Curves in the Möbius Space

by

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Preface

In this notebook we develop *Mathematica* tools for the Möbius differential geometry of curves. We construct Modules for the calculation of their Möbius invariants: the natural parameter, the conformal curvatures and the conformal Frenet frame for curves inthe 3-dimensional Möbius space. Frenet formulas for curves in the Möbius space are deduced. They are applied to the calculation and graphical presentation of osculating spheres and circles. We express the natural conformal parameter by Euclidean invariants and the conformal curvatures by the coordinates of a representative of the curve in the Euclidean 3-space. A short presentation of Euclidean curve theory is given in my paper [ECG] which together with a corresponding notebook EuCurves.nb may be downloaded from my homepage. To any generally curved curve in the Möbius n-sphere belongs a so called anti-curve uniquely and equivariantly defined by the last vector of the isotropic-orthogonal Frenet frame. The calculations and examples illustrate and complete the presentation of the subject in my yet not finished book [mdg], where one finds the general theory and more bibliographical dates. The two-dimensional case has been treated very detailed in my *Mathematica* notebook [mdg2D].

The notebook is designed for interactive work. Access to s. Wolfram's program Mathematica is a necessary condition for using it. Some experience in using Wolfram's programming language is recommended. An unexperienced user could learn this language working with the notebook using the tools collected in the Help menu. Before starting to evaluate the notebookk one has to prepare a working directory as described in the next Section Initialization. Then the work always begins calling the menu item Evaluation/Evaluate Initialization Cells. Information about a built-in Mathematica symbol one gets calling ?Symbol. Symbols and functions of Wolfram Language are detailed explained in the accompanying Help tool: Select the symbol and press F1 to read it, Example: Evaluate the next cell

In[•]:= ? Function

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Symbol                                                    $Failed

body & or Function[body] is a pure (or "anonymous")
  function. The formal parameters are # (or #1), #2, etc.
Function[x, body] is a pure function with
  a single formal parameter x.
Function[{x1, x2, …}, body] is a pure function
  with a list of formal parameters.
Function[params, body, attrs] is a pure function that is treated
  as having attributes attrs for purposes of evaluation.

Documentation Local » | Web »
    Attributes {HoldAll, Protected}
    Full Name System`Function

$Failed
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Out[•]=

Click the Link "Local" or "Web" to learn more. I recommend to use the Web. The functions and symbols introduced in the loaded packages are liste in Section 1. Other more specific functions are introduced in Initialization Cells of the notebook; they are explained in the context. Section 2 containes concepts of pseudo-Euclidean linear algebra necessary for treating differential geometry of submanifolds intheMöiusspace. Section 3 is the most important part of the notebook; the curve theory in the 3-dimensional Möbius space is developed here.
In the actual 4-th version of the notebook the Section 4 about curves of constant Möbius curvatures has been added. It containes a revised version of the now obsolete notebook [ccc]. Depending on the conformal constant curvatures each generally curved curve of constant curvatures has been assigned one of the following types: Euclidean, hyperbolic, or elliptic. They are Möbius equivalent to the curves of constant metric curvatures in the geometry of their type. Section 5 gives some interesting examples. It can be enlarged investigating the conformal igeometry of curves contained in A. Gray's package CURVES.m, see Subsection 1.8.

Finally I wish to express my sincere gratitude to the Mathematical Institute of the Humboldt University for giving me as a retired member access to use the software and the website of the Institute. In particular I thank Dr. J. Gehne and Mrs. H. Pahlisch for their help in solving some technical problems. Also I thank
S.. Wolfram, M. Trott and the colleagues of Wolfram's Technical Support Team for their interest and help in overcoming certain difficulties during my work applying Mathematica to differential geometry.

Keywords

Möbius group, generally curved curves, Möbius invariant natural parameter, Möbius invariant curvatures, anti-curve, Frenet frame, Frenet formulas, osculating circles, osculating spheres,

isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, Minkowski space, curves of constant Möbius curvatures, one-parametric subgroups of the Möbius group.

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Initialization

Before starting to work interactively with this notebook first time, read this section carefully and make the necessary preparation. Later it suffices to call the menu item "Evaluation. Evaluate Initialization Cells".

The Needed Packages

The Initialization

1. Lists of Symbols and their Usages

In this section one finds tables of the symbols introduced in the imported packages and in the Global Context.
To get the usages click on the name! If this does not work, enable Dynamic Updating in the Evaluation Menu.

1.1. Symbols in the Package vectorcalc.m

1.2. Symbols in the Package euvec.m

1.3. Symbols in the Package eudiffgeo.m

1.4. Symbols in the Package neuvec.m

1.5. Symbols in the Package mspher.m

1.6. Symbols in the Package circ.m

1.7. Symbols in the Package moeb2.m

1.8. Symbols in Alfred Gray's Package CURVES.m

1.9. Symbols in the Global Context

2. Pseudo-Euclidean Linear Algebra

The package neuvec.m contains interesting tools enhancing *Mathematica* with concepts of pseudo-Euclidean linear algebra. In this chapter we describe some of these tools and introduce some new functions which we need in Möbius differential geometry.

2.1. Test of some Functions of neuvec.m

2.2. Isotropic-Orthonormal Bases

2.3. Conformal Embeddings of Simply Connected Space Forms

2.3.1. Spheres of Radius r

2.3.2. The Euclidean Space

2.3.3. The Hyperbolic Space

2.3.4. The Map sphmap

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3. Curves in the Möbius Space

In this chapter we give a systematic presentation of the differential geometry of curves in the 3-dimensional Möbius space, see also [mdg].

3.1 Isotropic Representatives of Euclidean Curves

3.2. A Special 2nd Order Frame

3.2.1. Definition of the Frame

3.2.2. Examples

3.2.3. The Derivation Equations of the Frame mfre2[i}[mgcurve3D][t]

3.3. The Natural Parameter

3.4. A Special Third Order Frame

3.4.1. A Moving Frame of Third Order

3.4.2. Examples

3.5. TheMöbius-Geometric Frenet Frame for Curves in the Möbius Space

3.5.1. Frenet Frame and Conformal Curvatures

3.5.2. The Frenet Formulas of Möbius Geometry

3.5.3. Example. Helix

3.6. Osculating Circles and Spheres

Systematically the concepts of osculating spheres and circles belong to Möbius geometry. Then they are equivariantly defined under Euclidean transformations a fortiori. We begin this section with the Euclidean definitions of these concepts. Then we construct Modules for then in the framework of Möbius geometry.

3.6.1. Euclidean Definition of the Osculating Circle and the Osculating Sphere

3.6.2. Osculating Circles in Möbius Geometry

3.6.3. Osculating Spheres in Möbius Geometry

3.7. Anti-Curves

4. Curves of Constant Conformal Curvatures

In this section we describe the curves of constant curvatures as orbits of 1-parameter subgroups of the Möbius group, see section 3.2 of [MDG]. There is shown that any curve of constant Möbius curvatures k, h is Möbius equivalent to an orbit of the 1-parametric subgroup g(t) = exp[cc[k,h]t] of the Möbius group. In section 4.1 we calculate these subgroups, what leads to very large formulas, hardly to work with them manually. Nevertheless, by the help of *Mathematica* we may classify and visualize the curves of constant curvatures as shown in the following subsections. Since equivalent curves (under the action of the Möbius group) correspond to conjugated subgroups we obtain at the same time the classification of the 1-parametric subgroups of the Möbius group of S^3 in conjugacy classes.

4.1. The 1-Parametric Subgroups of the Möbius Group Generate the Curves of Constant Curvatures

4.1.1. The Matrix of the Frenet Equation with Constant Curvatures.

4.1.2. The Generating 1-Parameter Subgroup.

4. 1.3. Visualization in the Euclidean Space. Examples

4.2. Classification and Properties of the 1-Parameter Subgroups

4.2.1. The Classification Theorem

4.2.2. The Case h=0,k≠0

4.3. The Eigensystem of cc[k,h]

4.3.1. The Definitions

4.3.2 A General Test

4.3.3. The Case h = 0

4.4. Case 3, the Euclidean Case chB=0

4.5. Case 2, the Hyperbolic Case chB>0

4.6. Complete Systems of Representatives in the Hyperbolic Case

4.7. The Elliptic Case chB<0

4.7.1. The Elliptic Curves of Constant Curvature

4.7.2. Isotropic Representatives and Conformal Curvatures

4.8. Complete Systems of Representatives in the Elliptic Case

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5. Further Examples

5.1. Example. The Ellipse

5.2. Example Elliptic Helices

5.3. Example Torusknot

5.4. The Helix, its Curvatures and Anticurves

5.5. Example Horopter

5.6. Example. A Conical Spiral

5.7. Example. An Orbital Spiral

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Warning. This notebook is obsolete. Don’t open this notebook as a subnotebook of mdgcurves3D.nb: mismatch of definitions!

MGII

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MGIV

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Homepage

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