot X \cdot a^{-1} \text{ with } a \in GL(n, \mathbf{R}), X \in \mathfrak{g}l(n, \mathbf{R}). \tag{56}$$

The next example treats the Lie algebras of some important classes of linear Lie groups $G \subset GL(n, \mathbf{R})$. □

Example 4. Now we consider the n-dimensional real vector space W^n with a distinguished scalar product, as in Example 2. By (21) the linear group $GO(n)$ is defined as the group of all linear transformations preserving the scalar product. Obviously, formulas (52) - (56) are valid for any linear group as a subgroup of $GL(n, \mathbf{R})$. Let X belong to the Lie algebra $\mathfrak{go}(n) = T_e GO(n)$. Then the corresponding one-parameter subgroup belongs to $GO(n)$, and by (21) we have

$$\langle \mathrm{e}^{tX}\, \mathfrak{x}, \mathrm{e}^{tX}\, \mathfrak{y} \rangle = \langle \mathfrak{x}, \mathfrak{y} \rangle \text{ for all } \mathfrak{x}, \mathfrak{y} \in W^n.$$

Deriving this equation we obtain applying (55)

$$\frac{d\langle \mathrm{e}^{tX}\, \mathfrak{x}, \mathrm{e}^{tX}\, \mathfrak{y} \rangle}{dt} = \langle X\, \mathrm{e}^{tX}\, \mathfrak{x}, \mathrm{e}^{tX}\, \mathfrak{y} \rangle + \langle \mathrm{e}^{tX}\, \mathfrak{x}, X\, \mathrm{e}^{tX}\, \mathfrak{y} \rangle = 0. \tag{57}$$

Setting $t = 0$ we get a linear equation for X:

$$\langle X\mathfrak{x}, \mathfrak{y} \rangle + \langle \mathfrak{x}, X\mathfrak{y} \rangle = 0 \text{ for all } \mathfrak{x}, \mathfrak{y} \in W^n. \tag{58}$$

Conversely, if an element $X \in \mathfrak{g}l(n, \mathbf{R})$ satisfies (58), one may apply it for the vectors $\mathrm{e}^{tX}\, \mathfrak{x}, \mathrm{e}^{tX}\, \mathfrak{y}$, and it follows (57). The scalar product is constant, and the one-parameter subgroup belongs to $GO(n)$. Equation (58) characterizes the Lie algebra $\mathfrak{g}o(n)$ of $GO(n)$. Considering the matrix (ξ^i_j) of X with respect to a basis $(\mathfrak{a}_i)$ of W^n:

$$X\mathfrak{a}_j = \mathfrak{a}_k \xi^k_j,$$

we obtain a system of homogeneous linear equations defining the Lie subalgebra $\mathfrak{g}o(n) \subset \mathfrak{g}l(n, \mathbf{R})$:

$$\epsilon_{kj}\xi^k_i + \epsilon_{ik}\xi^k_j = 0, \tag{59}$$

where $(\epsilon_{kl}) = gr$ denotes the Gram matrix (20). The system (59) formally coincides with (23) for the Maurer-Cartan forms of $GO(n)$. This has to be expected: by definition (3) the structure form is a 1-form on $GO(n)$ with values in the Lie algebra $\mathfrak{g}o(n)$. If W^n is the Euclidean vector space and $(\mathfrak{a}_i)$ an orthonormal basis, gr is the unit matrix; formula (59) shows that the orthogonal Lie algebra $\mathfrak{o}(n)$ is represented as the space of all skew symmetric matrices of order n, the forms $(\omega^i_j)_{i<j}$, are a basis of the Maurer-Cartan forms of the orthogonal group $O(n)$. The structure equations (9) remain valid; on $O(n)$ one has to take into account the skew symmetry $\omega^i_j = -\omega^j_i$ implying $\omega^i_i = 0$ for all $i = 0, \ldots, n$. Analogue considerations are valid for the pseudo-orthogonal groups $O(k, n-k)$; formula (29) can be interpreted as the characterization of the Lie algebra $\mathfrak{o}(k, n-k)$. Finally, formula (17) gives the matrix representation of the affine Lie algebra $\mathfrak{a}(n)$, and, if the corresponding symmetry relations (29) of the part (ω^i_j) of the matrix are taken into account, also the Lie- algebras of the Euclidean resp. pseudo-Euclidean groups. □

1.3.5 Linear Isotropy Representations

Let the Lie group G act smoothly on the differentiable manifold M^n. Then the geometrical properties of the transformation group $[G, M^n]$ depends essentially on the linear isotropy representations:

Definition 3. Let $[G, M]$ be a Lie transformation group and

$$H_a = \{g \in G | g(a) = a\}$$

the *isotropy group* of the point $a \in M$. Then the map

$$\phi_a : h \in H_a \longmapsto \phi_a(h) := (dl_h)_a \in \boldsymbol{GL}(T_aM) \tag{60}$$

is a linear representation of H_a, named the *linear isotropy representation* of the transformation group $[G, M]$ at $a \in M$. □

Clearly, if $b = ga$ lies on the same G-orbit, the isotropy groups are isomorphic: $H_b = gH_ag^{-1}$, and the same is true for the isotropy representations:

$$\phi_b(h) = (dl_g)_a \circ \phi_a(g^{-1}hg) \circ (dl_g)^{-1}_a, \; h \in H_b, b = ga \in M.$$

Especially, all the linear isotropy representations ϕ_a at the points of a homogeneous space $M = G/H$ are equivalent, and one speaks about the linear isotropy representation of this space. One easily shows that the G-invariant tensor fields on a homogeneous space M are defined by the H_a-invariant tensors of the linear isotropy representation. We remember the following notations: If t is a tensor on the vector space V and $h : V \to W$ a linear isomorphism, the image h_*t is a well defined tensor on W. If $h : M^n \to L^n$ is a diffeomorphism and $t : x \in M^n \mapsto t(x)$ is a tensor field on M^n, then the image h_*t is defined as the tensor field on L^n by

$$h_*t : y \in L^n \longmapsto h_*t(y) := (dh_x)_*t(x), \text{ where } x = h^{-1}(y). \qquad (61)$$

A tensor field t on the space M^n of a transformation group $[G, M^n]$ is G-*invariant*, if $g_*t = t$ is true for any transformation $g \in G$.

Lemma 10. *Let $M^n \cong G/H_a$ be a homogeneous space. Then there exists a G-invariant tensor field $t = t(x)$ over M^n if there is a tensor t_a of the corresponding type at the origin $a \in M^n$ invariant under the linear isotropy representation: $(dh_a)_*t_a = t_a$ for all $h \in H_a$. If t_a is such a H_a-invariant tensor at the point a, then*

$$t : x = gH_a \in M^n \longmapsto t(x) := (dl_g)_{a*}t_a \qquad (62)$$

is the uniquely defined G-invariant tensor field on M^n with $t(a) = t_a$. □

Indeed, the invariance of t_a with respect to the linear isotropy representation implies that $t(x)$ does not depend on the transformation $g \in x = gH_a$ mapping a into x.

The basis forms ω^i play a central role in the differential geometry of homogeneous spaces. Therefore we return for the moment to the general theory considering the situation of the last subsection: Let

$$p : G \longrightarrow M^n = G/H, \; \omega = g^{-1}dg : TG \longrightarrow \mathfrak{g},$$

be the principal fibre bundle corresponding to the homogeneous space G/H of a Lie group G with Lie algebra $\mathfrak{g}$, $H = H_a$ the isotropy group of the point $a \in M^n$, ω the structure form on G. Let the Maurer-Cartan forms ω^i be a basis of the annihilator $\mathfrak{h}^\perp = [\omega^1, \dots, \omega^n]$ of the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the isotropy group H. The form

$$\theta := q \circ \omega : TG \longrightarrow \mathfrak{g}/\mathfrak{h}, \qquad (63)$$

where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the canonical projection, is named the *canonical form* of the homogeneous space G/H; the forms $\omega^i, i = 1, \dots, n$, are the components of the canonical form θ. Since $\omega^i(\mathfrak{h}) = 0$, the definition

$$\omega^i(X \bmod \mathfrak{h}) := \omega^i(X) \text{ for } X \bmod \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$$

yields a cobasis for the vector space $\mathfrak{g}/\mathfrak{h}$. The forms $\omega^i, i = 1, \dots n$, coincide with the *basis forms* of the homogeneous space $M^n = G/H$ defined in subsection 3. Using (51) we see that the Lie subalgebra $\mathfrak{h}$ remains invariant under the action of the adjoint representation of G restricted to H: $\mathrm{Ad}(h)\mathfrak{h} = \mathfrak{h}$ for all $h \in H$; therefore formula

$$\dot{\mathrm{Ad}}(h)(X \bmod \mathfrak{h}) := \mathrm{Ad}(h)(X) \bmod \mathfrak{h},\ h \in H, X \bmod \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}, \tag{64}$$

defines the *adjoint representation* $\dot{\mathrm{Ad}}$ of H on $\mathfrak{g}/\mathfrak{h}$. This definition implies

$$\dot{\mathrm{Ad}}(h) \circ q = q \circ \mathrm{Ad}(h),\ h \in H, \tag{65}$$

and equation (6) yields immediately: *The canonical form θ is a left invariant 1-form of type $\dot{\mathrm{Ad}}$ on G with values in $\mathfrak{g}/\mathfrak{h}$.* The latter means

$$R_h^{-1*}(\theta) = \dot{Ad}(h) \circ \theta \text{ for all } h \in H. \tag{66}$$

The geometrical meaning of the adjoint representation $\dot{\mathrm{Ad}}$ lies in the fact that it may serve as an algebraic model of the linear isotropy representation:

Lemma 11. *The linear isotropy representation of a homogeneous space G/H is equivalent to the adjoint representation $\dot{\mathrm{Ad}}$ of the isotropy group H on the factor space $\mathfrak{g}/\mathfrak{h}$.*

Proof. We identify $\mathfrak{g} = T_eG$. The differential dp_e of the projection $p : G \to G/H$ at the point e has the kernel $\mathfrak{h} = T_eH$, being the tangential space of the fibre $eH = H$ at e. Thus we have an uniquely defined linear isomorphism $\hat{p} : \mathfrak{g}/\mathfrak{h} \to T_aM$, $M = G/H$, satisfying

$$\hat{p}(X \bmod H) = dp_e(X),\ X \in \mathfrak{g}.$$

From this relation and (64) it follows for $h \in H$:

$$\hat{p}(\dot{Ad}(h)(X \bmod \mathfrak{h})) = \hat{p}(Ad(h)(X) \bmod \mathfrak{h}) = dp_e(Ad(h)(X)).$$

Now we use the definition (51) of the adjoint representation

$$\mathrm{Ad}(g) = dL_g \circ (dR_g^{-1})_e$$

and the obvious relations

$$p \circ R_h = p \text{ for } h \in H,\ p \circ L_g = l_g \circ p \text{ for } g \in G, \tag{67}$$

and obtain at the origin $a = p(e)$:

$$dp_e(Ad(h)(X)) = dp_e \circ dL_h \circ dR_{h^{-1}}(X) = dl_h(dp_e(X)) = dl_h(\hat{p}(X \bmod \mathfrak{h})),$$

what proves the lemma. □

1.3.6 Applications to Möbius Geometry

Again we return to Möbius geometry, $G = M(n), M^n = S^n$. At the end of subsection 2 we pointed out that the Maurer-Cartan forms $\omega^i := \omega_0^i$, $i = 1, \ldots, n$, span the annihilator of the isotropy algebra $\mathfrak{h}_a$; thus they are basis forms and the components of the canonical form θ of the Möbius space. The matrix of the elements $h \in H_a$ of the isotropy group are given by (1.11); one easily calculates

$$h(A, \mathfrak{a}, \lambda)^{-1} = h(A^{-1}, -A^{-1}\mathfrak{a}\lambda^{-1}, \lambda^{-1}). \tag{68}$$

For the structure form of the linear group one proves using matrix multiplication: Is $(\omega_j^i) \in \mathfrak{gl}(N, \mathbf{R})$ the matrix of an element of the linear Lie algebra and $(\gamma_k^i) \in GL(n, \mathbf{R})$ a group element g, then the matrix of $\mathrm{Ad}(g)(\omega)$ is

$$\mathrm{Ad}(g)(\omega) \doteq (\hat{\omega}_j^i) := (\gamma_k^i)(\omega_l^k)(\gamma_j^l)^{-1}. \tag{69}$$

Applying this and (68) for $g = h(A, \mathfrak{a}, \lambda)$ one obtains for the basis forms the transformation rule

$$\hat{\omega}^i = \gamma_k^i \omega^k \lambda \text{ with } A = (\gamma_k^i) \in O(n). \tag{70}$$

Since the representation $\dot{\mathrm{A}}\mathrm{d}$ is equivalent to the linear isotropy representation, it follows from Lemma 11:

Proposition 12. *The transformation rule of the basis forms of the Möbius space is given by (70). Therefore the symmetric bilinear form*

$$\varphi(\mathfrak{s}, \mathfrak{t}) := \sum_{i=1}^{n} \omega^i(\mathfrak{s})\omega^i(\mathfrak{t}),\ \mathfrak{s}, \mathfrak{t} \in \mathfrak{g}/\mathfrak{h}, \tag{71}$$

transforms under the action of $h = h(A, \mathfrak{a}, \lambda) \in H_a$ obeying the rule

$$\varphi(\dot{\mathrm{A}}\mathrm{d}(h)(\mathfrak{s}), \dot{\mathrm{A}}\mathrm{d}(h)(\mathfrak{t})) = \varphi(\mathfrak{s}, \mathfrak{t})\lambda^2 \tag{72}$$

The linear isotropy representation of the Möbius space has the kernel

$$K := \{h(I_n, \mathfrak{a}, 1) \in H_a | \mathfrak{a} \in \mathbf{R}^n\} \tag{73}$$

being isomorphic to the abelian group $[\mathbf{R}^n, +]$.

Proof. The transformation rule (72) immediately follows from (70) since A is an orthogonal matrix. Also by (70) it results that the dual basis (ω^i) of $\mathfrak{g}/\mathfrak{h}$ remains fixed iff A is the unit matrix and λ equals 1. The last statement follows from (1.11) by a direct calculation. □

From Proposition 12 and Lemma 10 the existence of a positive definite scalar product invariantly defined up to a positive factor on the Möbius space can be concluded: Obviously, such a scalar product φ_a exists on the tangential space T_aS^n at the origin $a = p(e)$, invariantly up to a positive factor under the action of the linear isotropy representation. Then the definition

$$\varphi_x(\mathfrak{s}, \mathfrak{t}) := \varphi_a(dl_g^{-1}(\mathfrak{s}), dl_g^{-1}(\mathfrak{t})) \text{ if } x = ga,\ \mathfrak{s}, \mathfrak{t} \in T_xS^n,$$

extends φ_a to a scalar product with the mentioned properties over S^n, thus defining a class of conformally equivalent Riemannian metrics on the n-sphere. This definition is a special case of (61).

We shall give another proof of this fact using the method of moving frames. This method has the advantage that the scalar product is given by an expression appropriate for explicit calculations, and basic for the Möbius differential geometry. Let $U \subset S^n$ be an open subset, and let $s(x) = (\mathfrak{c}_i(x)) \in M(n)$, $\hat{s}(x) = (\hat{\mathfrak{c}}_i(x))$, $x \in U$ be two moving frames defined on U. Then $h = \hat{s}^{-1}s \in H_a$ is an element of the isotropy group depending on $x \in U$, and we have the transformation rule of the moving frames

$$\hat{s}(x) = s(x).h^{-1}(x),\ x \in U, h(x) \in H_a. \tag{74}$$

We consider the local basis forms $\sigma^i = s^*\omega^i$, $\hat{\sigma^i} = \hat{s}^*\omega^i$, satisfying

$$d\mathfrak{c}_0 = \mathfrak{c}_i\sigma^i$$

and the corresponding equation for the differential $d\hat{\mathfrak{c}}_0$ on U. From (1.11) and (68) we get

$$\hat{\mathfrak{c}}_0 = \mathfrak{c}_0\lambda, \hat{\mathfrak{c}}_i = \mathfrak{c}_0\gamma_i^0 + \mathfrak{c}_j\gamma_i^j \text{ with } (\gamma_i^j) \in O(n).$$

Deriving the first equation we obtain from

$$d\hat{\mathfrak{c}}_0 = d\mathfrak{c}_0\lambda + \mathfrak{c}_0 d\lambda$$

for the transformed coefficients

$$\hat{\sigma}^i = \langle d\hat{\mathfrak{c}}_0, \hat{\mathfrak{c}}_i\rangle = \sigma^j\gamma_j^i\lambda \text{ with } (\gamma_i^j) \in O(n). \tag{75}$$

It follows

Proposition 13. *For any local isotropic-orthonormal moving frame on an open set $U \subset S^n$ with the basis forms $\sigma^i = \langle d\mathfrak{c}_0, \mathfrak{c}_i\rangle$ the bilinear form*

$$\varphi(\mathfrak{s}, \mathfrak{t}) := \langle d\mathfrak{c}_0, d\mathfrak{c}_0\rangle = \sum_{i=1}^{n} \sigma^i(\mathfrak{s})\sigma^i(\mathfrak{t}),\ \mathfrak{s}, \mathfrak{t} \in T_xS^n,$$

defines a positive definite scalar product on the tangential spaces T_xS^n up to a factor $\lambda^2 > 0$, depending on the choice of the frame $s(x) \in p^{-1}(x)$. Thus the angle α of the tangential vectors is correctly defined in the usual way:

$$\cos\alpha := \frac{\varphi(\mathfrak{s}, \mathfrak{t})}{|\mathfrak{s}||\mathfrak{t}|} \text{ with } |\mathfrak{s}| := \sqrt{\varphi(\mathfrak{s}, \mathfrak{s})}. \tag{76}$$

□

Usually one speaks about the *conformal geometry* of a Riemannian space as the theory of the properties invariant under arbitrary renormings of the Riemannian metric. In particular, if one takes the local moving frames with the properties $\mathfrak{c}_0 = \mathfrak{x} \in S_1^n$ and $\mathfrak{t}_i = \mathfrak{c}_i \in T_x S_1^n$, the induced Riemannian metric coincides with the standard Riemannian metric of constant curvature 1 on the unit n-sphere. By (1.11) and (75) it is clear that any positive renorming function on S^n may be realized by an appropriate choice of the moving frame $g = g(x)$. Therefore the conformal properties of the standard metric of the sphere S^n are also Möbius invariant properties. Clearly, not every renorming of the standard metric may be obtained by a Möbius transformation, depending on a finite number of parameters. It must be expected that the Möbius geometry as the theory of properties invariant under the action of the Möbius group is more special than the conformal geometry of the sphere.

By a theorem of A. Lichnérowicz, see § 30, p. 47 in [15], the linear isotropy representation is true, i.e. its kernel K is trivial, if on the homogeneous space $M^n = G/H$ with an effective action of G there exists a G-invariant linear connection. Thus we conclude:

Corollary 14. *On the Möbius space $S^n = M(n)/H_a$ does not exist neither a Möbius-invariant linear connection nor a Möbius-invariant pseudo-Riemannian structure.* □

Therefore the basis invariant of Möbius geometry is the class of Riemannian metrics Möbius equivalent to the standard metric on S^n, induced by its embedding as the unit sphere into the Euclidean space E^{n+1}.

1.3.7 Applications to the Space Forms E_c^n

In this subsection we calculate the structure forms and the canonical forms of the space forms E_c^n and relate them to the structure form of the Möbius group. To this aim we consider the bundles of orthonormal frames $O_c(n)$ over E_c^n and their embeddings into the Möbius group described in Proposition 2.2, (2.22), and (2.23). The vector $\mathfrak{c}_0 = \mathfrak{x}$ of the moving frame is uniquely defined by the embedings (2.3), (2.4), and (2.5) of the space forms. The $\mathfrak{c}_i, i = 1, \ldots, n$, form an orthonormal basis of the tangential space $T_{\mathfrak{x}} E_c^n$, and the complementary vector $\mathfrak{c}_{n+1}$ is defined by Proposition 2.2. The structure forms of the isometry groups of the space forms considered as subgroups of the Möbius group $M(n)$ are the restrictions of the structure form of $M(n)$ onto these subgroups. It follows in each of the cases

$$\omega^0 = -\langle d\mathfrak{x}, \mathfrak{c}_{n+1} \rangle = 0, \tag{77}$$

and from (42) we conclude

$$d\mathfrak{x} = \sum_{i=1}^{n} \mathfrak{c}_i \omega^i. \tag{78}$$

Since the isotropy group of the space forms is the orthogonal group $O(n)$ embedded into $M(n)$ by (24), we have $\lambda = 1$ in the transformation rules (70), (72), and (75). We conclude applying Proposition 12

Corollary 15. *On all n-dimensional space forms E^n_c the isotropy groups are isomorphic to the orthogonal group $O(n)$. The linear isotropy representations are the standard representation of $O(n)$ on the Euclidean vector space E^n; thus they are true. The bilinear form φ defined by (71), restricted to the bundle of orthonormal frames of the space forms E^n_c, defines an $O_c(n)$-invariant Riemannian metric on these spaces.* □

Now, differentiating the vectors $\mathfrak{c}_{n+1}$ defined by (23), we calculate the forms in the block τ of the restricted structure form (37) of $M(n)$:

$$\omega^0_i = \omega^i_{n+1} = \langle \mathfrak{c}_i, d\mathfrak{c}_{n+1} \rangle = -c\omega^i/2, \tag{79}$$

where $c = r^{-2}$, $c = -r^{-2}$, and $c = 0$ are the constant curvatures of the spherical, hyperbolical resp. Euclidean spaces E^n_c. Inserting (77) and (79) into the structure equations (38) - (41) of the Möbius group, one proves

Proposition 16. *The structure form of $M(n)$ restricted to the isometry group $O_c(n)$, $c = r^{-2}$, $c = -r^{-2}$, or $c = 0$, has the block matrix*

$$\begin{pmatrix} 0 & -(\omega^i c/2)' & 0 \\ (\omega^i) & (\omega^i_j) & -(\omega^i c/2) \\ 0 & (\omega^i)' & 0 \end{pmatrix}, \quad \text{with } \omega^i_j + \omega^j_i = 0,\ i,j = 1,\ldots,n. \tag{80}$$

The canonical form is given by (78) with $\theta = d\mathfrak{x}$. The structure equations of the spaces of constant curvature E^n_c *may be written in the following shape:*

$$d\omega^i + \omega^i_j \wedge \omega^j = 0, \tag{81}$$

$$d\omega^i_j + \omega^i_k \wedge \omega^k_j = c\omega^i \wedge \omega^j. \tag{82}$$

P r o o f. Formula (80) follows from (77) and (79). Inserting these equations into the structure equations of the Möbius group (38)-(41) one sees: (38) reduces to the trivial equation $0 = 0$, (81) and (82) result from (39) and (40), and (41) is equivalent to (81) if $c \neq 0$, and trivial else. Since the structure form of $O_c(n)$ is the restriction of that of $M(n)$ and exterior differentiation commutes with restrictions, Proposition 16 is proved. □

In difference to the Möbius geometry the differential geometry of the spaces of constant curvature is a special Riemannian geometry, and the structure equations (81), (82) are special cases of E. Cartan's structure equations of a Riemannian manifold, or more generally, a manifold with a linear connection. In particular, the left side of (82) are the components Ω^i_j of the *curvature form* of the linear connection, see e.g. S. Kobayashi, K. Nomizu [14], or R. Sulanke,P. Wintgen [30], Satz II.10.4, p.160:

$$\Omega^i_j := d\omega^i_j + \omega^i_k \wedge \omega^k_j = \frac{1}{2} R^i_{jkl} \omega^i \wedge \omega^j,\ R^i_{jkl} + R^i_{jlk} = 0. \tag{83}$$

The functions R^i_{jkl} are the coordinates of a tensor, the *curvature tensor* of the space. In case of the spaces of constant curvature from (82) it follows $R^i_{jkl} = \delta^i_j \delta_{kl} c$, the coordinates of the curvature tensor are zero or the same constant c, or, what is the same, the sectional curvature of these Riemannian spaces is constant. For details see the cited literature. E. Cartan's exposition of the Riemannian geometry is contained in the book [5], edited by S. P. Finikov in Russian language. The monography [37] of J. A. Wolf contains a detailed study of the spaces of constant curvature and their generalizations.

2

The Möbius Structure of an Immersion

2.1 E. Cartan's Method of Moving Frames

In the following we want to apply E. Cartan's method of moving frames [4] to the Möbius geometry of immersions $f : Y^m \to S^n$, $m = 1, \ldots, n-1$, into the sphere S^n. In general, two immersions $f_i : Y_i^m \to M^n := G/H$, $i = 1, 2$, are *G-equivalent*, if there exist a diffeomorphismus $\varphi : Y_1 \to Y_2$ and a transformation $g \in G$ such that $f_2 = g \circ f_1 \circ \varphi^{-1}$. Often one identifies the manifolds $Y_1 = Y_2 =: Y$ with the help of φ and assumes $\varphi = \mathrm{id}_Y$; in this case one speaks about the *G-congruence* of the immersions $f_i : Y^m \to M^n$, $i = 1, 2$. Clearly, these definitions yield equivalence relations in the classes of immersions under consideration. The first task is to characterize the equivalence classes by G-invariant geometric objects, like functions, tensor fields, or connections, defined on the manifolds Y^m. We do not want, and do not need here, to explain the theory of geometric objects in full generality; a very nice and contemporary presentation of this theory, appropriate for our purposes, is contained in the article [22] of S. E. Salvioli. Let us illustrate the defining property of a *geometric object*, more precisely, a geometric object field, with the example of a tensor field. A *tensor field* on the manifold Y^m is a section of a tensor bundle over Y^m:

$$y \in Y^m \longmapsto t(y) \in \mathfrak{T}^{r,s}(Y^m) \text{ with } p(t(y)) = y, \ (y \in Y^m).$$

Here $p : \mathfrak{T}^{r,s}(Y^m) \to Y^m$ denotes the projection in the bundle of r-times contravariant, s-times covariant tensors over Y^m. Examples of tensor fields are functions ($r = s = 0$), vector fields ($r = 1, s = 0$), Pfaffian forms ($r = 0, s = 1$), Riemannian metrics ($r = 0, s = 2$), curvature tensors, etc. Let $\mathcal{F}_m$ denote a class of immersions of m-dimensional manifolds $f : Y^m \to M^n$ into the homogeneous space M^n closed under diffeomorphisms and the action of G, $f \mapsto g \circ f, (g \in G)$. A map $f \in \mathcal{F}_m \mapsto t_f \in \Gamma(\mathfrak{T}^{r,s}(Y^m))$, where $\Gamma(\mathfrak{T}^{r,s}(Y^m))$ denotes the space of sections of the tensor bundle, is a *G-invariant*, if it is invariant under the action of G:

$$t_{g\circ f} = t_f \text{ for all } f \in \mathcal{F}_m,\ g \in G.$$

and equivariant under the action of diffeomorphisms $\varphi : Y^m \to \hat{Y}^m$:

$$t_{\hat{f}}(\hat{y}) = d\varphi_*(t_f(\varphi^{-1}(\hat{y}))),\ (\hat{f} = f \circ \varphi^{-1},\ \hat{y} \in \hat{Y}^m).$$

Here $d\varphi_*$ denotes the morphism of tensor bundles generated by the differential of φ. All manifolds and maps are supposed to be sufficiently often differentiable. The concepts of differentiable manifolds, tensor bundles, transformation groups etc. can be found in [30], in S. Kobayashi, K. Nomizu [14], [13], or other textbooks of differential geometry.

The main ideas of E. Cartan‘s method of moving frames [4] can be described as follows. Consider an immersion $f : Y^m \to M^n = G/H$ of an m-dimensional smooth manifold into the n-dimensional homogeneous space M^n, $0 < m < n$. Realize the group space G as a *space of moving frames*, as being done in chapter 1 for the Möbius group and the isometry groups of the space forms E^n_c. Let $p : G \to G/H$ be the canonical projection. Construct the *induced frame bundle* with the bundle space as the disjoint union

$$F_0(f) := \bigcup_{y\in Y^m} p^{-1}(f(y))). \tag{1}$$

In a canonical way, $F_0 = F_0(f)$ is provided with the structure of a principal fibre bundle $F_0(Y^m, p, H)$ over Y^m with structure group H and a bundle morphism $\hat{f}$ such that the following diagram is commutative:

$$\begin{array}{ccc} F_0 & \xrightarrow{\hat{f}} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Y^m & \xrightarrow{f} & M^n \end{array} \tag{2}$$

If f is injective, then also $\hat{f}$ is injective, and F_o can be identified with the subset $\{g \in G | p(g) \in f(Y^m)\}$. If the group space G is interpreted as a bundle of frames, then the fibre $p^{-1}(y)) \subset F_0$ at the point $y \in Y^m$ consists of all the frames at the point $f(y)$. The action of the isotropy group H on the fibres is preserved by this construction; $\hat{f}$ as a bundle morphism fulfils (2) and

$$R_h \circ \hat{f} = \hat{f} \circ R_h,\ h \in H. \tag{3}$$

Furthermore, the structure form ω on G induces by

$$\omega_0 := \hat{f}^*\omega := \omega \circ d\hat{f} \tag{4}$$

a 1-form on F_0 of typ $\mathrm{Ad}\,|_H$ with values in the Lie algebra $\mathfrak{g}$ of G, named *the structure form of order* 0 *of the immersion* f. In the paper [25] the structure just described is precised and subsumed under the concept of a G, H*-structure* $[F_0(Y^m, p, H), \omega_0]$.

Clearly, G-equivalent immersions and, a fortiori, G-congruent immersions induce isomorphic G, H-structures. On the other hand, in [29] we proved that to each G, H-structure on a connected manifold Y^m there exists an immersion of a certain covering manifold $\tilde{Y}^m$ of Y^m, $f : \tilde{Y}^m \to G/H$, whose induced G, H-structure is locally isomorphic to the given one; this immersion is uniquely defined up to G-congruence. Therefore, locally or for simply connected Y^m, the G-equivalence classes of immersions correspond bijectively to the isomorphy classes of G, H-structures, and the classification can be done studying the latter. E. Cartan's method of moving frames consists of a step by step adaption of the frames to the immersion, leading finally, in the most convenient case, to a uniquely defined *canonical frame*

$$s : y \in Y^m \longmapsto s(y) \in G \text{ with } p(s(y)) = y. \tag{5}$$

The canonical frames must be defined in a G-equivariant way, meaning that G-congruent immersions possess G-congruent canonical frames. Then the coefficients of the induced form $s * \omega_0 : TY^m \to \mathfrak{g}$ may serve as a complete system of invariants for the classification of the immersions under consideration. Unfortunately, in general there exist canonical frames only for special subclasses of immersions into G/H, and one has to find such subclasses. Often the adaption of frames may be carried out by successive natural reductions $F_{i+1} \subset F_i \subset F_0$ of the G, H-structure F_0, leading to smaller structure groups $H_0 = H \supset H_1 \ldots \supset H_k = \{e\}$, in case of a canonical frame. This reduction procedure has been carried out by E. Cartan in many special cases. The general theory and therewith the justification of the method has been given independentla by G. R. Jensen [11] and R. Sulanke, A. Švec [29], see also [25] for a global result. Another method to find G-invariants of the immersion is the following: Consider *local sections* of the induced frame bundle F_i defined on open subsets $U \subset Y^m$

$$s : y \in U \longmapsto s(y) \in F_i \text{ with } p(s(y)) = y, \tag{6}$$

and study the induced form $s^*\omega : TY^m|U \to \mathfrak{g}$. The components of $s^*\omega$ are 1-forms on U, and their transformation rules under a change of the local frames can be calculated using the adjoint representation of the group G restricted to the structure group H_i: If $\hat{s}(y) = s(y)h^{-1}(y)$ is the transformation of the local frames, we get

$$\begin{aligned} \hat{s}^*\omega &= \omega(d\hat{s}_y) \\ &= \omega(ds_y) \circ dR_{h^{-1}(y)} + \omega(s(y))(dh^{-1})_y \\ &= \mathrm{Ad}(h)(s^*\omega - h^{-1}dh_y), \end{aligned} \tag{7}$$

see formulas (3.9), (3.12) in [29]. Now, one often may construct G-invariant geometric objects using this transformation rule, and express them in terms of the components of the structure form. Then, if sufficiently many invariants are found, one conversely may construct the G, H-structure from these invariants; applying the immersion theorem in [29] one gets an existence and uniqueness theorem for the class of immersions under consideration.

The main task in solving the classification problems is the step-by-step reduction of the induced frame bundles. One has to describe the orbit structure of non-transitive transformation groups naturally related to the given class of immersions. Unfortunately, we don't have a general method to handle this task. Thus, the classification of immersions into a given homogeneous space is an interesting and, as a rule, not so simple task. E. Cartan's method of moving frames not only gives a general framework to attack these problems, but also yields, by the general immersion theorem, an existence and uniqueness theorem almost automatically. Its principal theoretical meaning consists of the fact that it bases the differential geometry of an homogeneous space $M^n = G/H$ on the structure of the pair of Lie groups $G \supset H$ defining the space. In the following I shall apply and illustrate this method in the differential geometry of the Möbius space. I have chosen just this space since 1. there does not exist an invariant metric nor an invariant linear connection – thus the classical methods of differential geometry, like absolute differentiation, are not applicable – and 2. the Möbius geometry is nearly related to the classical geometries of constant curvature, as pointed out in chapter 1, what gives the possibility to interpret the Möbius invariants in metric, in particular Euclidean terms, in this way investigating their geometrical meaning.

2.2 The Möbius Structure of an Immersion

In this section we consider immersions $f : Y^m \to S^n = M(n)/H_a$, $0 < m < n$, into the Möbius space. We use the denotations defined in chapter 1; $M(n)$ is the Möbius group, whose group space is identified with the manifold of all isotropic-orthonormal frames; $H_a \subset M(n)$ is the isotropy group of the *origin*, the point $a = [\mathfrak{a}_0]$, and $(\mathfrak{a}_i)$ a fixed isotropic-orthonormal basis of the pseudo-Euclidean vector space V^{n+2} of index 1, in which S^n is represented as the set of the 1-dimensional isotropic subspaces. Usually we will work with the vector model of the Möbius space, see subsection 1.1.2, and identify $S^n \equiv S^n_1$. Following the paper Ch. Schiemangk, R. Sulanke [23] we carry out the first reduction steps of the induced frame bundle of the immersion leading to the concept of its Möbius structure, common for all dimensions $m > 1$. The curves will be treated in the next section.

The *frame bundle of order* 0 of the immersion f is defined by (1.1) with $G = M(n)$. It is immersed into the group space by the bundle morphism $\hat{f}$ such that the commutative diagram (1.2) with $M^n = S^n = M(n)/H_a$ is valid. The fibre $p^{-1}(y)$ of F_0 at the point $y \in Y^m$ consists of all isotropic-orthonormal frames $(\mathfrak{c}_i) \in M(n)$ with the *basis point* $[\mathfrak{c}_0] = f(y)$. By (1.4) the structure form ω of the Möbius group induces the *structure form* ω_0 *of* *order* 0 *of the immersion* f with values in the Lie algebra $\mathfrak{m}(n) \cong \mathfrak{o}(1, n+1)$ of $M(n)$. Thus the $M(n), H_a$- structure $[F_0(Y^m, p, H_a), \omega_0]$ of the immersion f is constructed.

The *tangential space of the immersion* f *at the point* $y \in Y^m$,

$$T(y) := \operatorname{Im} df_y \subset T_{f(y)} S^n$$

is a well defined m-dimensional subspace; by Proposition 1.3.13 the *normal space of* f *at the point* $y \in Y^m$, $N(y) := T(y)^\perp$ is defined as its orthogonal complement, too. The bilinear form φ induces equally denoted forms on $T(y), N(y)$ defined up to a positive factor; thus the pullback $f^*\varphi$ of φ to the tangential bundle TY^m defines a Möbius invariant conformal class of Riemannian metrics on Y^m. The isotropy group H_a contains the orthogonal group $O(n)$ acting on $T_a S^n$ by the linear isotropy representation as in the n-dimensional Euclidean geometry, see formula (1.3.70). Therefore the following definition yields the *first order reduction* F_1 *of the frame bundle* F_0:

$$F_1(f) := \{z = (\mathfrak{c}_i) \in F_0 | dq_{\mathfrak{c}_0}(\mathfrak{c}_\alpha) \in T(p(z)) \text{ for } \alpha = 1, \ldots, m\} \tag{1}$$

see Corollary 1.1.5, $q : J \to S^n$ defined by (1.1.13). Since $p(z) = [\mathfrak{c}_0] = f(y)$ the notation $F_1(f)$ is justified; often we omit the argument f. The bundle F_1 is the subbundle of F_0 consisting of the isotropic-orthonormal frames adapted to the tangential bundle of the immersion f. The structure group of the principal fibre bundle is the subgroup $H_1 \subset H_a$ defined by

$$h(A, \mathfrak{a}, \lambda) \in H_1 \Longleftrightarrow A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ with } A_1 \in O(m),\ A_2 \in O(n-m), \tag{2}$$

cf. (1.1.11). Let again $p : F_1 \to Y^m$ denote the projection, and let $\hat{f}_1 := \hat{f}|F_1 : F_1 \to M(n)$ be the homomorphism of principal fibre bundles induced by $\hat{f}$; the corresponding homomorphism of the structure groups is the embedding $H_1 \hookrightarrow H_a$. Analogously to (1.4) we define the *first order structure form*:

$$\omega_1 := \hat{f}_1^* \omega = \omega \circ d\hat{f}_1 : TF_1 \to \mathfrak{g}, \tag{3}$$

being a 1-*form of type* $\operatorname{Ad}|H_1$ *with values in the Lie algebra* $\mathfrak{g}$. The latter means that the transformation rule

$$\omega_1 \circ dR_h^{-1} = \operatorname{Ad}(h) \circ \omega_1 = h\omega_1 h^{-1},\ h \in H_1, \tag{4}$$

is valid. The right side of the last equation is the matrix form of the adjoint representation for linear Lie groups. From the definitions it follows immediately that the following diagram is commutative:

$$\begin{array}{ccc} F_1 & \xrightarrow{\hat{f}_1} & M(n) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ Y^m & \xrightarrow{f} & S^n \end{array} \tag{5}$$

and formula

$$R_h \circ \hat{f}_1 = \hat{f}_1 \circ R_h,\ h \in H_1 \tag{6}$$

is fulfilled (see (1.2), (1.3)).

Thus the principal fibre bundle F_1 and the structure form ω_1 are the components of the $M(n), H_1$-structure $[F_1(Y^m, p, H_1), \omega_1]$, called *the first order frame bundle* of the immersion f. From definition (1) it follows immediately

$$F_1(g \circ f) = L_g F_1(f), \tag{7}$$

and by construction it is clear that equivalent immersions have isomorphic first order frame bundles.

The Maurer-Cartan forms

$$\omega^i, i = 1, \ldots, n,\ \omega^\kappa_\alpha, \alpha = 1, \ldots, m, \kappa = m+1, \ldots, n,$$

are a basis of the annihilator of the Lie subalgebra $\mathfrak{h}_1$ in the Lie algebra $\mathfrak{m}(n)$ of the Möbius group. The structure form $\omega_1 = \hat{f}_1^* \omega$ satisfies the same symmetry conditions (1.3.34-37) and the structure equations (1.3.38-41) as the structure form ω of $M(n)$. By abuse of notations we denote the corresponding forms induced on the frame bundle F_1 by the same letters. On F_1 we have $\omega^\kappa \equiv 0$, and the forms $\omega^\alpha, \alpha = 1, \ldots, m$, are a basis in the space of the forms annihilating the fibres $p^{-1}(y) \subset F_1,\ y \in Y^m$. Since also the forms ω^κ_α on F_1 annihilate these fibres, we obtain the *coefficients of first order* $c^\kappa_{\alpha\beta}$ as coordinates of ω^κ_α in the basis representation

$$\omega^\kappa_\alpha(z) = c^\kappa_{\alpha\beta}(z)\omega^\beta(z),\ z \in F_1. \tag{8}$$

Since $\omega^\kappa \equiv 0$ we get from (1.3.39)

$$0 \equiv d\omega^\kappa = -\omega^\kappa_\beta \wedge \omega^\beta = -c^\kappa_{\alpha\beta}\omega^\alpha \wedge \omega^\beta = -\sum_{1 \le \alpha < \beta \le m} (c^\kappa_{\alpha\beta} - c^\kappa_{\beta\alpha})\omega^\alpha \wedge \omega^\beta.$$

In the case of curves, $m = 1$, the last equation is trivial. We conclude the symmetry of the coefficients of first order:

$$c^\kappa_{\alpha\beta} = c^\kappa_{\beta\alpha} \text{ for } \alpha, \beta = 1, \ldots, m,\ \kappa = m+1, \ldots, n. \tag{9}$$

Summarizing we formulate

Proposition 1. *For any immersion $f : Y^m \to S^n$ of a manifold Y^m into the Möbius space S^n the equations (1), (3) define the frame bundle of first order F_1 and the structure form of first order ω_1 of the immersion. The bundle $F_1 = F_1(Y^m, H_1, p)$ is a principal fibre bundle over the basis space Y^m with structure group $H_1 \subset H$ defined by (2) and (1.1.11). By construction it is canonically immersed into the Möbius group $M(n)$ by the bundle morphism $\hat{f}_1$, such that (5) with $f \circ p = p \circ \hat{f}_1$ and (7) are valid. The structure form ω_1 is a* 1*-form on F_1 with values in the Lie algebra $\mathfrak{m}(n)$ of $M(n)$ of type* $\mathrm{Ad}\,|H_1$, *i.e. satisfying (4), such that* $[F_1(Y^m, H_1, p), \omega_1]$ *is a* $M(n), H_1$*-structure on* Y^m. *The* $M(n), H_1$*-structures of M(n)-equivalent immersions are isomorphic. The components* $\omega^\alpha, \alpha = 1, \ldots, m$ *are basis forms on* F_1, *one has* $\omega^\kappa = 0$, $\kappa = m+1, \ldots, n$, *and the forms* ω_α^κ *satisfy the symmetry relations (8),(9).* □

Applying Lemma 1.1.4 and Corollary 1.1.5 we conclude

Corollary 2. *Let* $q : J \to S_1^n$ *denote the projection (1.1.13) of the isotropic cone J onto the unit n-sphere and* $f : Y^m \to S_1^n$ *an immersion with the* $M(n), H_1$*-structure* $[F_1, \omega_1]$. *For any local section* $y \in U \mapsto s(y) = (\mathfrak{c}_i(y)) \in F_1$, $U \subset Y^m$ *an open set, one has:*

1. *For* $y = p(s(y)) \in U$ *is* $f(y) = q(\mathfrak{c}_0(y))$.
2. *For* $\alpha = 1, \ldots, m$ *the vectors* $\mathfrak{t}_\alpha := dq_{\mathfrak{c}_0}(\mathfrak{c}_\alpha)$ *are an orthogonal basis of the* tangential space $T_y := df_y(T_y Y^m) \subset T_{f(y)} S_1^n$ *of the immersion f at the point y.*
3. *The vectors* $\mathfrak{n}_\kappa := dq_{\mathfrak{c}_0}(\mathfrak{c}_\kappa)$, $\kappa = m+1, \ldots n$, *form an orthogonal basis of the* normal space $N_y := T_y^\perp \subset T_{f(y)} S_1^n$ *of the immersion f at y.*
4. *The metric*
$$\langle d\mathfrak{c}_0, d\mathfrak{c}_0 \rangle = \sum_{\alpha=1}^{m} (\omega^\alpha)^2$$
induced by the moving frame s(y) on U is conformal to the inner metric induced on Y^m *by the immersion f from the standard metric of* S_1^n. □

For the next reduction step we calculate the transformation rule of the coefficients of the first order $c_{\alpha\beta}^\kappa$ defined by (8). Let on the open set $U \subset Y^m$ be given two moving frames as local sections of the frame bundle F_1:

$$s(y) = (\mathfrak{c}_i(y)), \hat{s}(y) = (\hat{\mathfrak{c}}_i(y)), \hat{\mathfrak{c}}_i(y) = \mathfrak{c}_j(y) h_i^j(y), y \in U.$$

Since $h(y) = (h_i^j(y)) \in H_1$ belongs to the structure group of F_1, we may specify the transformation of the moving frames using (2) and (1.1.11) as follows:

$$\hat{\mathfrak{c}}_0 = \mathfrak{c}_0 \lambda^{-1}, \tag{10}$$
$$\hat{\mathfrak{c}}_\alpha = \mathfrak{c}_0 h_\alpha^0 + \mathfrak{c}_\beta h_\alpha^\beta, \alpha, \beta = 1, \ldots, m, (h_\alpha^\beta) \in \boldsymbol{O}(m), \tag{11}$$
$$\hat{\mathfrak{c}}_\kappa = \mathfrak{c}_0 h_\kappa^0 + \mathfrak{c}_\nu h_\kappa^\nu, \kappa, \nu = m+1, \ldots, n, (h_\kappa^\nu) \in \boldsymbol{O}(n-m), \tag{12}$$
$$\hat{\mathfrak{c}}_{n+1} = \mathfrak{c}_i h_{n+1}^i, i = 0, \ldots, n+1. \tag{13}$$

Here the sum convention has to be applied; also in the following the ranges of the indices are as indicated in these formulas, if not changed explicitly. We remember that the elements h_{n+1}^i in (13) are uniquely defined by the other matrix elements, see (1.1.11) and Lemma 1.2.1. Deriving (10) one obtains

$$d\hat{\mathfrak{c}}_0 = d\mathfrak{c}_0 \lambda^{-1} - \mathfrak{c}_0 \lambda^{-2} d\lambda \text{ and } \langle d\hat{\mathfrak{c}}_0, d\hat{\mathfrak{c}}_0 \rangle = \langle d\mathfrak{c}_0, d\mathfrak{c}_0 \rangle \lambda^{-2}. \tag{14}$$

By (11) we have $(h^\beta_\alpha) = (h^\alpha_\beta)^{-1} = (\overset{-1}{h}{}^\alpha_\beta)$. Applying (11),(14), the derivation equation (1.3.14), and $\omega^\kappa = 0$ on F_1, we get by the isotropic-orthonormal relations:

$$\hat{\omega}^\alpha = \langle d\hat{\mathfrak{e}}_0, \hat{\mathfrak{e}}_\alpha \rangle = \omega^\beta \, \overset{-1}{h}{}^\alpha_\beta \, \lambda^{-1}. \tag{15}$$

The transformation of the Maurer-Cartan forms ω^κ_α can be calculated using the derivation equations (1.3.15) and (11), (12):

$$\hat{\omega}^\kappa_\alpha = \langle d\hat{\mathfrak{e}}_\alpha, \hat{\mathfrak{e}}_\kappa \rangle = \langle d\mathfrak{e}_0 h^0_\alpha + d\mathfrak{e}_\delta h^\delta_\alpha + \mathfrak{e}_0 dh^0_\alpha + \mathfrak{e}_\delta dh^\delta_\alpha, \mathfrak{e}_0 h^0_\kappa + \mathfrak{e}_\nu h^\nu_\kappa \rangle.$$

We remember $\langle d\mathfrak{e}_0, \mathfrak{e}_\nu \rangle = \omega^\nu = 0$ for $\nu = m+1, \ldots, n$, and, by (1.3.35):

$$\langle d\mathfrak{e}_\delta, \mathfrak{e}_0 \rangle = -\omega^{n+1}_\delta = -\omega^\delta.$$

It follows

$$\hat{\omega}^\kappa_\alpha = \sum_{\nu=m+1}^{n} h^\nu_\kappa \sum_{\delta=1}^{m} \omega^\nu_\delta h^\delta_\alpha - h^0_\kappa \sum_{\gamma=1}^{m} \omega^\gamma h^\gamma_\alpha. \tag{16}$$

Now we use the definition (8) of the coefficients of the first order and the orthogonality of the matrices $(h^\alpha_\beta), (h^\kappa_\nu)$. Applying the inverse of the transformation formula (15) and comparing the components at the basis forms $\hat{\omega}^\beta$ one obtains the *transformation rule of the coefficients of the first order* :

$$\hat{c}^\kappa_{\alpha\beta} = \lambda(\overset{-1}{h}{}^\kappa_\nu \, h^\gamma_\alpha h^\delta_\beta c^\nu_{\gamma\delta} - h^0_\kappa \delta_{\alpha\beta}). \tag{17}$$

Here the sum convention has to be applied; $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Next we take the trace in (17):

$$\sum_{\alpha=1}^{m} \hat{c}^\kappa_{\alpha\alpha} = \lambda(\overset{-1}{h}{}^\kappa_\nu \sum_{\gamma=1}^{m} c^\nu_{\gamma\gamma} - h^0_\kappa m). \tag{18}$$

Considering the shape of the elements h^0_κ in the matrix $h(A, \mathfrak{a}, \lambda) \in H_1$ (see (2) and (1.1.11)), we conclude

Lemma 3. *It is always possible to find frames $z = (\mathfrak{e}_i) \in p^{-1}(y) \subset F_1$ such that*

$$\sum_{\alpha=1}^{m} c^\kappa_{\alpha\alpha} = 0 \text{ for } \kappa = m+1, \ldots, n. \tag{19}$$

The set $\tilde{F}_1 \subset F_1$ of all these frames is a principal fibre bundle over Y^m with the structure group $\tilde{H}_1 \subset H_1$ characterized by $h^0_\nu = 0$, $\nu = m+1, \ldots, n$, as a subgroup of H_1. The structure form $\tilde{\omega}_1 = \omega_1|\tilde{F}_1$ has coefficients of first order satisfying (19); the transformation rule (17) reduces to

$$\hat{c}^\kappa_{\alpha\beta} = \lambda \, \overset{-1}{h}{}^\kappa_\nu \, h^\gamma_\alpha h^\delta_\beta c^\nu_{\gamma\delta}, \; \lambda > 0, \, (h^\alpha_\gamma) \in \boldsymbol{O}(m), \, (h^\kappa_\nu) \in \boldsymbol{O}(n-m). \tag{20}$$

□

In the case of curves it results $\omega_1^\kappa \equiv 0$ on $\tilde{F}_1$; all the coefficients of first order are zero, and $\tilde{F}_1 = F_2$ is the bundle of frames of second order. The further reduction of this bundle will be carried out in the next section.

Till to the end of the section we assume $m > 1$, since the integrability conditions play an essential role. Analogously to the Euclidean invariant called the *length square of the second vectorial fundamental form* we introduce the function

$$z \in \tilde{F}_1 \longmapsto S^2(z) := \sum_{\alpha,\beta,\kappa} (c^\kappa_{\alpha\beta})^2. \tag{21}$$

From (20) we conclude using the orthogonality of the transformation matrices

$$S^2(\hat{z}) = S^2(R_h(z)) = \lambda^2 S^2(z),\ h = h(A, \mathfrak{a}, \lambda) \in \tilde{H}_1, z \in \tilde{F}_1. \tag{22}$$

Since $\lambda > 0$, the property $S^2(z) = 0$ does not depend on the choice of $z \in p^{-1}(y) \subset \tilde{F}_1$; it is a Möbius invariant property of the immersion f at the point $y \in Y^m$; for $m > 1$ such a point y is named an *umbilical point* of the immersion f. The immersion f is called *umbilical* if all its points are umbilical. This definition is justified by the following

Proposition 4. *Let $f : Y^m \to S^n$ be an umbilical immersion of a connected manifold $Y^m, m \geq 2$, into the n-sphere S^n. Then there exists an m-sphere $\Sigma^m \subset S^n$ with $f(Y^m) \subset \Sigma^m$ as an open submanifold.*

Proof. Consider a local section $s(y) = (\mathfrak{c}_i(y))$, $y \in U \subset Y^m$, of the frame bundle $\tilde{F}_1$ of the immersion f. We will show that the $(n-m)$-vector $\Pi := \mathfrak{c}_{m+1} \wedge \ldots \wedge \mathfrak{c}_n$ is constant on U:

$$d\Pi = \sum_{\kappa=m+1}^{n} \mathfrak{c}_{m+1} \wedge \ldots \wedge d\mathfrak{c}_\kappa \wedge \mathfrak{c}_{\kappa+1} \wedge \ldots \wedge \mathfrak{c}_n = 0. \tag{23}$$

Then, since Y^m is connected, the $n-m$-vector Π is constant on the manifold Y^m. Because $\Pi \neq 0$ is splitting it defines a fixed $n-m$-dimensional Euclidean subspace $W^{n-m} \subset V^{n+2}$: at each point y the vectors $\mathfrak{c}_\kappa$, $\kappa = m+1, \ldots, n$, span W. The orthogonal complement $W^\perp$ of W in V is a pseudo-Euclidean subspace defining a fixed m-sphere $\Sigma^m \subset S^n$ corresponding to the isotropic cone of W. From $\mathfrak{c}_0(y) \in W^\perp$ it follows $f(y) = q(\mathfrak{c}_0(y)) \in \Sigma^m$. Clearly $f(Y^m)$ as an immersion into a manifold of the same dimension must be an open subset.

It remains to show (23). We remember that on $\tilde{E}_1$ the forms ω^κ and, by our assumption, also the forms ω^κ_α, $\alpha = 1, \ldots, m$, $\kappa = m+1, \ldots, n$, vanish. Remembering further $\omega^i_0 = \omega^i$ we conclude from the integrability condition (1.3.40)

$$0 = d\omega^\kappa_\alpha = -\omega^0_\kappa \wedge \omega^\alpha.$$

Inserting the basis representation $\omega^0_\kappa = c_{\kappa\beta}\omega^\beta$ into the last formula and noting $m \geq 2$ one obtains $\omega^0_\kappa = 0$. It follows

$$d\mathfrak{c}_\kappa = \mathfrak{c}_\nu \omega^\nu_\kappa.$$

Since $(\omega^\nu\kappa)$ is skew symmetric each summand in (23) is zero.□

Example 1. In this example we show that any m-sphere $\Sigma^m \subset S^n$ consists of umbilical poins. Let $W^{m+2} \subset V^{n+2}$ denote the pseudo-Euclidean subspace whose isotropic onedimensional subspaces

$$x = [\mathfrak{x}] \in \Sigma^m \Longleftrightarrow \mathfrak{x} \in W^{m+2}, \mathfrak{x} \neq \mathfrak{o}, \langle \mathfrak{x}, \mathfrak{x} \rangle = 0,$$

are the points of the subsphere. Without restriction of generality we may assume that the vectors $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathfrak{a}_{n+1}$ form a basis of W^{m+2}. Let

$$\mathfrak{x} = \mathfrak{a}_0 t + \mathfrak{a}_\alpha u^\alpha + \mathfrak{a}_{n+1} s \in J \cap W^{m+2}$$

be the basis representation. All points $[\mathfrak{x}] \in \Sigma^m$ with exception of $[\mathfrak{a}_{n+1}]$ are contained in the image of the parameter representation of the representing vectors

$$\mathfrak{x} = \mathfrak{a}_0 + \mathfrak{a}_\alpha u^\alpha + \mathfrak{a}_{n+1} \sum_1^m (u^\alpha)^2/2, (u^\alpha) \in \mathbf{R^m}. \tag{24}$$

A moving frame of first order is obtained by the definitions

$$\mathfrak{c}_0 = \mathfrak{x}, \mathfrak{c}_\alpha = \partial\mathfrak{x}/\partial u^\alpha = \mathfrak{a}_\alpha + \mathfrak{a}_{n+1} u^\alpha, \mathfrak{c}_\kappa = \mathfrak{a}_\kappa, \mathfrak{c}_{n+1} = \mathfrak{a}_{n+1}.$$

The derivation equations are easily calculated:

$$\begin{aligned} d\mathfrak{c}_0 &= \mathfrak{c}_\alpha \omega^\alpha,\ \omega^\alpha = du^\alpha, \omega^\kappa = 0, \\ d\mathfrak{c}_\alpha &= \mathfrak{a}_{n+1} \omega^\alpha, \\ d\mathfrak{c}_\kappa &= d\mathfrak{c}_{n+1} = 0, \omega^\alpha_\kappa = -\omega^\kappa_\alpha = 0. \end{aligned}$$

Thus, all coefficients of first order vanish: $c^\kappa_{\alpha\beta} = 0$, all points are umbilical. Permuting the points $[\mathfrak{a}_0], [\mathfrak{a}_{n+1}]$ one sees that also the point $[\mathfrak{a}_{n+1}]$ is umbilical. We remark: From (24) and the derivation equations it follows

$$\langle d\mathfrak{x}, d\mathfrak{x} \rangle = \sum_1^m (\omega^\alpha)^2 = \sum_1^m (du^\alpha)^2.$$

Thus (24) can be considered as a Möbius-geometric model of the Euclidean m-space E^m, also for $m = n$. □

Corollary 5. *Let $f : Y^m \to S^n$ be an umbilical immersion of a compact, connected manifold $Y^m, m > 1$. Then $\Sigma^m := f(Y^m)$ is an m-sphere in S^n, and Y^m is diffeomorphic to S^m.*

P r o o f. By Proposition 4 there exists an m-sphere Σ^m containing $f(Y^m)$ as an open submanifold. As an image of a compact manifold $f(Y^m)$ is closed, too. Because Σ^m is connected, it follows $f(Y^m) = \Sigma^m$. Since f is an immersion of a compact, connected manifold onto a connected, simply connected manifold of the same dimension, it must be a covering, thus a diffeomorphism. □

As a consequence, any immersion of a connected, closed manifold $Y^m, m > 1$, not diffeomorphic to S^m, into the n-sphere must have non-umbilical points. Clearly, the reduction procedure for umbilical immersions is finished. The m-spheres are homogeneous submanifolds; their isotropy groups consists of all transformations $h \in M(n)$ preserving the pseudo-Euclidean subspace W^{m+2}; they are isomorphic to $M(m) \times O(n-m)$. [1]

Generically, any immersion into the n-sphere has umbilical points, but the set of non-umbilical points is open in Y^m. Unfortunately, the next reduction step is not applicable to umbilical points. Thus we now assume that the immersion $f : Y^m \to S^n, m > 1$, does not have umbilical points. We choose a constant $c > 0$ which may depend on the class of immersions under consideration. By the transformation rule (22), at each non-umbilical point $y \in Y^m$ it is possible to find a moving frame $z \in p^{-1}(y)$ with $S^2(z) = c$. Moreover, again by (22), it follows

Lemma 6. *Let $c > 0$ be constant and $f : Y^m \to S^n, m > 1$ an immersion without umbilical points. Then*

$$\hat{F}_1 := \{z \in \tilde{F}_1 | S^2(z) = c\} \tag{25}$$

is a reduction of the principal fibre bundle $\tilde{F}_1$ onto the subgroup $\hat{H}_1 \subset \tilde{H}_1$ characterized by the condition $\lambda = 1$. The vector $\mathfrak{c}_0(y)$ of the frame $z \in p^{-1}(y) \subset \hat{F}_1$ is a uniquely defined element of the positive isotropic cone

$$J_+ := \{\mathfrak{c} \in J | \langle \mathfrak{c}, \mathfrak{e}_0 \rangle < 0\},$$

named the canonical lift *of the immersion.*

□

From formulas (1.1.11), (2), and Lemma 3 we see that the matrices of the elements $h \in \hat{H}_1$ with respect to the fixed isotropic-orthonormal frame $(\mathfrak{a}_i)$ have the shape

$$h(A_1, A_2, \mathfrak{a}) = \begin{pmatrix} 1 & \mathfrak{a}' A_1 & \mathfrak{o}' & \langle \mathfrak{a}, \mathfrak{a} \rangle / 2 \\ \mathfrak{o} & A_1 & 0 & \mathfrak{a} \\ \mathfrak{o} & 0 & A_2 & \mathfrak{o} \\ 0 & \mathfrak{o}' & \mathfrak{o}' & 1 \end{pmatrix}, \quad \begin{array}{l} A_1 \in O(m), \\ A_2 \in O(n-m), \\ \mathfrak{a} \in \mathbf{R}^m. \end{array} \tag{26}$$

[1] Proposition 4 and Corollary 5 are published in [23]. There we used the term *M-flat* instead of *umbilical*.

The bundle $\hat{F}_1$ as a reduction of the bundle of moving frames is immersed into the group space; we have a commutative diagram (5) with F_1 replaced by the reduction $\hat{F}_1$. The structure form ω of $M(n)$ induces the structure form $\hat{\omega}_1 := \hat{f}_1^*\omega$ of the immersion f, being a 1-form of type $\mathrm{Ad}\,|\hat{H}_1$ on the principal fibre bundle $p : \hat{F}_1 \to Y^m$. We emphasize that, in the general case, this reduction is not yet the frame bundle of second order of the immersion, since the coefficients of first order are not yet classified. One has to study the tensorial transformarion rule (20) with $\lambda = 1$ to continue the classification. This will be done for some classes of immersions later on. For all non-umbilical immersions it is possible to carry out yet the next reduction step considering the fact that the Maurer-Cartan form $\omega^0 := \omega_0^0$ belongs to the annihilator of the Lie algebra of $\hat{H}_1$. Since (ω^α) are the basis forms of the annihilator, we have the basis representations

$$\omega^0(z) = b_\alpha(z)\omega^\alpha,\ (z \in \hat{F}_1). \tag{27}$$

We calculate the transformation rule for $\omega^0 = -\langle d\mathfrak{c}_0, \mathfrak{c}_{n+1}\rangle$. Using (26),

$$\hat{\mathfrak{c}}_{n+1} = \mathfrak{c}_0 \sum_1^m (a^\alpha)^2/2 + \mathfrak{c}_\alpha a^\alpha + \mathfrak{c}_{n+1}$$

with $\mathfrak{a} = (a^\alpha) = (h^\alpha_{n+1}) \in \mathbf{R}^m$ we obtain for $\hat{\omega}^0 = \omega^0(zh)$, $\omega^0 = \omega^0(z)$ regarding $\mathfrak{c}_0(zh) = \mathfrak{c}_0(z)$:

$$\begin{aligned}\hat{\omega}^0 &= -\langle d\mathfrak{c}_0, \hat{\mathfrak{c}}_{n+1}\rangle = \omega^0 - \textstyle\sum_1^m \omega^\alpha a^\alpha,\\ &= \textstyle\sum_1^m \omega^\alpha(b_\alpha - a^\alpha),\\ &= \textstyle\sum_{\alpha,\beta=1}^m \hat{\omega}^\beta h^\beta_\alpha(b_\alpha - a^\alpha).\end{aligned} \tag{28}$$

It follows that the definition

$$F_M := \{z \in \hat{F}_1 | \omega^0(z) = 0\} \tag{29}$$

makes sense. Summarizing we formulate applying proposition 1:

Proposition 7. *For any immersion $f : Y^m \to S^n, m > 1$, without umbilical points F_M is a reduction of the bundle of moving frames to the structure group $H_M = O(m) \times O(n-m)$, characterized by $\mathfrak{a} = \mathfrak{o}$ in $\hat{H}_1$. The restriction to F_M gives a commutative diagram corresponding to (5) and allows to define the* Möbius structure form *$\omega_M := \hat{f}_1^*\omega$, being a 1-form on F_M of type $\mathrm{Ad}\,|H_M$ with values in the Lie algebra $\mathfrak{m}(n)$ of the Möbius group. The $M(n), H_M$-structure $[\hat{F}_M(Y^m, p, \hat{H}_1), \hat{\omega}_M]$ is called the* Möbius structure *of the immersion f.* [2] *$M(n)$-equivalent immersions have isomorphic Möbius-structures.* □

[2] This term appears also in [23], but in this paper the $M(n), \hat{H}_1$-structure $[\hat{F}_1, \hat{\omega}_1]$ has been given this name; the last reduction step based on (28) has not been required in the definition given there.

The meaning of the Möbius structure lies in the fact that we reached almost the same situation as in the case of Riemannian immersions. We now formulate some obvious corollaries valid for immersions without umbilical points.

Corollary 8. *For all the frames $z = (\mathfrak{c}_i) \in p^{-1}(y) \subset F_M$ the vectors $\mathfrak{c}_0$, $\mathfrak{c}_{n+1}$ coincide respectively; they define the canonical lift*

$$\mathfrak{c}_0(y) := \mathfrak{c}_0 \in J_+, \text{ if } z = (\mathfrak{c}_i) \in p^{-1}(y) \subset F_M,$$

such that $f(y) = [\mathfrak{c}_0(y)]$. The point $[\mathfrak{c}_{n+1}(y)] \in S^n$ is called the counterpoint *of f at y.* □

We remark that the canonical lifts for distinct values of the constant c differ by a dilation only; this dependence is unessential. Since now

$$d\mathfrak{c}_0 = \mathfrak{c}_\alpha \omega^\alpha,$$

and the action of $h \in H_M$ transforms the vectors $\mathfrak{c}_\alpha$ by the orthogonal matrix $A_1 \in O(m)$, analogously the vectors $\mathfrak{c}_\kappa$ by the orthogonal matrix $A_2 \in O(n-m)$, we conclude

Corollary 9. *Let $f : Y^m \to S^n$, $m > 1$, be an immersion without umbilical points. Then for $z = (\mathfrak{c}_i) \in p^{-1}(y) \subset F_M$ the vector subspaces*

$$L^2(y) := [\mathfrak{c}_0, \mathfrak{c}_{n+1}], \tag{30}$$

$$T^m(y) := [\mathfrak{c}_1, \dots, \mathfrak{c}_m], \tag{31}$$

$$N^{n-m}(y) := [\mathfrak{c}_{m+1}, \dots, \mathfrak{c}_{n+1}] \tag{32}$$

do not depend on the choice of the frame z. They define a family of orthogonal decompositions of the vector space V^{n+2}:

$$y \in Y^m \longmapsto L^2(y) \oplus T^m(y) \oplus N^{n-m}(y) = V^{n+2} \tag{33}$$

depending $M(n)$-equivariantly from the immersion f. The bilinear form

$$\varphi(\mathfrak{s}, \mathfrak{t}) := \langle d\mathfrak{c}_0(\mathfrak{s}), d\mathfrak{c}_0(\mathfrak{t}) \rangle = \sum_1^m \omega^\alpha(\mathfrak{s})\omega^\alpha(\mathfrak{t}) \tag{34}$$

is a $M(n)$-invariantly defined Riemannian metric on Y^m. □

The space $L^2(y)$ is generated by the canonical representants of point and counterpoint of the immersion; it is a 2-dimensional pseudo-Euclidean subspace of V. Clearly, T(y) and N(y) are Euclidean subspaces. Taking into account Lemma 1.1.4 and Corollary 1.1.5 we name them the *tangential space* and the *normal space* of the immersion at the point $y \in Y^m$.

Corollary 10. *Under the assumptions of Corollary 9 let* $s(y) = (\mathfrak{c}_i(y))$, $y \in U \subset Y^m$, *be a local moving frame. Then the density*

$$dv(y) := \omega^1 \wedge \ldots \wedge \omega^m(y) \neq 0 \tag{35}$$

does not depend on the choice of s. *The density* dv *is a* $M(n)$*-invariantly defined volume density on* Y^m. *The field of symmetric bilinear maps* $y \in Y^m \mapsto \alpha_y$ *defined by*

$$\alpha_y : \mathfrak{s}, \mathfrak{t} \in T(y) \longmapsto \alpha_y(\mathfrak{s}, \mathfrak{t}) := \mathfrak{c}_\kappa \omega^\kappa_\beta(\mathfrak{s})\omega^\beta(\mathfrak{t}) \in N(y) \tag{36}$$

does also not depend on the choice of the moving frame s *in* F_M*; it is a M(n)-equivariantly defined bilinear map called the* second Möbius fundamental form *of the immersion* f. □

The Möbius structure of immersions and the obtained invariants are a good basis for solving Möbius geometric problems for many classes of immersions. Before specialising onto appropriate classes of immersions we shall treat the Möbius geometry of curves in the next section.

3

Möbius Geometry of Curves

In this chapter we study the Möbius Geometry of curves. To be precise we remember the defintion of a regular curve in differential geometry: A *regular curve* in the n-dimensional manifold M^n is an equivalence class of immersions $f : Y^1 \to M^n$ of connected 1-dimensional manifolds into M^n with respect to diffeomorphisms $\hat{f} = f \circ \varphi$ where $\varphi : \hat{Y}^1 \to Y^1$ denotes a diffeomorphism. As before we always assume that all functions or maps are *smooth*, i. e. as often continuously differentiable as necessary in the given context. *Regularity* means, that the immersion condition $df(t)/dt \neq 0$ is satisfied for all $t \in Y^1$, what is clear by our definition. Often one omits this condition and admits the existence of *singular points* t_0 of the smooth map f which are defined by $df(t)_{t=t_0} = 0$. If not mentioned explicitly, we don't admit singularities; under a curve we always understand a regular curve. The immersions $f(t)$ representing the curve are often called *parameter representations*, t a *parameter*, and the diffeomorphisms φ are *parameter transformations*

$$x = f(t) = f(\varphi(\hat{t})) = \hat{f}(\hat{t}) \in M^n \qquad (t \in Y^1, \hat{t} \in \hat{Y}^1).$$

The curve can be considered as the image of the immersion f being the same for a given equivalence class of immersions. This image is the object of geometric interest, the parameter representations are only tools to investigate the curve. A curve is *closed* if the manifolds Y^1 of the parameter representations defining the curve are diffeomorphic to the circle S^1. Since there exist only two types of connected 1-dimensional manifolds, namely the circle and the open, may be infinite, intervals $Y^1 \subset \mathbf{R}$, we have to distinguish closed and non-closed curves only. We mention that the immersions f are not supposed to be injective; double points are admitted. Often it is useful to interpret the parameter t as the time, and the parameter representation $x = f(t)$ as a motion of the *point x in the space M^n*. The curve is *oriented*, if the parameter manifolds are oriented and the equivalence is defined by orientation preserving diffeomorphisms only. For intervals Y^1 we have a natural orientation given by the $<$-relation; the orientation preserving maps are characterized by the momotony condition $d\varphi/dt > 0$. For $Y^1 \cong S^1$ the choice of an universal covering map $\beta : \mathbf{R} \to S^1$ defines the orientation of the curve by the orientation of the corresponding immersions $F = f \circ \beta : \mathbf{R} \to M^n$ - or by considering local parameter representations - the orientation of closed curves is reduced to the case of non closed curves.

In the first section we derive Frenet formulas for generally curved curves, this are those which infinitesimally do not lie in a subsphere. For them we find a natural parameter and a complete system of invariants: the *Möbius curvatures*. Furthermore we prove a criterion for a curve to lie entirely in a subsphere. The second section is devoted to the study of curves of constant curvatures. It starts with general considerations about such curves in the group space and in homogeneous spaces of a linear Lie group G explaining the relation of the subject with the exponential and 1-parameter subgroups of G. Then we calculate all curves of constant curvatures in the Möbius plane S^2 and the Möbius space S^3. Some remarks about the history of the field are collected in the last section of this chapter.

3.1 Fundamental Theorem for Curves in the Möbius Space S^n

Let Y^1 be an open interval or a manifold diffeomorphic to S^1. We identify the Möbius space S^n with its vector model and consider the isotropic cone projected onto S^n, $q : J \mapsto S^n$, see (1.1.13). A curve can be given by a sufficiently often differentiable map into the positive isotropic cone $J_+ \subset V^{n+2}$:

$$t \in Y^1 \longmapsto \mathfrak{x}(t) \in J_+$$

representing the smooth map into S^n, the *curve*

$$f : t \in Y^1 \longmapsto f(t) := q(\mathfrak{x}(t)) = [\mathfrak{x}(t)] \in S^n.$$

The map f is an immersion, if the regularity condition $f'(t) \neq \mathfrak{o}$ – here and in the following we often denote the derivations as $df/dt = f'$ – is satisfied for all $t \in Y^1$, and this is the case iff the *regularity condition* is valid:

$$\langle \mathfrak{x}'(t), \mathfrak{x}'(t) \rangle > 0. \tag{1}$$

By Lemma 1.1.4, the kernel of $dq_{\mathfrak{x}}$ is $[\mathfrak{x}]$; therefore $f'(t) = dq(\mathfrak{x}'(t))$ does not vanish iff $\mathfrak{x}(t), \mathfrak{x}'(t)$ are linearely independent. Since $\mathfrak{x} \in J_+$ implies $\langle \mathfrak{x}(t), \mathfrak{x}'(t) \rangle = 0$, and in the pseudo-Euclidean vector space of index 1 the only vectors orthogonal to $\mathfrak{x} \in J_+$ are proportional to $\mathfrak{x}$ or spacelike, condition (1) follows. Points at which (1) is satisfied are said to be *regular*, and *singular* else. A curve is named *regular* if all its points are regular.

Let $\mathfrak{x}(t)$ be an arbitrary representative for the curve $f(t) = [\mathfrak{x}(t)]$. We define the *osculating space* $B_k(t)$ *of order* $k = 0, \ldots, n+1$, as the linear hull of the derivatives up to order k:

$$B_k(t) := [\mathfrak{x}(t), \mathfrak{x}'(t), \ldots, \mathfrak{x}^{(k)}(t)].$$

One easily proves that $B_k(t)$ does not depend neither on the representative nor on the parameter representation of the curve. Indeed, if

$$\mathfrak{y}(t) = \mathfrak{x}(t)\mu(t), \mu(t) \neq 0,$$

denotes another representative, the Leibniz identities yield

$$\mathfrak{y}' = \mathfrak{x}'\mu + \mathfrak{x}\mu',\ \mathfrak{y}^{(2)} = \mathfrak{x}^{(2)}\mu + 2\mathfrak{x}'\mu' + \mathfrak{x}\mu^{(2)}, \ldots.$$

Under a parameter transformation $\mathfrak{y}(s) = \mathfrak{x}(t(s))$ the chain rule and its generalizations yield

$$\frac{d\mathfrak{y}}{ds} = \frac{d\mathfrak{x}}{dt}\frac{dt}{ds},\ \frac{d^2\mathfrak{y}}{ds^2} = \frac{d^2\mathfrak{x}}{dt^2}(\frac{dt}{ds})^2 + \frac{d\mathfrak{x}}{dt}\frac{d^2t}{ds^2}, \ldots.$$

From these formulas the assertion follows. Clearly one has $B_k(t) \subset B_{k+1}(t)$. A point $f(t)$ of the curve is called *k-flat* , if $0 < k < n+1$ and $k+1 = \dim B_k(t) = \dim B_{k+1}(t)$ is satisfied. A regular curve is called *generally curved*, if it does not have k-flat points for $k = 1, \ldots, n$. The following Lemma implies that a regular curve does not have 1-flat points:

Lemma 1. *The osculating spaces $B_0(t), B_1(t)$ at the points of a regular curve in S^n are isotropic, while all the other osculating spaces $B_k(t), k \geq 2$, are pseudo-Euclidean.*

Proof. The first property follows from $\langle\mathfrak{x}, \mathfrak{x}\rangle = 0, \langle\mathfrak{x}, \mathfrak{x}'\rangle = 0$. From $\langle\mathfrak{x}, \mathfrak{x}'\rangle = 0$ we obtain deriving

$$\langle\mathfrak{x}', \mathfrak{x}'\rangle + \langle\mathfrak{x}, \mathfrak{x}''\rangle = 0.$$

By (1), the regularity implies $\langle\mathfrak{x}, \mathfrak{x}''\rangle < 0$. Therefore $\mathfrak{x}''(t)$ is not contained in the isotropic 2-space $B_1(t)$, we have $\dim B_2(t) = 3$, $B_2(t)$ is a non isotropic space containing the isotropic vector $\mathfrak{x}$, it must be pseudo-Euclidean. Since all osculating spaces $B_k(t), k > 2$, contain $B_2(t)$, they are pseudo-Euclidean, too. □

It is well known that the $(k-2)$-dimensional spheres $\Sigma^{k-2} \subset S^n$ correspond equivariantly and bijectively to the k-dimensional pseudo-Euclidean subspaces $W^k \subset V^{n+2}$, see e.g. [19], section 2.7. The spheres $\Sigma^{k-1}(t), k > 1$, corresponding to the $(k+1)$-dimensional osculating subspaces $B_k(t)$, are named the *osculating* $(k-1)$*-spheres* of the curve. Obviously it follows

Corollary 2. *For any point $f(t)$ of a regular curve in S^n the osculating circle $\Sigma^1(t)$ corresponding to $B_2(t)$ is well defined. At a k-flat point only the osculating l-spheres for $l < k$ are defined. For a generally curved immersion $f(t)$ there exists a uniquely defined family of flags of osculating subspaces*

$$B_0(t) \subset B_1(t) \subset B_2(t) \subset \ldots \subset B_{n+1}(t) = V^{n+2}$$

which defines a corresponding family of flags of osculating subspheres

$$f(t) \in \Sigma^1(t) \subset \Sigma^2(t) \subset \ldots \subset \Sigma^n(t) = S^n.$$

where $\Sigma^k(t)$ *denotes the k-sphere defined by* $B_{k+1}(t)$. □

The name "osculating $(k-1)$-sphere" can be justified as follows. Consider the osculating space $B_k(t_0)$ at a fixed point of the curve. Let the orthogonal complement $B_k(t_0)^\perp = [\mathfrak{n}_1, \ldots, \mathfrak{n}_{n-k+1}]$ be the span of the *normal vectors* $\mathfrak{n}_\kappa$ at this point. Then the osculating $(k-1)$-sphere is represented by the isotropic vectors $\mathfrak{y}$ satisfying the system of implicit equations

$$F(\mathfrak{y}) := \langle \mathfrak{y}, \mathfrak{y} \rangle = 0,\ F_\kappa(\mathfrak{y}) := \langle \mathfrak{n}_\kappa, \mathfrak{y} \rangle = 0 \text{ for } \kappa = 1, \ldots, n-k+1$$

Inserting in these equations the parameter representation of the curve $\mathfrak{y} = \mathfrak{x}(t)$ we obtain $n-k+2$ functions of t:

$$f_0(t) = F(\mathfrak{x}(t)), f_\kappa(t) = F_\kappa(\mathfrak{x}(t)),$$

the first of which vanishes identically, and the others vanish at least up to order k at the point t_0, since

$$f_\kappa^{(\alpha)}(t_0) = \langle \mathfrak{n}_\kappa, \mathfrak{x}^{(\alpha)}(t_0) \rangle = 0 \text{ for } \alpha = 0, \ldots, k,\ , \kappa = 1, \ldots, n-k+1,$$

by the definition of the normal vectors. This is the meaning of the osculation of order k: the curve deviates from its osculating $(k-1)$-sphere $S^{k-1}(t_0)$ in the neighbourhood of the point $[\mathfrak{x}(t_0)]$ of order $k+1$ or higher:

$$f_\kappa(t) = \langle \mathfrak{n}_\kappa, \mathfrak{x}^{(k+1)}(t_0) \rangle \frac{t^{k+1}}{(k+1)!} + O(t^{k+2}).$$

Obviously, the osculating $(k-1)$-sphere is uniquely defined by these conditions if the point is not k-flat. The geometric meaning of k-flatness is expressed also in the following

Proposition 3. *A regular curve* $f : Y^1 \to S^n$ *is k-flat,* $k > 1$, *at all points* $t \in Y^1$ *if and only if it is contained in a* $(k-1)$*-sphere* $S^{k-1} \subset S^n$, *and not in any subsphere of lower dimension.*

Proof. We consider the $(l+1)$-vector

$$\Pi_l(t) := \mathfrak{x}(t) \wedge \mathfrak{x}'(t) \wedge \ldots \wedge \mathfrak{x}^{(l)}(t).$$

If $\Pi_l(t) \neq 0$ it defines the osculating space $B_l(t)$:

$$\mathfrak{z} \in B_l(t) \Longleftrightarrow \Pi_l(t) \wedge \mathfrak{z} = 0.$$

Clearly, the curve $f(t)$ is k-flat at the point $t \in Y^1$, iff $\Pi_k(t) \neq 0$ and $\Pi_{k+1}(t) = 0$ are satisfied. Deriving $\Pi_l(t)$ we obtain applying the product rule:

$$\frac{d\Pi_l}{dt} = \Pi_{l-1} \wedge \mathfrak{x}^{(l+1)}.$$

If $f(t)$ is k-flat at t, we have $\mathfrak{x}^{(k+1)} \in B_k(t)$. It follows: If the curve is k-flat at each point, there exists a real function $h(t)$ such that

$$\frac{d\Pi_k(t)}{dt} = \Pi_k(t)h(t).$$

Now it is a well known fact that if the derivative of the vector function $\Pi_k(t)$ is proportional to the vector function itself, as indicated in the last equation, the vector function is proportional to a constant vector: there exists a real function $\mu(t)$ such that

$$\Pi_k(t) = \Pi_k(t_0)\mu(t),$$

where t_0 is an arbitrary fixed point of the curve. Since proportional l-vectors define the same vector subspace, it follows that the osculating subspaces $B_k(t) = B_k(t_0)$ are constant along the curve, and thus the osculating $(k-1)$-sphere S^{k-1} is constant, too: $S^{k-1}(t) = S^{k-1}(t_0)$. Since $\mathfrak{x}(t)$ belongs to $B_k(t_0)$, f(t) belongs to $S^{k-1}(t_0)$. The inverse statement follows immediately. □

Now our aim is using E. Cartan's method to construct an M(n)-equivariant canonical moving frame $(\mathfrak{c}_i(t))$ for generally curved curves. By the Möbius structure of the immersions the first steps of the reduction procedure are already done, see Lemma 2.2.3 and the remarks thereafter. The $M(n), \tilde{H}_1$-structure $[\tilde{F}_1, \tilde{\omega}_1]$ coincides with the second order reduction of the frame bundle, since all the first order coefficients are zero; indeed, by Lemma 2.2.3 we have

$$\omega_1^2(z) = \ldots = \omega_1^n(z) = 0, \; z \in \tilde{F}_1.$$

Therefore for any frame of the second order we have

$$d\mathfrak{c}_1 = \mathfrak{c}_0\omega_1^0 + \mathfrak{c}_{n+1}\omega_1^{n+1},$$

and it follows for the osculating spaces

$$B_0(t) = [\mathfrak{c}_0], B_1(t) = [\mathfrak{c}_0, \mathfrak{c}_1], B_2(t) = [\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_{n+1}], \tag{2}$$

and these conditions characterize the second order frames. Thus we obtained the frame bundle of second order of the curve $F_2 = \tilde{F}_1, \omega_2 = \tilde{\omega}_1$, with structure group $H_2 = \tilde{H}_1$.

We shall give now a direct, more elementary approach to the curve theory not using the Möbius structure. Let us start with the *tangential frames* F_1 being the set of all isotropic-orthonormal frames $z = (\mathfrak{c}_i) \in p^{-1}(t)$ tangential to the curve at the point $f(t) = [\mathfrak{x}(t)]$:

$$\mathfrak{c}_0 := \mathfrak{x}(t), \; p^{-1}(t) = \{z = (\mathfrak{c}_i) \in M(n) | [\mathfrak{c}_0] = f(t), [\mathfrak{c}_0, \mathfrak{c}_1] = B_1(t)\}.$$

Then F_1 is the disjoint union of the *fibres* $p^{-1}(t)$, where p denotes the *canonical projection* $p : F_1 \to Y^1$. The defining condition implies that for $z \in F_1$ we have

$$\mathfrak{c}_0'(t) \in B_1(t) = [\mathfrak{c}_0, \mathfrak{c}_1],$$

and it follows for the first derivation equation

$$d\mathfrak{c}_0 = \mathfrak{c}_0 \omega^0 + \mathfrak{c}_1 \omega^1.$$

The basis form is $\omega^1 = \langle d\mathfrak{c}_0, \mathfrak{c}_1 \rangle$; we assume that Y^1 and the sphere S^n are oriented. A frame is *positively oriented*, if its matrix representation has the determinant 1, and the vectors $\mathfrak{c}_0, \mathfrak{c}_{n+1}$ belong to the positive half J_+ of the isotropic cone. Only parameter transformations $s = s(t)$ with positive derivative $ds/dt > 0$ are admitted. We require that $\mathfrak{c}_1$ corresponds to the given orientation, what means

$$\langle \mathfrak{c}_0', \mathfrak{c}_1 \rangle > 0.$$

If $(\hat{\mathfrak{c}}_i)$ is another oriented frame of first order over $t \in Y^1$, it follows

$$\hat{\mathfrak{c}}_0 = \mathfrak{c}_0 \lambda^{-1},\ \hat{\mathfrak{c}}_1 = \mathfrak{c}_0 b_1 + \mathfrak{c}_1 h_1^1,$$

$$1 = \langle \hat{\mathfrak{c}}_1, \hat{\mathfrak{c}}_1 \rangle = (h_1^1)^2,$$

and it follows $h_1^1 = 1$. The structure group H_1 of F_1 preserves the defining conditions; it consists of all $h = h(A, \mathfrak{a}, \lambda) \in H_0$ with matrices of the shape

$$h(A, \mathfrak{a}, \lambda) \doteq \begin{pmatrix} \lambda^{-1} & b^1 & b^2 \dots b^n & c \\ 0 & 1 & 0 \dots 0 & a^1 \\ \vdots & \vdots & A_2 & \vdots \\ 0 & 0 & & a^n \\ 0 & 0 & 0 \dots 0 & \lambda \end{pmatrix} \text{ with } \lambda > 0, \mathfrak{a} = (a^i) \in \mathbf{R}^n,$$

and

$$A_2 \in SO(n-1),\ A = \begin{pmatrix} 1 & \mathfrak{o}' \\ \mathfrak{o} & A_2 \end{pmatrix},\ \mathfrak{b} = A^{-1}\mathfrak{a}/\lambda = (b^i) \in \mathbf{R}^n, c = \frac{\langle \mathfrak{a}, \mathfrak{a} \rangle}{2\lambda}.$$

Denote by $\hat{z} = (\hat{\mathfrak{c}}_i) = z \cdot h \in F_1,\ h \in H_1$, the transformed frame. The first transformed vectors are

$$\hat{\mathfrak{c}}_0 = \mathfrak{c}_0/\lambda,\ \hat{\mathfrak{c}}_1 = \mathfrak{c}_0 b^1 + \mathfrak{c}_1.$$

Deriving these formulas we obtain the transformation of the basis form ω^1 (see also (2.2.15)):

$$\hat{\omega}^1 = \omega^1(zh) = \langle d\hat{\mathfrak{c}}_0, \hat{\mathfrak{c}}_1 \rangle = \omega^1(z)\lambda^{-1}. \tag{3}$$

Considering with abuse of notations the corresponding forms ω_j^i as linear forms on the Lie algebra $\mathfrak{h}_0$ of the isotropy subgroup H_0, the equations $\omega_1^k = 0$, $k = 2, \dots, n$, characterize the Lie algebra $\mathfrak{h}_1$ of H_1 as a subalgebra of $\mathfrak{h}_0$. For the next reduction we have to calculate the transformation of the corresponding *coefficients of the first order* $c_1^k(z)$ of the basis representations

$$\omega_1^k(z) = c_1^k \omega^1(z) = \langle d\mathfrak{c}_1, \mathfrak{c}_k \rangle.$$

Taking into account

$$d\mathfrak{c}_1 = \mathfrak{c}_0 \omega_1^0 + \sum_{k=2}^{n} \mathfrak{c}_k \omega_1^k + \mathfrak{c}_{n+1} \omega^1,$$

the analogous equation for $d\hat{\mathfrak{c}}_1$, and

$$\hat{\mathfrak{c}}_k = \mathfrak{c}_0 b^k + \sum_{j=2}^{n} \mathfrak{c}_j h_k^j, \ (h_k^j) = A_2 \in SO(n-1),$$

one gets by a straightforward calculation

$$\hat{\omega}_1^k = \langle d\hat{\mathfrak{c}}_1, \hat{\mathfrak{c}}_k \rangle = \sum_{j=2}^{n} \omega_1^j h_k^j - \omega^1 b^k.$$

Using (3) it follows the transformation rule for the coefficients of the first order:

$$\hat{c}_1^k = \lambda (\sum_{j=2}^{n} c_1^j h_k^j - b^k).$$

Since the numbers $b^k \in \mathbf{R}$ may be given arbitrary values we conclude: *At any point $t \in Y^1$ there exist frames $z \in p^{-1}(t) \subset F_1$ for which the conditions*

$$\omega_1^2(z) = \ldots = \omega_1^n(z) = 0$$

are valid. These frames are the *second order frames*, again characterized by the conditions (2). The set of all these frames is defined to be the frame bundle $F_2 \subset F_1$ of second order. It is a reduction of F_1 to the subgroup $H_2 \subset H_1$ permuting the elements of the fibres $p^{-1}(t) \cap F_2$. The matrices of the transformations $h \in H_2$ have the shape

$$h \doteq \begin{pmatrix} 1/\lambda & a/\lambda & \mathfrak{o}' & a^2/(2\lambda) \\ 0 & 1 & \mathfrak{o}' & a \\ \mathfrak{o} & 0 & A_2 & \mathfrak{o} \\ 0 & 0 & \mathfrak{o}' & \lambda \end{pmatrix} \text{ with } A_2 \in O(n-1), \lambda > 0, a = h_1^{n+1} \in \mathbf{R}, \quad (4)$$

with respect to the fixed isotropic-orthonormal basis $(\mathfrak{a}_i)$, see also Lemma 2.2.3. Since on F_2 all the coefficients of the first order are zero, there do not exist invariants of the second order. Also we do not have a natural, i. e. invariant, parameter of the second order, see (3). The structure form ω_2 on F_2 is defined as the restriction $\omega_2 = \omega_1 | F_2$; it has the matrix representation

$$\omega_2 = \begin{pmatrix} \omega^0 & \omega_1^0 & \omega_2^0 & \omega_3^0 & \dots & \omega_n^0 & 0 \\ \omega^1 & 0 & 0 & 0 & \dots & 0 & \omega_1^0 \\ 0 & 0 & 0 & \omega_3^2 & \dots & \omega_n^2 & \omega_2^0 \\ 0 & 0 & \omega_2^3 & 0 & \dots & \omega_n^3 & \omega_3^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \omega_2^n & \omega_3^n & \dots & 0 & \omega_n^0 \\ 0 & \omega^1 & 0 & 0 & \dots & 0 & -\omega^0 \end{pmatrix} \tag{5}$$

with $\omega_\nu^\kappa + \omega_\kappa^\nu = 0$ for $\kappa, \nu = 1, \dots, n$.

This follows immediately from $\omega_1^\nu(z) = 0$ for $z \in F_2$ and the symmetry relations (1.3.34-36) of the structure form ω of $M(n)$. For the next reduction we consider the action of H_2 on F_2. The corresponding transformation of the frames $z = (\mathfrak{c}_i) \mapsto zh = (\hat{\mathfrak{c}}_i)$ is

$$\begin{aligned} \hat{\mathfrak{c}}_0 &= \mathfrak{c}_0/\lambda, \\ \hat{\mathfrak{c}}_1 &= \mathfrak{c}_0 a/\lambda + \mathfrak{c}_1, \\ \hat{\mathfrak{c}}_\nu &= \mathfrak{c}_\kappa h_\nu^\kappa, \\ \hat{\mathfrak{c}}_{n+1} &= \mathfrak{c}_0 a^2/(2\lambda) + \mathfrak{c}_1 a + \mathfrak{c}_{n+1}\lambda, \end{aligned} \tag{6}$$

with $(h_\nu^\kappa) \doteq A_2 \in O(n-1)$. For the moving frames of second order we have

$$\mathfrak{c}_0 = \mathfrak{x},\ d\mathfrak{c}_0 = \mathfrak{c}_0\omega^0 + \mathfrak{c}_1\omega^1,\ d\mathfrak{c}_1 = \mathfrak{c}_0\omega_1^0 + \mathfrak{c}_{n+1}\omega^1. \tag{7}$$

Since the group H_2 in H_1 is characterized by $h_\kappa^0 = 0$, $\kappa = 2, \dots, n$, we have to consider the coefficients of the second order c_κ obtained by the basis decompositions

$$\omega_\kappa^0 = c_\kappa \omega^1,\ \kappa = 2, \dots, n. \tag{8}$$

Applying (5) and (6) on $\hat{\omega}_\kappa^0 = -\langle d\hat{\mathfrak{c}}_\kappa, \hat{\mathfrak{c}}_{n+1}\rangle$ one calculates the transformation rule for the coefficients of second order

$$\hat{c}_\kappa = c_\kappa(zh) = c_\nu(z) h_\kappa^\nu \lambda^2,\ \nu, \kappa = 2, \dots, n. \tag{9}$$

Now we have to distinguish two cases:

- *Curves of type* A_2^0: All coefficients of second order are zero,

$$\omega_\kappa^0 = 0, \kappa = 2, \dots, n,$$

- *Curves of type* A_2: For any $t \in Y^1$ there exists a $z \in p^{-1}(t) \subset F_2$ with $\omega_k^0(z) \neq 0$ for at least one $\kappa = 2, \dots, n$.

Of course, this distinction is not a full classification. Generically for almost all curves there will exist points $t \in Y^1$ such that all the coefficients of second order $c_\kappa(z)$ vanish for all frames $z \in p^{-1}(t) \subset F_2$, For these points we have

$$d\mathfrak{c}_{n+1} = \mathfrak{c}_1\omega_1^0 - \mathfrak{c}_{n+1}\omega^0 \tag{10}$$

we conclude $d\mathfrak{c}_{n+1} \in B_2(t)$; these points are 2-flat. Applying Proposition 3 we obtain

Corollary 4. *The curve defined by* $f(t) = [\mathfrak{x}(t)]$ *is contained in a circle* $\Sigma^1 \subset S^n$ *iff it is of type* A_2^0*, i.e. if all its points are 2-flat*[1]. □

Now we consider immersions of type A_2. Again we assume that not only Y^1 but also the sphere S^n is oriented. In the transformation rule (9) the frames are positively oriented only if $(h^\nu_\kappa) \in SO(n-1)$, i.e. if the orientation of the *normal space* $B_2^\perp$ is preserved. We conclude

Lemma 5. . *For any point* $t \in Y^1$ *of a curve of type* A_2 *there exist positively oriented frames* $z = (\mathfrak{c}_i) \in p^{-1}(t) \subset F_2$ *such that* $c_2(z) = \pm 1$ *and* $c_\kappa(z) = 0$ *for* $\kappa > 2$*. Moreover, in the case* $n > 2$ *the first condition can be sharpened to* $c_2(z) = 1$.

Proof. Consider the vector $(c_\kappa) \in \mathbf{R}^{n-1}$; it is transformed under a change of the moving frame by equation (9). For $n > 2$ one always may find a matrix $(h^\nu_\kappa) \in SO(n-1)$ such that the transformed vector, normed by a factor $\lambda > 0$, equals $(1, 0, \dots, 0)$. In the case $n = 2$ we have only one coefficient $c_2(z)$; one gets $c_2(z) = \pm 1$ by norming with an appropriate $\lambda > 0$. The sign of $c_2(z)$ remains invariant. □

The *frame bundle of third order* F_3 of the curve of type A_2 is the subset of all frames $z \in F_2$ fulfilling the conditions mentioned in Lemma 5. It is a principal fibre bundle over Y^1 with structure group $H_3 \cong \mathbf{R} \times SO(n-2)$ consisting of all matrices (4) with the additional conditions

$$\lambda = 1 \text{ and } h^2_\nu = h^\nu_2 = \delta^2_\nu \text{ for } \nu = 2, \dots, n. \tag{11}$$

Thus we obtained the *third order structure* $[F_3(Y^1, H_3, p), \omega_3]$ *of the immersion* f, where ω_3 is the restriction of ω_2 to $F_3 \subset F_2$; its matrix representation is given by (5) with the additional conditions

$$\omega^0_2 = \pm\omega^1,\ \omega^0_3 = \dots = \omega^0_n = 0. \tag{12}$$

The geometrical meaning of these conditions is that $\mathfrak{c}_2$ is a normal vector of $B_2(t)$ in $B_3(t)$ and that $\mathfrak{c}_{n+1}$ may be normed such that

$$\langle \frac{d\mathfrak{c}_{n+1}}{dt}, \mathfrak{c}_2 \rangle = \pm 1$$

is fulfilled. Below we shall construct such a moving frame in terms of the derivatives of a representing isotropic vector $\mathfrak{x}(t)$ of the curve.

The main point now is that we may introduce a natural parameter. The transformation rule (3) shows that $\omega^1(z)$ does not depend on $z \in p^{-1}(t) \subset F_3$. Integrating $z^*\omega^1$ for an arbitrary *moving frame*, that is a local or better global section $z = z(t) \in F_3$, $p(z(t)) = t$, we may formulate

[1] In my paper [26] I called this points M-flat; this term is obsolete now.

Lemma 6. *For any curve of type A_2 there exists a* natural parameter *s invariantly defined up to an integration constant by*

$$s(t) = \int_{t_0}^{t} z^*\omega^1, \ \text{or } ds = z^*\omega^1, \tag{13}$$

where $z(t)$ is a local section of the third order frame bundle F_3. □

Our next aim is to calculate the natural parameter in terms of a representative $\mathfrak{x}(t)$ of the regular curve. The representation $f(t) = [\mathfrak{x}(t)]$ is named a *normed representation* of the curve, if $\langle \mathfrak{x}'(t), \mathfrak{x}'(t)\rangle = 1$. Setting, for an arbitrary representation,

$$\mu(t) := \sqrt{\langle \mathfrak{x}'(t), \mathfrak{x}'(t)\rangle}, \tag{14}$$

one easily proves that $\mathfrak{y}(t) := \mathfrak{x}(t)/\mu(t)$ is a normed representation of the same regular curve; apply (1) and $\langle \mathfrak{x}(t), \mathfrak{x}(t)\rangle = 0$, $\langle \mathfrak{x}(t), \mathfrak{x}'(t)\rangle = 0$. The advantage of the normed representations is the simple form of the scalar products of the derivatives, leading to

Lemma 7. *Let $f(t) = [\mathfrak{x}(t)] \in S^n$ be a normed representation of a regular curve. Then the following equations are true:*

$$\begin{aligned} &\langle \mathfrak{x}(t), \mathfrak{x}(t)\rangle = 0,\ \langle \mathfrak{x}'(t), \mathfrak{x}'(t)\rangle = 1,\ \langle \mathfrak{x}(t), \mathfrak{x}'(t)\rangle = 0,\\ &\langle \mathfrak{x}'(t), \mathfrak{x}''(t)\rangle = 0,\ \langle \mathfrak{x}(t), \mathfrak{x}''(t)\rangle = -1,\ \langle \mathfrak{x}, \mathfrak{x}^{(3)}\rangle = 0. \end{aligned} \tag{15}$$

The definitions

$$\mathfrak{c}_0 := \mathfrak{x},\ \mathfrak{c}_1 := \mathfrak{x}',\ \mathfrak{c}_{n+1} := \mathfrak{x}'' + \mathfrak{x}\langle \mathfrak{x}'', \mathfrak{x}''\rangle/2, \tag{16}$$

$(\mathfrak{c}_\kappa(t))$, $\kappa = 2, \dots, n$, any orthonormal basis of $B_2(t)^\perp$, yield a section of the second order frame bundle F_2 of the curve.

P r o o f. By successive derivation of the defining first two conditions one obtains (15), and these identities imply that $(\mathfrak{c}_i(t))$ is an isotropic-orthonormal basis of V^{n+2}. For the derivation equation one calculates

$$\begin{aligned} d\mathfrak{c}_0 &= \mathfrak{c}_1\omega^1,\ \omega^1 = dt,\\ d\mathfrak{c}_1 &= -\mathfrak{c}_0(\langle \mathfrak{x}'', \mathfrak{x}''\rangle/2)\omega^1 + \mathfrak{c}_{n+1}\omega^1,\\ d\mathfrak{c}_\kappa &= \mathfrak{c}_0\omega^0_\kappa + \mathfrak{c}_\lambda\omega^\lambda_\kappa,\ \lambda, \kappa = 2, \dots, n,\\ d\mathfrak{c}_{n+1} &= -\mathfrak{c}_1(\langle \mathfrak{x}'', \mathfrak{x}''\rangle/2)\omega^1 + \textstyle\sum_2^n \mathfrak{c}_\kappa\omega^0_\kappa. \end{aligned} \tag{17}$$

Comparing (17) with (5) confirms the assertion.□

Now we have to find the vector $\mathfrak{c}_2(t)$ and the transforming factor $\lambda(t)$ such that (12) is satisfied. At the same time we get a formula for the calculation of the natural conformal parameter of the curve. Let $\mathfrak{x}(t)$ be a normed representation of the curve and take $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_{n+1}$ as defined by (16). We choose the vector $\mathfrak{c}_2(t) \in B_2^\perp \cap B_3(t)$; since a curve of type A_2 does not have 2-flat points, it is uniquely defined up to sign. In case $n = 2$ it is uniquely defined because we consider positively oriented frames $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3)$ only. We derive the last equation (16) and obtain

$$\frac{d\mathfrak{c}_{n+1}}{dt} = \mathfrak{x}^{(3)} + \mathfrak{x}'\langle \mathfrak{x}'', \mathfrak{x}''\rangle/2 + \mathfrak{x}\langle \mathfrak{x}'', \mathfrak{x}^{(3)}\rangle. \tag{18}$$

In case $n > 2$ we find a uniquely defined $\mathfrak{c}_2(t) \in B_3(t)$ such that

$$c_2(t) = \langle \frac{d\mathfrak{c}_{n+1}(t)}{dt}, \mathfrak{c}_2(t)\rangle = \langle \mathfrak{x}^{(3)}(t), \mathfrak{c}_2(t)\rangle > 0, \tag{19}$$

since for a non 2-flat point we have $\mathfrak{x}^{(3)}(t) \notin B_2(t)$. For $n > 2$ we complete the basis $(\mathfrak{c}_0(t), \mathfrak{c}_1(t), \mathfrak{c}_2(t), \mathfrak{c}_{n+1}(t))$ of $B_3(t)$ by an orthonormal basis of $B_3^\perp$ to a positively oriented basis of V^{n+2}. Comparing (17) and (18) we get

$$\omega_2^0 = \langle \frac{d\mathfrak{c}_{n+1}}{dt}, \mathfrak{c}_2\rangle dt = \langle \mathfrak{x}^{(3)}, \mathfrak{c}_2\rangle \omega^1, \tag{20}$$

$$\omega_\kappa^0 = \langle \frac{d\mathfrak{c}_{n+1}}{dt}, \mathfrak{c}_\kappa\rangle dt = 0,\ \kappa = 3, \dots, n. \tag{21}$$

We decompose $\mathfrak{x}^{(3)}(t)$ in the basis $(\mathfrak{c}_0(t), \mathfrak{c}_1(t), \mathfrak{c}_2(t), \mathfrak{c}_{n+1}(t))$ of $B_3(t)$ and calculate using (8) and (20)

$$c_2 = \langle \mathfrak{x}^{(3)}, \mathfrak{c}_2\rangle = \pm\sqrt{\langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)}\rangle - \langle \mathfrak{x}'', \mathfrak{x}''\rangle^2}, \tag{22}$$

where in case $n > 2$ by (19) the positive root has to be taken. For the calculation one uses the last equation (15). Since $\mathfrak{x}^{(3)}(t) \neq \mathfrak{o}$ may not be proportional to the isotropic vector $\mathfrak{x}$, the last equation implies $\langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)}\rangle > 0$. Now the norming condition $(\hat{c}_2)^2 = 1$ of Lemma 5 together with (6), (9) and (21) leads to the definition of the transforming parameter $\lambda > 0$:

$$\lambda = (\langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)}\rangle - \langle \mathfrak{x}'', \mathfrak{x}''\rangle^2)^{-\frac{1}{4}}. \tag{23}$$

Inserting this value into the transformation rule (3) of ω^1 we obtain the Möbius-invariant 1-form of the frame bundle F_3 of the curve by *H. Liebmann's formula*, see [16]:

$$ds = \hat{\omega}^1 = (\langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)}\rangle - \langle \mathfrak{x}'', \mathfrak{x}''\rangle^2)^{\frac{1}{4}}\, dt, \tag{24}$$

resuming we formulate

Proposition 8. *Let $\mathfrak{x}(t)$ be a normed representation of a curve $f(t) = [\mathfrak{x}(t)] \in S^n$ without 2-flat points. Then a moving isotropic orthonormal frame $(\mathfrak{c}_i(t))$, where $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_{n+1}$ are defined by (16), and*

$$\mathfrak{c}_2(t) = (\mathfrak{x}\langle \mathfrak{x}'', \mathfrak{x}^{(3)}\rangle + \mathfrak{x}'\langle \mathfrak{x}'', \mathfrak{x}''\rangle + \mathfrak{x}^{(3)})\lambda^2(t)$$

is satisfied, is a special moving frame of the second order, fulfilling

$$\omega_\kappa^0 = \langle \frac{d\mathfrak{c}_{n+1}}{dt}, \mathfrak{c}_\kappa \rangle = 0 \ \mathit{for}\ \kappa = 3, \ldots, n.$$

The frame defined by

$$\hat{\mathfrak{c}}_0 = \mathfrak{c}_0/\lambda, \hat{\mathfrak{c}}_{n+1} = \mathfrak{c}_{n+1}\lambda,\ \hat{\mathfrak{c}}_j = \mathfrak{c}_j \ \mathit{for}\ j = 1, \ldots, n,$$

is a section of the frame bundle F_3 of third order of the curve defined behind Lemma 5. The formula (24) *defines the natural Möbius invariant parameter of the curve up to an integration constant.*

Proof. The formula for $\mathfrak{c}_2$ follows from $\mathfrak{c}_2 \in B_3 \cap B_2^\perp$ using the relations (15). To prove (12) it remains to show

$$\hat{\omega}_2^0 = \hat{\omega}^1.$$

Indeed, applying (22), (23), and (3) we conclude

$$\hat{\omega}_2^0 = \langle d\hat{c}_{n+1}, \hat{\mathfrak{c}}_2 \rangle = \langle d\mathfrak{c}_{n+1}, \mathfrak{c}_2 \rangle \lambda = \omega_2^0 \lambda = c_2 \lambda^2 \hat{\omega}^1 = \hat{\omega}^1.$$

The last statement follows from the invariance of $\hat{\omega}^1$ under the action of the structure group H_3 with $\lambda = 1$, see (11). □

Remark 1. Let $\boldsymbol{x}(t) \in \boldsymbol{E}^n$ be a regular curve in the Euclidean space with arc length t. By (1.2.7) the vector

$$\mathfrak{x}(t) = \mathfrak{a}_0 \langle \boldsymbol{x}, \boldsymbol{x} \rangle / 2 + \boldsymbol{x} + \mathfrak{a}_{n+1} \tag{25}$$

is an isotropic representative of the corresponding curve immersed into the Möbius space S^n. Deriving one obtaines easily

$$\langle \mathfrak{x}^{(k)}, \mathfrak{x}^{(k)} \rangle = \langle \boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k)} \rangle \text{ for } k = 1, 2 \ldots.$$

Thus the representation $\mathfrak{x}(t)$ of the curve is normed, in formula (24) the isotropic representation may be replaced by the Euclidean representation, and the scalar product at the right-hand side may be considered as the Euclidean. From formulas (20) – (22) it follows

$$c_2^2 = \langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)} \rangle - \langle \mathfrak{x}'', \mathfrak{x}'' \rangle^2 \geq 0,$$

where equality takes place iff the point $f(t) = [\mathfrak{x}(t)]$ is 2-flat. Therefore the scalar $c_2(t)$ may be interpreted as a measure for the deviation of the curve from its osculating circle in the neighbourhood of the given point, in dependence of the parameter t. The natural parameter is characterized by the property that this deviation is the same, namely 1, for each point of the curve. Moreover, in the case $n = 2$ the sign of $c_2(t)$ indicates whether the curve turns right or left in the neighbourhood of $f(t)$.

Example 1. Let $\boldsymbol{x} = \boldsymbol{x}(t) \in E^n$ be a parameter representation of a curve in the Euclidean space, t its arc length. We want to express c_2, and thus the natural Möbius-invariant parameter, by Euclidean invariants of the curve. To this aim we again consider the immersion (25) into the Möbius space S^n. Let $\mathfrak{b}_1(t), \mathfrak{b}_2(t), \mathfrak{b}_3(t)$ be the first three vectors of an Euclidean Frenet frame of the curve. Forming the first three derivatives of $\mathfrak{x}$ with respect to the arc length t we obtain using the Frenet formulas:

$$\begin{aligned}
&\mathfrak{x}' = \mathfrak{a}_0\langle \boldsymbol{x}, \boldsymbol{x}'\rangle + \boldsymbol{x}' = \mathfrak{a}_0\langle \boldsymbol{x}, \boldsymbol{x}'\rangle + \mathfrak{b}_1,\\
&\mathfrak{x}'' = \mathfrak{a}_0(1 + \langle \boldsymbol{x}, \boldsymbol{x}''\rangle) + \mathfrak{b}_1' = \mathfrak{a}_0(1 + \langle \boldsymbol{x}, \boldsymbol{x}''\rangle) + \mathfrak{b}_2 k,\\
&\mathfrak{x}^{(3)} = \mathfrak{a}_0\langle \boldsymbol{x}, \boldsymbol{x}^{(3)}\rangle + \mathfrak{b}_2' k + \mathfrak{b}_2 k',\\
&\text{or}\\
&\mathfrak{x}^{(3)} = \mathfrak{a}_0\langle \boldsymbol{x}, \boldsymbol{x}^{(3)}\rangle - \mathfrak{b}_1 k^2 + \mathfrak{b}_2 k' + \mathfrak{b}_3 k\kappa,
\end{aligned} \tag{26}$$

where k and κ denote the Euclidean first and second curvatures of the curve (curvature and torsion in case $n = 3$). An easy calculation using (24) and the orthogonality $\mathfrak{a}_0 \in E^{n\perp}$ yields a formula which I found also (for dimension 3) with another proof in T. Takasu's book [34], p.123:

$$ds = (k^2\kappa^2 + k'^2)^{\frac{1}{4}} dt. \tag{27}$$

Compare this result with the geometric interpretation discussed in Remark 1. Especially we have $ds = 0$ iff the curve is a segment of a circle or a line; in these cases the natural parameter is not defined. We remember that the curves (and all immersions) are supposed to be smooth, i.e. of class C_∞, what is essential for the statements in this example. In dimensions 2 and 3 equation (27) corresponds to formulas for the natural parameter (named there Eigenparameter) in the paper of H. Liebmann [16]. There the author used the curvature radius $r = 1/k$, the torsion radius $\rho = 1/\kappa$, and the radius of the osculating sphere

$$R = \sqrt{r^2 + \rho^2(r')^2}.$$

In the case $n = 3$, $k\kappa \neq 0$, his formula is

$$ds = \sqrt{\frac{R}{r^2\rho}}dt$$

which can easily be shown to be in accordance with (27). □

Now we return to the reduction procedure. We first remark that the form $\hat{\omega}^0$ for the frame $(\hat{\mathfrak{c}}_i)$ does not vanish, in general:

$$\hat{\omega}^0 = -\langle d\hat{\mathfrak{c}}_0, \hat{\mathfrak{c}}_{n+1}\rangle = -\langle d\mathfrak{c}_0/\lambda - \mathfrak{c}_0 d\lambda/\lambda^2, \mathfrak{c}_{n+1}\lambda\rangle = \omega^0 - d\lambda/\lambda = -d\lambda/\lambda,$$

since for the frame $(\mathfrak{c}_i(t))$ we have $\omega^0 = 0$.

Let $(\mathfrak{c}_i(t))$ be an arbitrary moving frame of the third order, for example the frame $(\hat{\mathfrak{c}}_i(t))$ considered in Proposition 8. By (11), the Lie algebra $\mathfrak{h}_3$ of the structure group H_3 is characterized as a subalgebra of the Lie algebra $\mathfrak{h}_2$ of H_2 as the annihilator of the Maurer-Cartan forms

$$\omega^0 = 0, \omega_2^\kappa = 0,\ \kappa = 3, \dots, n,\ (\omega^0, \omega_2^\kappa \in \mathfrak{h}_2^*). \tag{28}$$

Thus the coefficients of the third order we have to consider are c_0, c_2^κ defined by

$$\omega^0 = c_0(z)\omega^1,\ \omega_2^\kappa = c_2^\kappa(z)\omega^1,\ (z \in F_3). \tag{29}$$

The matrices of the transformations $h \in H_3$ of the frames of third order have the shape (4) with the conditions (11). Thus the vectors $\hat{\mathfrak{c}}_i(t),\ \mathfrak{c}_j(t)$ of two moving frames $\hat{z}(t), z(t) \in F_3$ are transformed by the rules

$$\begin{aligned}
\hat{\mathfrak{c}}_0(t) &= \mathfrak{c}_0(t),\\
\hat{\mathfrak{c}}_1(t) &= \mathfrak{c}_0(t)a + \mathfrak{c}_1(t) \text{ with } a \in \mathbf{R},\\
\hat{\mathfrak{c}}_2(t) &= \mathfrak{c}_2(t),\\
\hat{\mathfrak{c}}_\kappa(t) &= \mathfrak{c}_\mu(t)h_\kappa^\mu \text{ for } \kappa, \mu = 3, \dots, n,\ (h_\kappa^\mu) \in \boldsymbol{SO}(n-2),\\
\hat{\mathfrak{c}}_{n+1}(t) &= \mathfrak{c}_0(t)a^2/2 + \mathfrak{c}_1(t)a + \mathfrak{c}_{n+1}(t).
\end{aligned}$$

For the transformation of the form ω^0 we calculate

$$\begin{aligned}
\hat{\omega}^0 &= -\langle d\hat{\mathfrak{c}}_0, \hat{\mathfrak{c}}_{n+1}\rangle = -\langle d\mathfrak{c}_0, \hat{\mathfrak{c}}_{n+1}\rangle,\\
\hat{\omega}^0 &= -\langle \mathfrak{c}_0\omega^0 + \mathfrak{c}_1\omega^1, \mathfrak{c}_0 a^2/2 + \mathfrak{c}_1 a + \mathfrak{c}_{n+1}\rangle.
\end{aligned}$$

Since by (3) and $\lambda = 1$ we have $\hat{\omega}^1 = \omega^1$ it results $\hat{\omega}^0 = \omega^0 - a\omega^1$ and we find a uniquely defined $a \in \mathbf{R}$ solving

$$\hat{c}_0 = c_0 - a = 0. \tag{30}$$

Therefore always exist frames $z \in p^{-1}(t) \subset F_3$ with $\omega^0 = 0$. In dimensions $n = 2, 3$ the manifold of these frames is a canonical section of the bundle F_3; the reduction is finished, since a is fixed by (30), and there are no other free parameters in the structure group $H_3 = \mathbf{R} \times SO(n-2)$. We consider these cases in the next examples:

Example 2. For a curve without 2-flat points in the sphere $f(t) \in S^2$ the vectors $\mathfrak{c}_0(s), \mathfrak{c}_1(s), \mathfrak{c}_3(s)$, s the natural parameter, are uniquely defined during the reduction procedure. We complete them by the spacelike unit vector $\mathfrak{c}_2(s)$ to the uniquely defined positively oriented, isotropic-orthonormal *Frenet frame* $(\mathfrak{c}_0(s), \mathfrak{c}_1(s), \mathfrak{c}_2(s), \mathfrak{c}_3(s))$. Then, following the reduction steps, (5), (12), and (30), the *Frenet formulas for a spherical curve without 2-flat points* are obtained:

$$\begin{aligned}
\mathfrak{c}_0' &= \mathfrak{c}_1,\\
\mathfrak{c}_1' &= \mathfrak{c}_0 k + \mathfrak{c}_3,\\
\mathfrak{c}_2' &= \mathfrak{c}_0 \epsilon,\\
\mathfrak{c}_3' &= \mathfrak{c}_1 k + \mathfrak{c}_2 \epsilon.
\end{aligned}$$

Here $k = k(s)$ denotes the *Möbius curvature* of the spherical curve and $\epsilon = \pm 1$ shows whether the curve turns right or left. □

Example 3. For a curve without 3-flat points in the 3-sphere $f(t) \in S^3$ the vectors $\mathfrak{c}_0(s), \mathfrak{c}_1(s), \mathfrak{c}_2(s), \mathfrak{c}_4(s)$, s the natural parameter, are uniquely defined during the reduction procedure. We complete them by the spacelike unit vector $\mathfrak{c}_3(s)$ to the uniquely defined positively oriented, isotropic-orthonormal *Frenet frame* $(\mathfrak{c}_0(s), \mathfrak{c}_1(s), \mathfrak{c}_2(s), \mathfrak{c}_3(s), \mathfrak{c}_4(s))$. Then, following the reduction steps, (5), (12), and (30), the *Frenet formulas for a spherical curve without 3-flat points* are obtained:

$$\begin{aligned}
\mathfrak{c}_0' &= \mathfrak{c}_1, \\
\mathfrak{c}_1' &= \mathfrak{c}_0 k_1 + \mathfrak{c}_4, \\
\mathfrak{c}_2' &= \mathfrak{c}_0 + \mathfrak{c}_3 k_2, \\
\mathfrak{c}_3' &= -\mathfrak{c}_2 k_2, \\
\mathfrak{c}_4' &= \mathfrak{c}_1 k_1 + \mathfrak{c}_2.
\end{aligned}$$

Here $k_i = k_i(s)$ denote the *first and second Möbius curvatures* of the curve and the sign of k_2 shows whether the curve turns right or left. □

Continuing the reduction we assume now $n \geq 3$. Since $\hat{\mathfrak{c}}_2 = \mathfrak{c}_2$, the only remaining transformation is

$$\hat{\mathfrak{c}}_\kappa = \sum_{\nu=3}^{n} \mathfrak{c}_\nu h_\kappa^\nu, \; (h_\kappa^\nu) \in SO(n-2)).$$

By (5) and (12) we obtain

$$d\hat{\mathfrak{c}}_2 = \sum_{\kappa=3}^{n} \hat{\mathfrak{c}}_\kappa \hat{\omega}_2^\kappa = d\mathfrak{c}_2 = \sum_{\nu=3}^{n} \mathfrak{c}_\nu \omega_2^\nu = \sum_{\kappa,\nu=3}^{n} \hat{\mathfrak{c}}_\kappa h_\nu^\kappa \omega_2^\nu.$$

comparing the coefficients at $\hat{\mathfrak{c}}_\kappa$ we get

$$\hat{\omega}_2^\kappa = \sum_{\nu=3}^{n} h_\nu^\kappa \omega_2^\nu$$

and the same rule is valid also for the coefficients of third order

$$\hat{c}_2^\kappa = \sum_{\nu=3}^{n} h_\nu^\kappa c_2^\nu. \tag{31}$$

Using the orthogonality of the transformation matrix we may achieve $\hat{c}_2^\kappa = 0$ for $\kappa = 4, \ldots, n$. It follows

Proposition 9. *Let $\mathfrak{x}(t)$ represent a regular curve $f(t) = [\mathfrak{x}(t)] \in S^n$, $n \geq 3$, without 3-flat points. Then we may find frames $z \in p^{-1}(t) \subset F_3$ such that*

$$\omega^0(z) = \omega_2^4(z) = \ldots = \omega_2^n(z) = 0. \tag{32}$$

For this frames $\omega_2^3(z) = c_2^3(z)\omega^1 \neq 0$ is satisfied, and it exists a uniquely defined vector $\mathfrak{c}_3(t)$ such that $c_2^3 > 0$, if $n > 3$, or $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4)$ is positively *oriented in case $n = 3$. The set of frames with these properties is defined to be* the frame bundle $F_4 \subset F_3$ of order 4 *of the curve. It is a reduction of the frame bundle F_3 onto the structure group $H_4 \cong SO(n-3)$. Together with the structure form $\omega_4 = \omega_3|TF_4$ it is a* $M(n), H_4$-structure $[F_4(Y^1, p, H_4), \omega_4]$. *$M(n)$-equivalent immersions have isomorphic $M(n), H_4$-structures.* □

Now the construction of the Frenet frame can be accomplished like in the n-dimensional Euclidean space by induction. We consider a curve without $(n-1)$-flat points $f(s) = [\mathfrak{c}_0(s)] \in S^n$, $n > 3$, s the natural parameter, $z(s) = (\mathfrak{c}_i(s))$ a section of the $M(n), H_k$-structure F_k of the immersion, $k \geq 4$, $H_k \cong SO(n-k+1)$ acting on $B_k^\perp = [\mathfrak{c}_k, \ldots, \mathfrak{c}_n]$. By Proposition 9 and examples 2, 3, we may starting the induction with $k = 4$, $n \geq 4$. We assume that the vectors $(\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2, \ldots \mathfrak{c}_{k-1}, \mathfrak{c}_{n+1})$ are uniquely and $M(n)$-equivariantly defined. We have to define the vector $\mathfrak{c}_k$ of the frame equivariantly depending on the immersion f. The moving frame will be defined in such a way that the k-th osculating space of a curve without $(n-1)$-flat points fulfils

$$B_k = [\mathfrak{x}_0, \mathfrak{x}', \ldots, \mathfrak{x}^{(k)}] = [\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{k-1}, \mathfrak{c}_{n+1}] \text{ for } k = 2, \ldots, n. \tag{33}$$

We state that there exists a uniquely and $M(n)$-equivariantly defined *Frenet frame* of the immersion such that for $n \geq 3$ the *Frenet formulas* are satisfied:

$$\begin{aligned} \mathfrak{c}_0' &= \mathfrak{c}_1, \\ \mathfrak{c}_1' &= \mathfrak{c}_0 k_1 + \mathfrak{c}_{n+1}, \\ \mathfrak{c}_2' &= \mathfrak{c}_0 + \mathfrak{c}_3 k_2, \\ \mathfrak{c}_\kappa' &= -\mathfrak{c}_{\kappa-1} k_{\kappa-1} + \mathfrak{c}_{\kappa+1} k_\kappa, \ \kappa = 3, \ldots, n-1, \\ \mathfrak{c}_n' &= -\mathfrak{c}_{n-1} k_{n-1}, \\ \mathfrak{c}_{n+1}' &= \mathfrak{c}_1 k_1 + \mathfrak{c}_2. \end{aligned} \tag{34}$$

Here $k_\kappa > 0$ for $\kappa = 2, \ldots, n-2$ must be required; the sign of k_{n-1} decides whether the curve turns left or right. By Example 3, the statement is true for $n = 3$. We assume $n > 3$ and construct the vectors $\mathfrak{c}_\kappa$ inductively. By Proposition 9, the derivation equations (34) are satisfied for the vectors $\mathfrak{c}_0, \ldots, \mathfrak{c}_3, \mathfrak{c}_{n+1}$. Thus we starting the induction with $k = 4$. If $k = n+1$ we are ready; thus we assume $k < n+1$. The already found vectors $\mathfrak{c}_0, \mathfrak{c}_1, \ldots, \mathfrak{c}_{k-1}, \mathfrak{c}_{n+1}$ of the frame spanning B_k satisfy the derivation equations (34) as far as they contain these vectors only. The structure group $H_k \cong SO(n-k+1)$ acts with the standard orthogonal action in the *k-th normal spaces* $B_k^\perp(s)$ of the immersion, forming a vector bundle associated to F_k. Therefore we may put the vector $\mathfrak{c}_k$ into the one-dimensional space $B_{k+1} \cap B_k^\perp$; it is defined up to sign. Since the immersion does not have $(n-1)$-flat points, the derivative $\mathfrak{c}_{k-1}'$ does not lie in B_k for $k < n$, and it follows

$$\langle \mathfrak{c}_{k-1}', \mathfrak{c}_k \rangle \neq 0, \ 2 < k < n.$$

If $k < n$, we may choose $\mathfrak{c}_k$ such that

$$k_{k-1} := \langle \mathfrak{c}_{k-1}', \mathfrak{c}_k \rangle > 0, \ 1 < k < n. \tag{35}$$

If $k = n$, the vector $\mathfrak{c}_n$ is uniquely defined as the last completing vector of the frame, which is positively oriented. In this case the sign of $k_{n-1} := \langle \mathfrak{c}'_{n-1}, \mathfrak{c}_n \rangle$ reflects whether the curve turns left or right; for n-flat points we have $k_{n-1}(s) = 0$. Thus the next vector $\mathfrak{c}_k$ of the frame and the next curvature k_{k-1} are defined, and by the construction equivariantly connected with the curve. If $k = n+1$ one calculates using the symmety properties of the structure form ω the second last equation (34) and the existence and uniqueness of the Frenet frame is proved.The functions $k_\kappa(s)$ are named the*Möbius curvatures* of the curve. We state the *fundamental theorem of Möbius curve geometry:*

Theorem 10. *Let $f(s) = [\mathfrak{x}(s)] \in S^n$, $n \geq 3$, $s \in Y^1$ the Möbius invariant natural parameter, be an oriented curve without $(n-1)$-flat points in the n-dimensional oriented Möbius space. Then there exist a uniquely defined* Frenet frame $(\mathfrak{c}_i)$ *satisfying the conditions*

1. *$z(s) = (\mathfrak{c}_i)$ is a section for each $M(n), H_k$-structure $[F_k, \omega_k]$, $0 \leq k \leq n$, of the immersion f.*
2. *The frame satisfies the Frenet formulas (34).*
3. *The curvatures k_κ satisfy (35).*

The correspondence $f(s) \mapsto z(s)$ is $M(n)$-equivariant. The differential $\omega^1 = ds$ and the curvatures k_κ, $\kappa = 1, \ldots, n-1$, are a complete system of Möbius invariants in the class of smooth curves without $(n-1)$-flat points. For any system $(k_\kappa), \kappa = 1, \ldots, n-1$, of smooth functions on Y^1 satisfying $k_\kappa(s) > 0$ for $\kappa = 2, \ldots, n-2$ there locally, for open intervals $U \subset Y^1$, exist immersions $f : U \to S^n$ without $(n-1)$-flat points with the given functions as curvatures and s as its natural parameter, uniquely defined up to Möbius transformations. If Y^1 is simply connected, such immersions exist globally for $U = Y^1$.

Proof. The existence and uniqueness of the Frenet frame has been proved already by the construction before the theorem; the equivariance is an immediate consequence of the construction, which is based on $M(n)$-invariant properties. We have to show the existence of the immersion with the given functions as curvatures. To this aim we consider (34) as a system of linear differential equations for the frame $(\mathfrak{c}_i(s))$. Let $(\mathfrak{c}_i(s))$ be the uniquely defined smooth solution with starting conditions $(\mathfrak{c}_i(s_0)) = (\mathfrak{a}_i)$, the fixed isotropic-orthonormal frame. Using the Frenet formulas (34) one easily verifies that the solution is isotropic-orthonormal at any point s, since the scalar products of the vectors are constant:

$$\frac{d\langle \mathfrak{c}_i, \mathfrak{c}_j \rangle}{ds} = \langle \mathfrak{c}'_i, \mathfrak{c}_j \rangle + \langle \mathfrak{c}_i, \mathfrak{c}'_j \rangle = 0.$$

In particular, we have $\langle \mathfrak{c}_0(s), \mathfrak{c}_0(s) \rangle = 0$; thus $\mathfrak{x}(s) := \mathfrak{c}_0(s)$ is isotropic and not the null vector, since $\langle \mathfrak{c}_0(s), \mathfrak{c}_{n+1}(s) \rangle = -1$. It is the representing vector of the smooth immersion $f(s) := [\mathfrak{x}(s)] \in S^n$. The regularity condition follows by the first equation of (34). Using the Frenet formulas one calculates

$$\mathfrak{x}' = \mathfrak{c}_1,\ \mathfrak{x}'' = \mathfrak{c}_1' = \mathfrak{c}_0 k_1 + \mathfrak{c}_{n+1},\ \mathfrak{x}^{(3)} = \mathfrak{c}_0 k_1' + \mathfrak{c}_1 2k_1 + \mathfrak{c}_2, \tag{36}$$

and it results

$$\langle \mathfrak{x}^{(3)}, \mathfrak{x}^{(3)} \rangle - \langle \mathfrak{x}'', \mathfrak{x}'' \rangle^2 = 1. \tag{37}$$

Equation (24) shows that s is the natural parameter of the curve f. Since the Frenet frame is uniquely defined by the curve and the Frenet formulas, the solution $z(s)$ satisfying the Frenet formulas must be the Frenet frame of the curve. Since now (35) is satisfied, also (33) follows: the immersion does not have $(n-1)$-flat points. Thus at least the local existence of an immersion with the rquired properties is proved. Let $U \subset Y^1$ be a neighbourhood of s_0 for which the solution $z(s)$, $s \in U$, exists, and let $\hat{z}(s)$, $s \in U$, be another solution. Then $\hat{z}(s_0)$ is an isotropic-orthonormal frame, and there exists a uniquely defined Möbius transformation $g \in M(n)$ with $g(z(s_0)) = \hat{z}(s_0)$. We consider the family of isotropic-orthonormal frames $w(s) := g(z(s))$. For the vectors forming $w(s) = (g(\mathfrak{c}_i(s))$ the differential equation (34) is fulfilled, since

$$(g\mathfrak{c}_i)' = g(\mathfrak{c}_i') = g(\mathfrak{c}_j)\xi_i^j,$$

where $(\xi_i^j) = (\omega_i^j(s)/ds) \in \mathfrak{m}(n)$ denotes the matrix of the coefficients at the right side of (34). By the uniqueness theorem for differential equations the solutions $\hat{z}(s)$ and $w(s)$, having the same starting values for $s = s_0$, coincide. The local existence and uniqueness of the immersions up to Möbius transformations is proved. Now let Y^1 be simply connected, i. e. an open interval or diffeomorphic to $\mathbf{R}$. We consider a closed interval $W \subset Y^1$. We may cover W by a finite number of open neighbourhoods U_ι, $\iota = 1, \dots N$, for which solutions of (34) exist. Assume $s_0 \in U = U_1$ and let $\tilde{U} \supset U$ be a maximal connected open subset of Y^1 onto which $z(s)$ may be continued as a solution of (34). We state

$$\tilde{U} \supset U(W) = \bigcup_1^N U_\iota.$$

Indeed, if that would not be the case, we could find an U_ι with $\tilde{U} \cap U_\iota \neq \emptyset$ such that U_ι is not contained in $\tilde{U}$. Let $s_\iota \in \tilde{U} \cap U_\iota$ be a fixed element and consider the maximal solution $\tilde{z} : \tilde{U} \to M(n)$. We take the solution z_ι on U_ι with starting value $z_\iota(s_\iota) = \tilde{z}(s_\iota)$. By the local existence and uniqueness property this solution coincides with the restriction $\tilde{z}|\tilde{U} \cap U_\iota$. Therefore we could continue $\tilde{z}$ onto $\tilde{U} \cup U_\iota$ in contradiction to the maximality of $\tilde{z}$. Since W is an arbitrary closed interval in Y^1 we conclude $\tilde{U} = Y^1$, and Theorem 10 is proved. □

REMARK 2. In the case $Y^1 \cong S^1$ the continuation of a local solution onto an open subset, say $Y^1 \setminus \{0\}$ may lead to a solution for which the right and the left limit of $z(s)$, $s \to \pm 0$, are distinct. In this case there does not exist a global immersion of Y^1 with the wanted properties.

3.2 Curves of Constant Curvatures

In this section we shall find the curves of constant curvatures in the Möbius plane S^2 and the Möbius space S^3. The relations to the curves of constant curvatures in the Riemannian space forms E^n_c will be discussed. As a heuristic principle we remark that the curves of constant curvatures in a homogeneous space G/H are exactly the orbits of the 1-parameter subgroups $g(t) \in G$. Indeed, let

$$t \in \mathbf{R} \longmapsto g(t) \in G,\ g(t+s) = g(t)g(s),$$

be a smooth homomorphism of the additive group of real numbers into G defining the 1-parameter subgroup $g(t) \in G$. Consider its orbit $x(t) = g(t)aH$. Then for an arbitrarily fixed number $t_0 \in \mathbf{R}$ the transformation $g(t_0)$ transforms the curve $x(t)$ into itself:

$$g(t_0)x(t) = g(t_0)g(t)aH = g(t+t_0)aH = x(t+t_0).$$

Therefore any G-invariant function $k(t)$ of the curve satisfies $k(t+t_0) = k(t)$ for all $t, t_0 \in \mathbf{R}$, thus it must be constant. Now we consider the situation described in the Fundamental Theorem 1.10. Let $x(t)$ be a curve in S^n with constant curvatures k_i, $i = 1, \ldots, n-1$, and $z(t), p(z(t)) = x(t)$, its uniquely defined Frenet frame. Since the curvatures are constant we may assume that $x(t), z(t)$ are defined for all $t \in \mathbf{R}$. Then for each point $x(t)$ let $g(t) \in G$ be the uniquely defined transformation mapping $z(0)$ into $z(t) = g(t)z(0)$. Now we fix an arbitrary $t_0 \in \mathbf{R}$ and consider the curve $\hat{x}(t) := g(t_0)x(t)$.For its Frenet frame one has $\hat{z}(t) = g(t_0)z(t)$. The Frenet frame $z(t)$ of the curve $x(t)$ satisfies $z(t+t_0) = g(t+t_0)z(0)$; for $t = 0$ it has the same starting value $g(t_0)z(0)$ like $\hat{z}(t)$ and has the same constant curvatures. By the Fundamental Theorem both curves coincide and it follows

$$\hat{z}(t) = g(t_0)z(t) = g(t_0)g(t)z(0) = z(t+t_0) = g(t+t_0)z(0).$$

Since the group acts simply transitively over the manifold of the frames we conclude $g(t+t_0) = g(t)g(t_0)$, and the $g(t)$ form a 1-parameter subgroup of G with the orbit $g(t)x(0) = x(t)$. From these considerations it becomes clear that the curves of constant curvatures of one of the space forms must also be curves of constant curvatures of the Möbius space containing the conformal models of the space forms, since their isometry groups are subgroups of the Möbius group. A general detailed approach to the curves of constant curvatures of linear homogeneous spaces is contained in the next Subsection 3.3.

3.2.1 Curves of Constant Curvatures in S^2

In this subsection we consider the Möbius geometry of the sphere S^2. Clearly, the Euclidean geometry is a subgeometry of the Möbius geometry. The curves of constant curvature of the Euclidean plane are the lines and the circles. All these curves are locally Möbius equivalent; they are curves of type A_2^0, see Proposition 1.3. By the local equivalence of the curves of this class we don't have Möbius invariants and the curves are of constant curvature in the trivial sense. For the curves of type A_2 in the sphere S^2 the Frenet fornulas contain a curvature k. We will show that the curves of type A_2 of constant curvature are the loxodromes. A ship navigates along a loxodrome if its compass course is constant: a *loxodrome* is a line on the sphere which intersects all the meridians under a constant angle. The *meridians* may be defined as the circles of the circle bundle through two points.

The loxodromes are orbits of a 1-parameter subgroup of the Euclidean conformal group with the matrix representation in orthonormal coordinates in the Euclidean plane

$$g(t,a) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mathrm{e}^{at} \,. \tag{1}$$

Their orbits in the Euclidean plane are well known under the name of the *logarithmic spirals*[1]. Figure 3.1 shows a part of the spiral being the orbit of the point $\{1,0\}$ with parameter $a = 0.2$. The parameter representation of the orbit of the point $(b,0)$ of the x-axes in orthonormal Cartesian coordinates is

$$\begin{aligned} x(t) &= b\,\mathrm{e}^{at}\cos t \\ y(t) &= b\,\mathrm{e}^{at}\sin t \end{aligned} \qquad (a > 0, b > 0). \tag{2}$$

The logarithmic spirals are *isogonal trajectories* of the bundle of lines through the origin. The tangent function of the intersection angle α is

$$\tan(\alpha) = \frac{1}{a}.$$

In polar coordinates ρ, φ its equation is $\rho = b\,\mathrm{e}^{a\varphi}$. The arc length of the curve can be calculated easily as

$$s(t) = \frac{1}{a}\sqrt{b^2(1+a^2)}\,\mathrm{e}^{at} \,. \tag{3}$$

The integration constant is chosen in such a way that the origin as the limit of the spiral for $t \to -\infty$ corresponds to $s = 0$; $s(t) > 0$ is the length of the arc from the origin till to the point with parameter value t. The Euclidean curvature radius as a function of the angle t is

$$r(t) = \sqrt{b^2(1+a^2)}\,\mathrm{e}^{at} > 0. \tag{4}$$

The curvature is always positive; inverting the orientation of the curve (or the direction of the rotation) yields spirals of negative curvature. Eliminating t from (3),(4) yields the curvature radius as a function of the arc length:

[1] See Weisstein, Eric W. "Logarithmic Spiral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LogarithmicSpiral.html

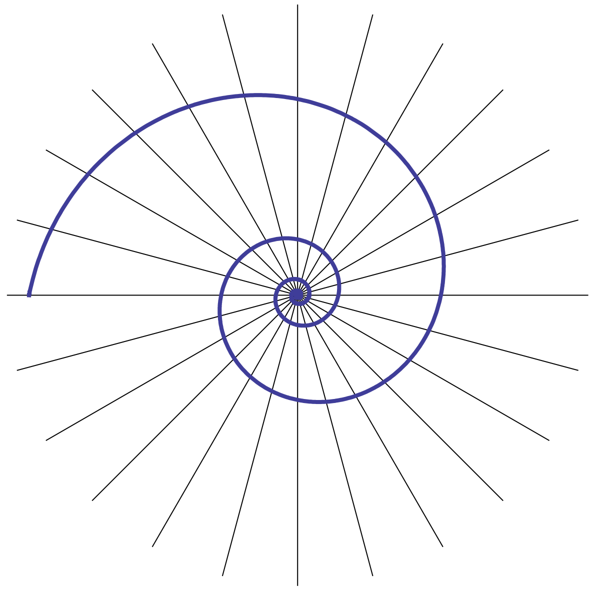

Fig. 3.1. A logaritmic spiral in E^2.

$$r(s) = as, \tag{5}$$

sometimes named the *natural equation* or *Cesàro equation* of the plane curve.

By stereographic projection of the Euclidean plane into the sphere one obtains the *loxodromes* as images of the logarithmic spirals. Using the conformal embedding (1.2.7) of the Euclidean plane into the Möbius sphere one obtaines loxodromes differing from the former by a reflection only.

Our aim is now to calculate the natural conformal parameter and the Möbius curvature of the logarithmic spirals or, equivalently, the loxodromes. Since the logarithmic spirals for distinct values of $b > 0$ distinguish by dilations only, they are conformally equivalent. Therefore we assume $b = 1$ from now on. The calculation have been carried out in the Mathematica notebook loxodromes.nb [32] which can be downloaded from my homepage. This notebook contains modules for the calculation of the Euclidean arc length and curvatures, e.g. formulas (3) - (5) and graphical tools for creating pictures of curves and surfaces. For the logarithmic spirals the Möbius-equivariant moving frame is calculated. The natural conformal parameter S as a function of the Euclidean arc length s is

$$S = \frac{\log s}{\sqrt{1 + a^2}}. \tag{6}$$

Inserting (3) one sees that S is a linear function of the group parameter t, the rotation angle. The constant conformal curvature of the spiral with parameter

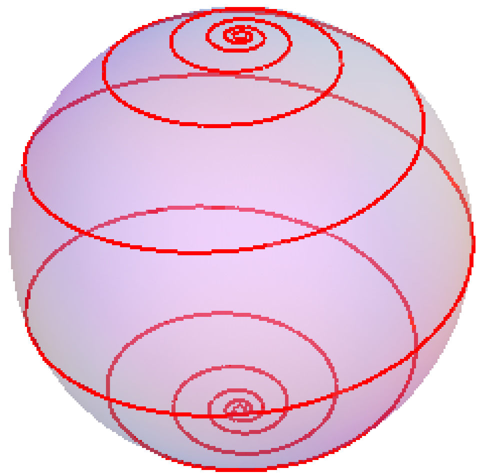

Fig. 3.2. A loxodrome on S^2, $a = 0.12$.

$a > 0$ is

$$cfk(a) = \frac{a^2 - 1}{2a}. \tag{7}$$

If a runs from 0 to $+\infty$ the conformal curvature runs monotonously from $-\infty$ to $+\infty$. Therefore the logarithmis spirals (2) with $b = 1$ (or the corresponding loxodromes) are a complete system of pairwise not conformally equivalent, positively oriented curves of constant conformal curvature. For $a = 1$ we obtain the spiral with conformal curvature 0, For $a > 1, a \to \infty$, the spirals becomes steeper, the intersection angle with the maridians tends to 0, and the spiral approaches a meridian. For $a < 1, a \to 0$, the intersection angle tends to $\pi/2$, the spiral becomes denser on the sphere and finally collapses into the equator. Remember that equator and meridians as circles are curves of type A_0^2, and not generally curved. More pictures and tools to explore the dependency of the shape of the spiral and its curvature may be found in the notebook mentioned above.

3.2.2 Curves of Constant Curvatures in S^3

In my paper [26], Corollary 2.1, I proved the following proposition solving a linear differential equation of order 4 for the isotropic representative of the curve, which has been deduced from the Frenet formulas, see Example 1.3.

Theorem 1. *Let k, h be two real numbers and set $chB(k,h) = 1 + kh^2/2$. For any such numbers there exists a curve $y(s) \in S^3$ of type A_2 with constant Möbius geometric curvatures $k1 = k, k2 = h$ and natural parameter $s \in \mathbf{R}$ uniquely defined up to Möbius transformations. These curves have the following properties:*

a) For $h = 0$ the curve is a loxodrome lying in a sphere $S^2 \subset S^3$.
b) If $chB(k,h) = 0$ the curve is Möbius equivalent to a helix in the Euclidean space E^3.
c) If $chB(k,h) > 0$ and $h \neq 0$ the curve is Möbius equivalent to a curve of constant metric curvatures in the hyperbolic space $E^3(-c^2)$.
d) If $chB(k,h) < 0$ the curve is Möbius equivalent to a curve of constant metric curvatures in the elliptic space $E^3(c^2)$.

Proof. Here I will prove the theorem using the fact that the curves of constant curvatures are orbits of 1-parametric subgroups of the Möbius group, see the beginning of this section and Proposition 8 in Section 3. By the Fundamental Theorem 3.1.10 the curves of type A_2 in the Möbius space S^3 form a Frenet class of curves, see Definition 2 in Section 3. The matrix of the Frenet formulas in Example 1.3 is

$$cc(k,h) = \begin{pmatrix} 0 & k & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & -h & 1 \\ 0 & 0 & h & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad k, h \in \mathbf{R}. \tag{8}$$

One easily checks that the matrix $cc(k,h)$ satisfies the equation

$$cc(k,h)' \cdot gr + gr \cdot cc(k,h) = 0,$$

where gr denotes the Gram matrix (1.3.33) in isotropic-orthogonal coordinates. By (1.3.59) this means that $cc(k,h)$ is an element of the Lie algebra of the Möbius group. By Section 3, Proposition 8, we get all curves of constant curvatures as orbits of the 1-parametric subgroups

$$g[k,h](t) := \exp(cc(k,h)t). \tag{9}$$

It follows

Proposition 2. *Let $x_0 = [\mathfrak{a}_0]$ be the origin of the Möbius space S^3 with respect to a fixed isotropic orthogonal coordinate system. Then the orbits $x[k,h](t)$ defined by*

$$\gamma[k,h](t) := g[k,h](t)x_0 \tag{10}$$

are curves $\gamma[k,h]$ of constant curvatures k,h in the Möbius space S^3, t is the natural conformal parameter. These curves form a complete system of representatives in the class of the curves of constant curvatures of type A_2. The column vectors of the matrix $g[k,h](t)$ are the vectors of the Frenet frame of the curve $\gamma[k,h]$. In particular the first column

$$\mathfrak{x}(t) := g[k,h](t) \cdot \mathfrak{a}_0$$

is the canonical representing vector function of the curve $\gamma[k,h]$. □

The expression of the matrix g[k,h](t) is too large to be handled manually or to be printed here. The calculations (and this expression) may be found in my Mathematica notebook [33], Section 2.1.1, Proposition. The next section of the notebook there contains a visualisation program for curves in the Möbius space S^3. It works in two steps: the first step constructs the curve in S^3 corresponding to the representing isotropic vector family. The second step shows this curve by stereographic projection into the Euclidean space E^3, which can be plotted by Mathematica's graphical tools. Thus we are able to show conformal images of all curves in S^3. The figure 3.3 is such an image.

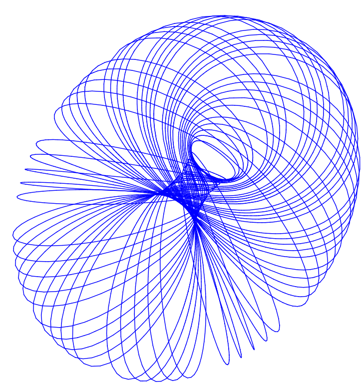

Fig. 3.3. Conformal image of the curve $\gamma[-2, 1.5]$.

The curves with $h = 0$ are m-flat, they lie on a subsphere and are conformally equivalent to the loxodromes. In the notebook [33], Section 2.2.2, we constructed a parameter representation for the corresponding spherical curves of constant conformal curvature k. The figure 3.4 is produced with this representation.

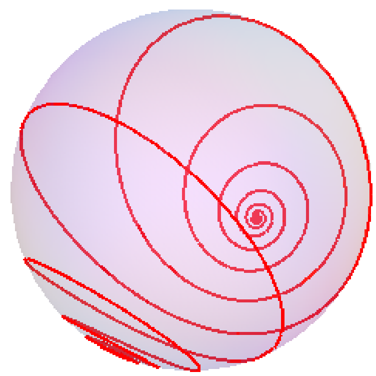

Fig. 3.4. Conformal image of the curve $\gamma[-5, 0]$.

As seen on this figure (and is proved in the notebook) the curve converges to two distinct points for $t \to \pm\infty$ and is an isogonal trajectory of the bundle of circles through these two points. Taking the limit points as North- and South Pole of the sphere one gets the curve in the usual form of a loxodrome.

The proof of Theorem 1 is carried out in great detail in the notebook [33]. The invariant $chB(k.h)$ is the pseudo-orthogonal scalar square of a non zero vector fixed under the action of the subgroup $g[k, h](t)$ in the 5-dimensional pseudo-Euclidean vector space; this vector is uniquely defined up to a constant scalar factor. In the case $chB(k, h) = 0$ the fixed vector is isotropic and defines a fixed point of the Möbius space whose complement gives a conformal model of the Euclidean space, invariant under the action of the group $g[k, h]$. Since the vector and not only its direction is fixed, the group is a subgroup of the Euclidean group and the curve has metric constant curvatures: it is a helix. In case c) the fixed vector is spacelike; its orthogonal complement is a 4-dimensional pseudo-Euclidean vector space whose hypersphere of imaginary radius is a model of the hyperbolic space invariant under the group $g[k, h]$. The orbits are curves of constant curvatures in the hyperbolic geometry. Finally, in case d) a timelike vector and its orthogonal complement, a 4-dimensional Euclidean vector space, and also its unit 3-sphere are invariant under the action of $g[k,h]$. The group is a subgroup of the isometry group of the 3-sphere and the orbits are curves of constant spherical curvatures. □

Moreover, in the notebook [33] is shown that any subgroup $g[k,h]$ with $h \neq 0$ lies in a uniquely defined maximal abelian subgroup being 2-dimensional and explicitly calculated. The orbits of these 2-dimensional subgroups are homogeneous surfaces named *Dupin cyclides*. We shall classify them and describe their properties later in the chapter about surface theory, see also my paper [27]. Any Dupin cyclide can be generated as an envelope of a 1-parametric sphere familiy (or plane family) in a twofold way. The elements of these families are transformed into each other by the transformation of the generating abelian subgroup, and the curves of constant curvatures are isogonal trajectories of the characteristics of these envelopes being circles or lines. The simplest example is the circular cylinder and the helices on it. The following figures created in [33] give examples corresponding to the cases b), c), d) in Theorem 1; they show a Dupin cyclide and a curve of constant curvature in each case.

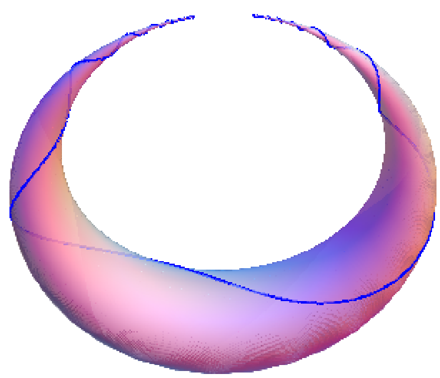

Fig. 3.5. $chB(k,h) = 0$: curve $\gamma[-0.295858, 1.3]$.

For $t \to \pm\infty$ the two ends of the curve and the cyclide come together; the common limit point has the coordinates $(0, 0, 1.833848)$. Applying a spherical reflection with this point as center one gets a circular cylinder with a helix as the image of figure 3.5:

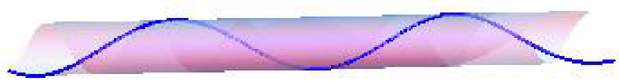

Fig. 3.6. Part of the spherically reflected Figure 3.5.

Figure 3.7 shows a spherical curve of constant curvatures and figure 3.8 a curve of constant curvatures in the hyperbolic space, both embedded in the corresponding Dupin cyclides.

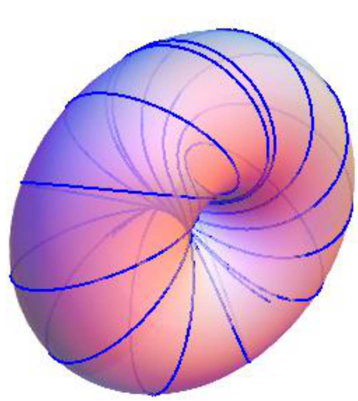

Fig. 3.7. $chB(k,h) = -8$: curve $\gamma[-2, 1.5]$.

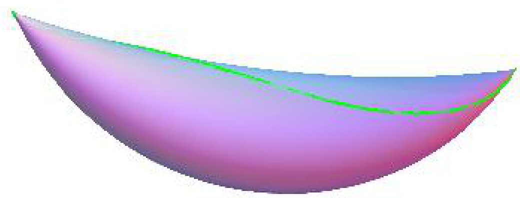

Fig. 3.8. $chB(k,h) = 3$: curve $\gamma[1,1]$.

Example 1. We consider a curve of constant curvature of the hyperbolic type: $chB[k,h] > 0$. In this case the curve tends to limit points for $t \to \pm\infty$, see figure 3.8, being fixed points of the generating 1-parametric transformation group. They are also limit points of the corresponding Dupin cyclide and appear as singularities of this surface, but they don't belong to it. In the Möbius 3-sphere they are given as the points corresponding to the eigenvectors to real eigenvalues of the transformation. If one takes one of the fixed point as the infinite point and the other as origin of a conformal model of the Euclidean space, the curve appears as an isogonal trajectory of the generators of a circular cone, see figure 3.9. Such curves we name *3D-spirals*. Its

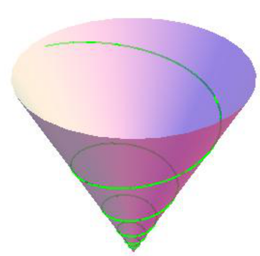

Fig. 3.9. A 3D-spiral.

projection into the x,y-plane (the plane orthogonal to the axis of the circular cone through its vertex) is a logarithmic spiral. Erecting the parallels to the z-axis on the points of the logarithmic spiral we obtain a *spiral cylinder* being a homogeneous surface, too, see [27]. It is an orbit of a two-dimensional solvable subgroup of the Möbius group. The details of the proofs are contained in my notebook [33] in which also the figures of this subsection are created. The 3D-spirals are the intersections of two homogeneous surfaces of the Möbius space, a circular cone and a spiral cylinder, see figure 3.10. □

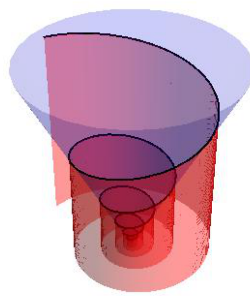

Fig. 3.10. A 3D-spiral as intersection.

3.2.3 Historical Notes

Originally Möbius transformations appeared as rational transformations of the complex plane. The natural parameter of plane curves has been found by G. Pick [20] in 1914. In the year 1923 H. Liebmann published a paper [16] containing the natural parameter and formulas for the conformal curvatures also for curves in the Möbius space S^3. Another not so elementary approach to the same subject was developed by E. Vessiot [36] in 1925. The monograph [3] of W. Blaschke and G. Thomsen contains a broad presentation of the geometry of plane curves (chapter 3), but only a short and incomplete overview of some results about curves in S^3 (chapter 6, § 60). A collection of elementary results not only for curves can be found in the voluminous monograph of T. Takasu [34]. From another point of view, namely that of conformal transformations of Riemannian manifolds, A. Fialkow [7] developed a conformal geometry of submanifolds of Riemannian spaces with tensorial methods. The natural conformal parameter which he defined is of higher order than the Möbius geometric natural parameter. In 1959 L. L. Verbitzkij [35] published a short and precice differential geometry of curves of the n-dimensional Möbius space. The paper contains the invariant natural parameter and the conformal curvatures of the curve. Our presentation in this book is a completed and corrected version of the paper R. Sulanke [26] published 1981. A geometric interpretation of the conformal curvatures as limits of invariants of pairs of osculating spheres appeared in the year 2001, see A. Montesinos Amilibia, M. C. Romero Fuster, and E. Sanabria Codesal [17].

3.3 Curves in Linear Homogeneous spaces

In this section we continue the considerations of section 1.3. The Möbius space and the space forms introduced in Chapter 1, or more generally, all Cayley-Klein spaces have the following property: they are homogeneous spaces $M \cong G/H$ whose defining group G is a subgroup of a real linear group. $G \subset GL(V^N)$. We name such spaces *linear homogeneous spaces*. In the first subsection we consider curves in linear group spaces, and in the second we apply these considerations to calculating the curves of constant curvatures in linear homogeneous spaces. Furthermore we describe curves of constant curvatures in some special linear homogeneous spaces.

3.3.1 Curves in Group Spaces

An advantage of the linear homogeneous spaces is that E. Cartan's method of moving frames can be applied in its elementary form: If $z_0 = (\mathfrak{a}_i)$ denotes a fixed frame (i.e. an ordered basis) of the vector space V^N, then the group G can be realized as the subgroup of matrices $(\gamma_i^j(g)) \in GL(N, \mathbf{R})$, defined by the coordinate representation of the *moving frame*

$$z(g) = (\mathfrak{c}_j) := (g(\mathfrak{a}_j)) \text{ with } g(\mathfrak{a}_j) = \mathfrak{a}_i \gamma_j^i.$$

The structure form ω of G has the matrix representation defined by

$$\omega = g^{-1}dg = (\omega_j^i) = (\gamma_k^i(g))^{-1}(d\gamma_j^k) \in \mathfrak{g} \subset \mathfrak{gl}(N, \mathbf{R}). \tag{1}$$

Here $\mathfrak{g}$ denotes the Lie algebra of the group G being a subalgebra of the general linear Lie algebra $\mathfrak{gl}(N, \mathbf{R})$ (see Example 1.3.3). Equivalently, the Maurer-Cartan forms ω_j^i of G can be defined as the coefficients of the basis decomposition of the differentials of the moving frame $z(g) = (\mathfrak{c}_j)$ with respect to itself:

$$d\mathfrak{c}_j = \mathfrak{c}_i \omega_j^i. \tag{2}$$

Now consider a smooth function $g(t) = (\gamma_j^i(t)) \in GL(N, \mathbf{R}), t \in \mathbf{R}$, and set for the components of the induced structure form

$$\omega_j^i(t, dt) = k_j^i(t)dt. \tag{3}$$

We obtain the derivation formulas

$$\frac{d\mathfrak{c}_j}{dt} = \mathfrak{c}_i k_j^i(t), \tag{4}$$

a special case of which are the Frenet formulas (3.1.34). Conversely, if the real functions $k^i_j(t)$ are given, equations (4) are a sytem of linear differential equations for curves in the matrix space $\mathcal{M}_N(\mathbf{R})$. Using the existence and uniqueness theorem for systems of linear differential equations one easily proves

Proposition 1. *Let $G \subset GL(N, \mathbf{R})$ be a Lie subgroup and $\mathfrak{g}$ its Lie algebra. Given a continuous matrix function defined on the possibly infinite interval Y^1:*

$$t \in Y^1 \longmapsto (k^i_j(t)) \in \mathcal{M}_N(\mathbf{R})$$

there exists for any starting conditions $\mathfrak{c}_j(t_0) = \mathfrak{b}_j$, $\mathfrak{b}_j \in V^N$ fixed vectors, a uniquely defined solution of the differential equation (4). The solutions $(\mathfrak{c}_j(t))$ are defined for all $t \in Y^1$; they are $(r+1)$-times continously differentiable if the functions $k^i_j(t)$ are r-times continuously differentable. The solution $g(t) = (\mathfrak{c}_j(t))$ belongs to the Lie group G iff the starting element $(\mathfrak{b}_j)$ belongs to G and the coefficient matrix (k^i_j) belongs to the Lie algebra $\mathfrak{g}$ of G.

Proof. The first and the second statement follow from the general theory of differential equations, see e. g. E. Kamke [12] § V.19. Recursively one obtains the continuous differentiability of the solution deriving equation (4). For the last statement we show first

Lemma 2. *The solution $(\mathfrak{c}_j(t))$ belongs to the general linear group $GL(N, \mathbf{R})$ iff the starting value $(\mathfrak{b}_j)$ has this property. Therefore the solution either lies entirely in the general linear group, or in its complement.*

Proof. Set $h(t) := \det(\gamma^i_j)$. Deriving the exterior product

$$\mathfrak{c}_1 \wedge \ldots \wedge \mathfrak{c}_N = \mathfrak{a}_1 \wedge \ldots \wedge \mathfrak{a}_N h(t),$$

where $(\mathfrak{a}_j)$ denotes a fixed basis of V^N, one obtains the differential equation for $h(t)$:

$$\frac{dh}{dt} = h(t)a(t) \text{ with } a(t) = \sum_0^N k^i_i(t). \tag{5}$$

The solution of this equation is

$$h(t) = h(t_0) \exp\Big(\int_{t_0}^t a(t)dt\Big),$$

what proves the lemma. □

If G is a Lie subgroup of $GL(N, \mathbf{R})$ and $X(t) := (k^i_j(t))$ belongs to its Lie algebra $\mathfrak{g}$, then for any $g \in G$ the element gX is tangential to G. Therefore, if the assumptions are satisfied, the starting value belongs to G and all the tangential vectors of the solution are tangential to G; thus the solution lies in G. The converse is trivial. □

Now we consider immersions $g(t) = (\gamma^i_j(t))$ of Y^1 into the group space of the linear Lie group G. Since G acts on itself simply transitively by the *left translations*

$$L_a : g \in G \longmapsto L_a(g) := ag \in G \qquad (a \in G),$$

it is a special case of a linear homogeneous space. The immersion condition means that the induced structure form doesn't have zeros on Y^1. By (1) and (3) this is equivalent to

$$(k^i_j(t)) = (\gamma^i_l)^{-1}(\frac{d\gamma^l_j}{dt}(t)) \neq 0 \text{ for all } t \in Y^1. \tag{6}$$

The coefficients $k^i_j(t)$ we name the *relative curvatures* of the immersion. The left invariance of the structure form implies the invariance of the relative curvatures under left translations, and Proposition 1 implies

Corollary 3. *Let $t \in Y^1 \longmapsto (k^i_j(t)) \neq 0$ be an r-times continuously differentiable function with values in the Lie algebra $\mathfrak{g}$ of the linear Lie group G. Then the solutions of* (4) *are immersions having the properties mentioned in Proposition* 1*. All these immersions are G-equivalent under left translations. If $g(t)$ denotes the solution with the starting value $g(t_0) = e$ the solution with starting value $a \in G$ is given by $L_a(g(t))$.* $\square$

We consider a parameter transformation

$$\varphi : \hat{Y}^1 \to Y^1,\ s \in \hat{Y}^1 \mapsto \hat{g}(s) = (\hat{\mathfrak{c}}_j(s)) := (\mathfrak{c}_j(\varphi(s))) \in G$$

of the curve in G defined by the immersion $g : t \in Y^1 \mapsto g(t) \in G$. One easily calculates: *The relative curvatures of the immersion $\hat{g}(s)$ are proportional to the relative curvatures of $g(t)$:*

$$\frac{d\hat{\mathfrak{c}}_j(s)}{ds} = \hat{\mathfrak{c}}_i(s)\hat{k}^i_j(s) \text{ with } \hat{k}^i_j(s) = k^i_j(\varphi(s))\frac{d\varphi(s)}{ds}. \tag{7}$$

Defining

$$k(t) := (\sum_{i,j}(k^i_j(t))^2)^{1/2},$$

it follows that the ratios

$$c^i_j(t) := k^i_j(t)/k(t)$$

which may be called the *curvatures* of the given curve in the group space G, don't depend on the parameter representation, i.e. the immersion, representing the curve. A parameter s is called a *natural parameter* if the equation $k(s) = 1$ is satisfied for all $s \in \hat{Y}^1$. Clearly, if $g(t)$ is any immersion representing the curve, one can find a natural parameter by integration

$$s(t) := \int_{t_0}^{t} k(t)dt = \varphi^{-1}(t)$$

and inverting this function. From this considerations one easily proves

Proposition 4. *For any curve in the group space of the linear Lie group G there exists a natural parameter s defined up to an addtive constant. The relative curvatures of the immersion with the natural parameter equal the curvatures of the curve. An arbitrary immersion $g(t)$ has a natural parameter $s = t$ iff for their relative curvatures the condition*

$$k(t) = (\sum_{i,j} (k^i_j(t))^2)^{1/2} = 1, \qquad (t \in Y^1), \tag{8}$$

is satisfied. Conversely, given the curvature matrix $(k^i_j(t)) \in \mathfrak{g}$ on Y^1 satisfying (8)*, the solutions of* (4) *represent all the curves in the group space G having t as a natural parameter and the functions $k^i_j(t)$ as curvatures.* □

Thus, the curvatures define the curve up to left translations. Note that the functions $k^i_j(t)$ may not be given arbitrarily; besides of (8) the relations ensuring $(k^i_j(t)) \in \mathfrak{g}$ must be fulfilled. We remark that the norming of the relative curvatures could be reached also by other conditions, e.g. if one considers a class of curves for which the property $\omega^1_2(t) \neq 0$ is known for all representing immersions, one could fix s by the condition $\omega^1_2(s) = ds$ up to an additive constant.

The curves of constant curvature in the group space of the linear group G can be found easily using the matrix exponential

$$\exp(X) := \sum_0^\infty \frac{X^\nu}{\nu!}. \tag{9}$$

The well known properties of the matrix exponential can be found in textbooks of matrix calculus, or in the classical book C. Chevalley [6]. They are summarized in the following

Proposition 5. *Let G be a linear Lie group with Lie algebra $\mathfrak{g}$. The matrix power series* (10) *converges for all $X \in \mathcal{M}(N, \mathbf{C})$ and has values in the general linear group $GL(N, \mathbf{C})$. The function* exp *has the following properties:*

1. *If* $\mathrm{tr}(X)$ *denotes the trace of the matrix X, one has*
$$\det(\exp(X)) = \exp(\mathrm{tr}(X)). \tag{10}$$
2. *There exists a neighbourhood $U \subset \mathfrak{g}$ of $0 \in \mathfrak{g}$ for which the restriction* $\exp|U$ *is an analytic diffeomorphism onto the neighbourhood $W = \exp U \subset G$ of the unit element $e \in G$.*
3. *For commuting X, Y, $[X, Y] = 0$, one has* $\exp X \exp Y = \exp Y \exp X$.
4. *Fix an element $X \neq 0$ in $\mathfrak{g}$. Then the map $t \in \mathbf{R} \mapsto \exp tX \in G$ is a parameter representation of a* 1*-parametric subgroup of* G*. Any* 1*-parametric subgroup of G can be obtained this way.*

5. *The* 1*-parametric subgroup corresponding to* $X \in \mathfrak{g}$ *satisfies the differential equation*

$$\frac{d \exp tX}{dt} = \exp tX \cdot X. \tag{11}$$

□

As a corollary of Proposition 5 one easily proves that the matrix exponential is a special case of the exponential defined in (1.3.49) for general Lie groups.

Now we compare equations (4) and (11). Considering the vectors $\mathfrak{c}_j$ as column vectors of the matrix $\exp tX$ and assuming the relative curvatures as constant, it follows

Corollary 6. *Let* $X := (k^i_j) \in \mathfrak{g}$*,* $X \neq 0$*, be constant. Then* $g(t) := \exp tX$*,* $t \in \mathbf{R}$*, is the solution of* (4) *with starting condition* $g(0) = e$*; it represents a curve of constant curvatures in the group space* G*. The parameter* t *is a constant multiple of a natural parameter. Every curve of constant curvatures in* G *is obtained by a left translation of the* 1*-parameter subgroup corresponding to the curvature matrix* X*. More exactly: All the curves of constant curvatures with curvature matrix* $X \neq 0$ *are represented by immersions* $f(t) = ag(t)$ *being solutions of (4) with starting condition* $f(0) = a$*.* □

3.3.2 Curves in Linear Homogeneous Spaces

Now let $t \in Y^1 \to x(t) \in M^n = G^r/H^{n-r}$ be an immersion representing a curve in the linear homogeneous space M. The immersion $t \in Y^1 \to g(t) \in G$ is called a *lift* of the given immersion $x(t)$ if

$$p(g(t)) = g(t)H = x(t) \qquad (t \in Y^1) \tag{12}$$

is satisfied. With other words, the lift is nothing else as a moving frame along the curve.

Definition 1. A lift $g(t)$ of the immersion is named a *Frenet lift* or a *Frenet moving frame*, if its relative curvatures are G-invariants of the corresponding oriented curve $x(t) = p(g(t))$. □

In general, Frenet lifts don't need to exist for arbitrary curves. However in the classical geometries for large classes of curves there exist canonically defined moving frames, which are special cases of Frenet frames. Their derivation equations (4) are the Frenet formula e. g. in Euclidean geometry:

Example 1. The *affine group* $A(n)$ and the *Euclidean group* $E(n)$ are linear groups: one obtaines isomorphic embeddings into the general linear group $GL(n+1, \mathbf{R})$, if in affine coordinates with respect to a fixed basis $\mathfrak{a}_i$, $i = 1, \ldots, n$, and the fixed origin $o \in A^n$ the affine transformation

$$f(C,\mathfrak{c}) : \mathfrak{x} \in A^n \longmapsto f(C,\mathfrak{c})(\mathfrak{x}) := C\mathfrak{x} + \mathfrak{c} \in A^n, \qquad (\mathfrak{c} \in \mathbf{R}^n, C \in GL(n,\mathbf{R})),$$

is mapped to the matrix by

$$f(C,\mathfrak{c}) \in A(n) \longmapsto \begin{pmatrix} 1 & \mathfrak{o}' \\ \mathfrak{c} & C \end{pmatrix} \in GL(n+1,\mathbf{R}).$$

The frames are here the column vectors of the matrix; the zeroth: $(1,\mathfrak{c})' = (1,\gamma^1,\ldots\gamma^n)$ represents the point with coordinates (γ^i) in A^n, and the other n columns $(0,\mathfrak{c}_j)' = (0,\gamma^1_j,\ldots,\gamma^n_j)'$, $j = 1,\ldots,n$, are a basis of the vector space W^n of A^n, $W^n \subset V^{n+1}$ defined by $\gamma^0 = 0$. For Euclidean spaces these bases are supposed to be orthonormal; for other restrictions of affine geometries one has to consider the corresponding assumptions for the bases, e, g. pseudo-orthonormal or symplectic. The affine Lie algebra $\mathfrak{a}(n)$ is represented by matrices of the shape

$$\begin{pmatrix} 0 & \mathfrak{o}' \\ \mathfrak{b} & B \end{pmatrix} \in \mathfrak{a}(n) \Longleftrightarrow \mathfrak{b} \in \mathbf{R}^n \text{ and } B \in \mathcal{M}(n,\mathbf{R}).$$

For the Euclidean Lie algebra $\mathfrak{e}(n)$ represented in orthonormal coordinates additionally the skew symmetry of the matrix B must be required:

$$\begin{pmatrix} 0 & \mathfrak{o}' \\ \mathfrak{b} & B \end{pmatrix} \in \mathfrak{e}(n) \Longleftrightarrow B + B' = 0.$$

We remember some basic facts of the Euclidean curve theory. A detailed exposition of the n-dimensional Euclidean curve theory may be found on my homepage. Furthermore there one finds a Mathematica notebook [31] with many programs for calculations of the Euclidean invariants and for graphical presentations of the curves, see the section Mathematica Notebooks and Comments/Euclidean Curve Theory.

Let $x(s)$ be an immersion of Y^1 into the Euclidean space E^n, s its natural parameter, the arc length, and $(\mathfrak{c}_j(s))$ the canonical frame of the curve. The relative curvatures of the moving frame are given by the curvatures $k_1(s),\ldots,k_{n-1}(s)$ for generally curved immersions as the *Frenet formulas for curves in the Euclidean space* E^n show:

$$\begin{aligned}
&\frac{dx}{ds} = \mathfrak{c}_1,\\
&\frac{d\mathfrak{c}_1}{ds} = \mathfrak{c}_2 k_1(s),\\
&\frac{d\mathfrak{c}_2}{ds} = -\mathfrak{c}_1 k_1(s) + \mathfrak{c}_3 k_2(s),\\
&\cdots\cdots\cdots,\\
&\frac{d\mathfrak{c}_j}{ds} = -\mathfrak{c}_{j-1} k_{j-1}(s) + \mathfrak{c}_{j+1} k_j(s),
\end{aligned}$$

$$\begin{aligned}&\cdots\cdots\cdots,\\ &\frac{d\mathfrak{c}_{n-1}}{ds} = -\mathfrak{c}_{n-2}k_{n-2}(s) + \mathfrak{c}_n k_{n-1}(s),\\ &\frac{d\mathfrak{c}_n}{ds} = -\mathfrak{c}_{n-1}k_{n-1}(s).\end{aligned}$$

The curvature matrix for the class of generally curved curves in the Euclidean space E^n can not be given arbitrarily; it has the well defined shape

$$F(k_1,\dots,k_{n-1}) := \begin{pmatrix} 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & k_1 & 0 & \dots & \dots & \dots \\ 0 & -k_1 & 0 & k_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & k_{j-1} & 0 & k_j & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -k_{n-2} & 0 & k_{n-1} \\ 0 & \dots & \dots & \dots & \dots & -k_{n-1} & 0 \end{pmatrix}. \tag{13}$$

Furthermore, for this class of curves it is well known that the curvatures satisfy the necessary conditions

$$k_j(s) > 0 \text{ for } j = 1,\dots,n-2. \tag{14}$$

The shape (13) and the conditions (14) are obtained during the adaption of the moving frame by E. Cartan's or another geometrical method. Finishing the adaption one finds a Frenet frame of the curve such that the derivation equations with (13) and (14) are satisfied. Then Proposition 1 and Corollary 3 show that given the curvatures $k_j(s), j = 1,\dots n-1$, such that (14) is satisfied there exists one and up to Euclidean motions only one moving frame having the curvature matrix (13); by the shape of the Frenet formulas it is clear that s is the arc length and $k_j(s)$ are the curvatures of the curve defined by the immersion $x(s) \in E^n$; as a component of the moving frame it is defined uniquely up to an Euclidean motion. This is the content of the so-called *Fundamental Theorem of Euclidean curve theory* which states that a generally curved curve in the n-dimensional Euclidean space is characterized by its $n-1$ curvatures as functions of the arc length. □

Another example is the fundamental theorem in the Möbius geometry of curves, s. (1.34) and Theorem 1.10. This and other existence and uniqueness theorems of differential geometry of curves show that the curvature matrix for a given class of Frenet curves $(k^i_j(t))$ is a function $F(k_1, k_2, \dots, k_m)$ of a complete invariant system $k_1,\dots,k_m$, mostly named the curvatures of the curve. For formulating a general fundamental theorem for curves in linear homogeneous spaces we need

Definition 2. A *Frenet class of curves* in the linear homogeneous space G/H is defined to be a class of curves satisfying the following conditions:

1. For any curve $\gamma \in \Gamma$ is given an m-tupel of G-invariants $k_i(t), i = 1, \dots, m$, defined on an open interval $t \in Y^1$, and satisfying some necessary conditions $\mathcal{R}$ (see e.g. (14)).
2. There is given a matrix function $F : \mathbf{R}^m \to \mathfrak{g} \subset \mathcal{M}(N, \mathbf{R})$ such that any $\gamma \in \Gamma$ has a Frenet moving frame $g(t)$ with relative curvatures $F(k_1(t), \dots, k_m(t))$, $k_i(t)$ G-invariants of γ.
3. For any m-tupel $(k_1, \dots, k_m)$ satisfying the conditions $\mathcal{R}$ the elements $F(k_1(t), \dots, k_m(t))$ do not belong to the Lie algebra $\mathfrak{h}$ of the isotropy subgroup H.
4. Γ is closed under the action of G: with $a \in G$ and $\gamma \in \Gamma$ one has also $l_a(\gamma) \in \Gamma$.

□

Theorem 7. *Let $G \subset GL(N, \mathbf{R})$ be a linear Lie group and $\mathfrak{g}$ the Lie algebra of G. We consider a Frenet class Γ of oriented curves in the linear homogeneous space G/H. Then for any Frenet frame of γ the parameter t is uniquely defined up to an additive constant, and for any m-tupel of functions $(k_i(t)), i = 1, \dots, m$, satisfying the necessary conditions $\mathcal{R}$, there exists one and up to a transformation $l_a, a \in G$, only one curve $\gamma \in \Gamma$ having the given functions as its curvatures.* [1] *For any element $a \in G$ the immersion $ag(t)$ is a Frenet lift of the immersion $l_a(x(t))$.*

Proof. For a curve $\gamma \in \Gamma$ let $(k^i_j(t))$ be the matrix of the relative curvatures of a Frenet frame $g(t)$. We remember that equation (4) can be written symbolically as

$$\frac{dg}{dt} = g(t)X(t) \qquad (X(t) = (k^i_j(t)) \in \mathfrak{g}). \tag{15}$$

The immersion condition for the lift implies $X(t) \neq 0$. Since the k^i_j by Definition 1 of the Frenet frame are invariants of the curve (or the same constants for all $\gamma \in \Gamma$), by (7) the only parameter transformation φ preserving them are those with $d\varphi/dt = 1$ and the first statement follows. Now let the curvatures $k_i(t), i = 1, \dots, m$, satisfying the conditions $\mathcal{R}$ be given and consider equations (4) with the coefficient matrix $(k^i_j(t)) = F(k_1(t), \dots, k_m(t))$. Applying Proposition 1 we find the corresponding immersions $g(t)$ into the group space as solutions of (4). By condition 3 of Definition 2 the projections $x(t) = p(g(t))$ are immersions into the homogeneous space G/H; indeed, the differential dp_g has the kernel $g\mathfrak{h} \subset T_gG$. Having in mind the uniqueness property of the solutions $g(t)$ and the rule

$$p \circ L_a = l_a \circ p \qquad (a \in G) \tag{16}$$

[1] In this sense one names the m-tupel $(k_1(t), \dots, k_m(t))$ a *complete system of G-invariants* for the curves in Γ.

the uniqueness of the curve up to transformations l_a and the last statement are proved. Since equations (4) are fulfilled for the Frenet frame $g(t)$ of the curve $x(t)$, the functions $k_i(t)$ are invariants of γ characterizing the curve.□

Now we consider the curves of constant curvatures. By the general properties of the solutions of (4) with constant coefficients the solutions are defined for all $t \in Y^1 = \mathbf{R}$. We remark first

Proposition 8. *Any orbit of a 1-parameter subgroup of the Lie group G in the homogeneous space G/H is a fixed point or a curve of constant curvatures. Conversely, any curve of constant curvatures in a Frenet class Γ of curves in a linear homogeneous space is part of an orbit of a 1-parameter subgroup of G.*

Proof. Let $g(t) = \exp(Xt)$ be the 1-parameter subgroup corresponding to $X \in \mathfrak{g}$. We consider the vector field on G/H generated by the action of $g(t)$:

$$\tilde{X}(x) := \frac{d \exp(Xt)x}{dt}|_{t=0} \in T_x(G/H). \tag{17}$$

The zeros of this field are the fixed points of the action, and its integral curves are the 1-dimensional orbits. Now consider an orbit $x(t) = g(t)x_0$. Since it is homogeneous:

$$x(t+s) = g(t+s)x_0 = g(t)g(s)x_0 = g(t)x(s),$$

all G-invariants of the orbit must be constant. Conversely, let $x(t)$ be a parameter representation of a curve $\gamma \in \Gamma$ with constant curvatures $k_1, \dots, k_m$. We may asssume that $x(t)$ is defined for all $t \in \mathbf{R}$. First we consider a curve γ whose Frenet frame $z(t)$ has the property $z(0) = e \in G$. This Frenet frame is the uniquely defined solution of (4) with constant coefficients $F(k_1, \dots, k_m)$ and starting condition $z(0) = e$. Fix any $s \in \mathbf{R}$ and consider the immersion $w(t) := L_{z(s)}(z(t)) = z(s)z(t)$. Since the curvatures are invariant under left translations and constant, $w(t)$ is the solution of (4) with the starting condition $w(0) = z(s)$. On the other hand, the immersion $t \mapsto z(s+t)$ has the same property. By the uniqueness of the solution we obtain $z(s+t) = z(s)z(t)$: the Frenet frame itself is a 1-parameter subgroup[2]. Projecting $z(t) = z(t) \cdot e$ and applying (16) one gets $x(t) = l_{z(t)}x(0)$ is an orbit of the subgroup $z(t)$. For an arbitrary curve γ of constant curvatures $y(t)$ in G/H let $a = v(0) \in G$ be the element of its Frenet frame $v(t)$ at $y(0)$ Then $a^{-1}v(t)$ is the solution (4) with starting condition $z(0) = e$, and it follows $a^{-1}v(t) = z(t)$. The projection

$$y(t) = p(v(t)) = p((az(t)a^{-1} \cdot a). = az(t)a^{-1}y(0)$$

shows that $y(t)$ is the orbit of a conjugated subgroup to $z(t)$. □

Proposition 8 shows the possibility to find all curves of constant curvatures in a linear homogeneous space: One has to find all Frenet classes of curves in

[2] This statement follows also from Corollary 6

the space and in each class to solve the differential equation (4) with constant coefficients. Since G-equivalent curves of constant curvatures correspond to conjugated 1-parameter subgroups, the classification of the curves can serve as a tool for finding the conjugacy classes of the subgroups. By the easily to prove well known formula

$$a \cdot \exp X \cdot a^{-1} = \exp(\mathrm{Ad}(a)(X)) \qquad (a \in G, X \in \mathfrak{g}) \tag{18}$$

this is equivalent to the classification of the elements of the Lie algebra $\mathfrak{g}$ under the action of the *adjoint representation* of the Lie group G defined by

$$\mathrm{Ad}(g)(X) := dL_g \circ dR_{g^{-1}}(X) \qquad (g \in G, X \in \mathfrak{g} = T_eG). \tag{19}$$

In the case of linear groups this amounts to the classification of the matrices $X \in \mathfrak{g}$ (or linear endomorphisms of V^N) under the action of G:

$$\mathrm{Ad}(g)(X) = g \cdot X \cdot g^{-1} \text{ for } G \text{ linear.} \tag{20}$$

Comparing Cayley-Klein Geometries we often meet the following situation: The homogeneous space $M^n = G/H$ is an orbit $M^n = Gx_0 \subset X^N = G_1/H_1$ in the homogeneous space X^N, where G is a Lie subgroup of the larger Lie group G_1; in this case we name the transformation group $[G, M^n]$ a *subgeometry* of $[G_1, X^N]$. Since any 1-parameter subgroup of G is also a 1-parameter subgroup of G_1 it follows

Corollary 9. *The curves of constant curvatures of a subgeometry G/H of G_1/H_1, $G \subset G_1$, are also curves of constant curvatures in the larger homogeneous space under the canonical embedding*

$$\iota : x = gH \in G/H \mapsto \iota(x) = gH_1 \in G_1/H_1, \qquad (g \in G).$$

□

References

1. M. A. Akivis and V. V. Goldberg. *Conformal Differential Geometry and its Generalizations*. John Wiley & Sons, Inc., New York etc., 1996.
2. W. Blaschke and H. Reichardt. *Einführung in die Differentialgeometrie*. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1960.
3. W. Blaschke and G. Thomsen. *Vorlesungen über Differentialgeometrie III, Differentialgeometrie der Kreise und Kugeln*. Grundlehren der mat. Wiss. Bd. 29. Springer-Verlag, Berlin, 1929.
4. E. Cartan. *La théorie des groupes fini et la géométrie différentielle, traitées par la méthode du repère mobile*. Gauthier-Villars, Paris, 1937.
5. E. Cartan. *Riemannian Geometry in Orthogonal Frames*. Izd. Moskovskogo Universiteta, Moscow, 1960. Russian. Edited by S. P. Finikov.
6. C. Chevalley. *Theory of Lie Groups*, volume I. Princeton University Press, Princeton, 1946.
7. A. Fialkow. The conformal theory of curves. *Trans. Amer. Math. Soc.*, 51:435–501, 1942.
8. H. Freudenthal and H. De Vries. *Linear Lie Groups*. Academic Press, New York, 1969.
9. S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, New York, San Francisco, London, 1978.
10. Udo Hertrich-Jeromin. *Introduction to Möbius Differential Geometry*. London Math. Soc. L. N. Series 300. Cambridge University Press, Cambridge, 2003.
11. G. R. Jensen. *Higher Order Contact of Submanifolds of Homogeneous Spaces*, volume 610 of *L. N. in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1977.
12. E. Kamke. *Differentialgleichungen reeller Funktionen*. Akademische Verlagsgesellschaft Geest und Portig K.-G., Leipzig, 1952.
13. S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry II*. Interscience Publishers, New-York, London, 1963.
14. S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry I*. Interscience Publishers, New-York, London, Sydney, 1969.
15. André Lichnérowicz. *Géométrie des Groupes de Transformations*. Dunod, Paris, 1958.
16. H. Liebmann. Beiträge zur inversionsgeometrie der kurven. *Sitzungsberichte der Mathematisch-Physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München*, 53:79 – 94, 1923.

17. A. Montesinos Amilibia, M.C. Romero Fuster, and E. Sanabria Codesal. Conformal curvatures of curves in $\mathbf{R}^{n+1}$. *Indag. Math., New Ser.*, 12(3):369–382, 2001.
18. A. L. Onishchik and R. Sulanke. *Algebra und Geometrie I.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1986. 2. Auflage.
19. A. L. Onishchik and R. Sulanke. *Projective and Cayley-Klein Geometries.* Springer-Verlag, Berlin, Heidelberg, 2006.
20. G. Pick. Zur theorie der konformen abbildung kreisförmiger bereiche. *Rendiconti del Circolo Matematico di Palermo I*, 37:341–344, 1914.
21. P. K. Raschewski. *Riemannsche Geometrie und Tensoranalysis.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. Deutsche Übersetzung; Original: Gostechisdat, Moskau 1953; 2. Aufl. 1964.
22. Sarah E. Salvioli. On the theory of geometric objects. *Journal of Differential Geometry*, 7:257–278, 1972.
23. Ch. Schiemangk and R. Sulanke. Submanifolds of the möbius space. *Math. Nachr.*, 96:165 – 183, 1980.
24. R. W. Sharpe. *Differential Geometry*, volume 166 of *Graduate Texts in Mathemarics.* Springer-Verlag, New York, Berlin, Heidelberg,..., 1997.
25. R. Sulanke. On e. cartan's method of moving frames. *Colloquia Mathem. Soc. J. Bolyai. Differential Geometry, Budapest 1979*, 31:681–704, 1979.
26. R. Sulanke. Submanifolds of the möbius space, ii. frenet formulas and curves of constant curvatures. *Math. Nachrichten*, 100:235–247, 1981.
27. R. Sulanke. Möbius geometry v. homogeneous surfaces in the möbius space. *Coll. Math. Soc. J. Bolyai 46. Topics in Differential Geometry. Debrecen 1984*, 46:1141–1154, 1984.
28. R. Sulanke. Submanifolds of the möbius space iv. conformal invariants of immersions into spaces of constant curvature. *Potsdamer Forschungen, Reihe B*, 43:21 – 26, 1984.
29. R. Sulanke and A. Švec. Zur differentialgeometrie der untermannigfaltigkeiten eines kleinschen raumes. *Beitrï£¡e zur Algebra und Geometrie*, 10:63–85, 1980.
30. R. Sulanke and P. Wintgen. *Differentialgeometrie und Faserbündel.* VEB Deutscher Verlag der Wissenschaften, Birkhäuser, Berlin, Basel, 1972.
31. Rolf Sulanke. *Euclidean Curve Theory.* Mathematica Notebook, http://www-irm.mathematik.hu-berlin.de/~sulanke/diffgeo/euklid/eucurves.tgz, 2009.
32. Rolf Sulanke. *Loxodromes.* Mathematica Notebook, http://www-irm.mathematik.hu-berlin.de/~sulanke/Notebooks/loxodromes.nb, 2009.
33. Rolf Sulanke. *Curves of Constant Curvatures in Möbius Geometry.* Mathematica Notebook, http://www-irm.mathematik.hu-berlin.de/~sulanke/Notebooks/cccmdg.nb, 2010.
34. T. Takasu. *Differentialgeometrie in den Kugelräumen I. Konforme Differentialgeometrie von Liouville und Möbius.* Tokyo, 1938.
35. L.L. Verbitzkij. Foundations of curve theory in the n-dimensional conformal space (russ.). *Izv. Vysch. Utsch. Saved. Matematika*, 6:26–37, 1959.
36. E. Vessiot. Enveloppes de sphéres et courbes gauches. *J. École Polytechnique 2. ser.*, 25:43–91, 1925.
37. Joseph A. Wolf. *Spaces of Constant Curvature.* University of California, Berkeley, 1972.

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