Curves in the Möbius Plane

by

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Mathematica v. 8.0.1, v.11.3.

Summary

In this notebook we develop *Mathematica* tools for the Möbius differential geometry of plane curves. We construct Modules for the calculation of their Möbius invariants: the natural parameter and the conformal curvature. Frenet frames and Frenet formulas for curves in the Möbius plane are deduced. We express the natural conformal parameter and the conformal curvature by Euclidean invariants of the curve. A short presentation of Euclidean curve theory is given in my paper [ECG] which together with a corresponding notebook EuCurves.nb may be downloaded from my homepage. To any generally curved curve in the Möbius n-sphere belongs a so called anti-curve uniquely and equivariantly defined by the last vector of the isotropic-orthogonal Frenet frame. The osculating circles of the plane curve are related to the adapted frames, Modules to calculate and plot these circles are presented. The calculations and examples illustrate and complete the presentation of the subject in my yet not finished book [mdg].
The revision leads to many corrections and improvements. The Mathematica packages necessary for working with the notebook remained unchanged, they may be downloaded here: moebpack.zip.

Before starting the revision I tested the first edition of this notebook (finished 2012) with the actual Mathematica v. 11.3. It came out, that the evaluations need often more time than earlier; some evaluations, in particular some simplifications of symbolic expressions by the function FullSimplify, couldn't be completed in a reasonable time, or Mathematica exited without result. Thus I decided to work out the revision with Mathematica v. 8.0.1. Finishing the revision I tested the notebook again with the two mentioned versions of Mathematica. Unfortunately, the first test results are confirmed, see the evaluation results described in red text cells in subsections 3.2.6, 3.2.7. The used Mathematica versions are installed on the same workstation with an Intel Core i7 processor and 16 GB Ram under Linux open SUSE Leap 42.3. The third test, if mentioned, results noticed after "Windows 10", is run on a laptop with an Intel Core i7 processor and 8 GB Ram, with Mathematica v. 11.3 under Windows 10..

Keywords

Möbius group, generally curved plane curves, Möbius invariant natural parameter, Möbius invariant curvature, anti-curve, Frenet frame, osculating circles, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, Minkowski space.

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Initialization

1. Lists of Symbols and their Usages

In this section one finds tables of the symbols introduced in the imported packages and in the Global Context.
To get the usages click on the name! If this does not work, enable Dynamic Updating in the Evaluation Menu.

1.1. Symbols in the Package vectorcalc.m

1.2. Symbols in the Package euvec.m

1.3. Symbols in the Package eudiffgeo.m

1.4. Symbols in the Package neuvec.m

1.5. Symbols in the Package mspher.m

1.6. Symbols in the Package mcirc.m

1.7. Symbols in the Package moeb2.m

1.8. Symbols in the Global Context

2. Pseudo-Euclidean Linear Algebra

The packages neuvec.m and moeb2.m contain important tools enhancing *Mathematica* with concepts of pseudo-Euclidean linear algebra. In this section we describe some of these tools and introduce some new functions which we need in Möbius differential geometry.

2.1. Test of some Functions of neuvec.m

2.1.1. Some Often Used Functions

2.1.2. New Function: indexorder

2.2. Isotropic-Orthonormal Bases

2.3. The Function sphrefl

2.4. Conformal Embeddings of Simply Connected Space Forms

2.3.1. Spheres of Radius r

2.3.2. The Euclidean Space

2.3.3. The Hyperbolic Space

2.3.4. The Spherical Map sphmap

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3. Curves in Dimension 2

3.1 Isotropic Representatives of Euclidean Curves

3.2. The Frenet Frame

3.2.1. A Moving Frame of 2nd Order

3.2.2. The Derivation Equations of the 2nd Order Frame

3.2.3. Example. The Circle

3.2.4. Curves in the Euclidean Plane Parametrized by their Arc Length

3.2.5. The Natural Parameter. Pick's Invariant

3.2.6 The Next Reduction Step

3.2.7. The Last Reduction Step

3.3.. A Curvature Formula

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4. Examples

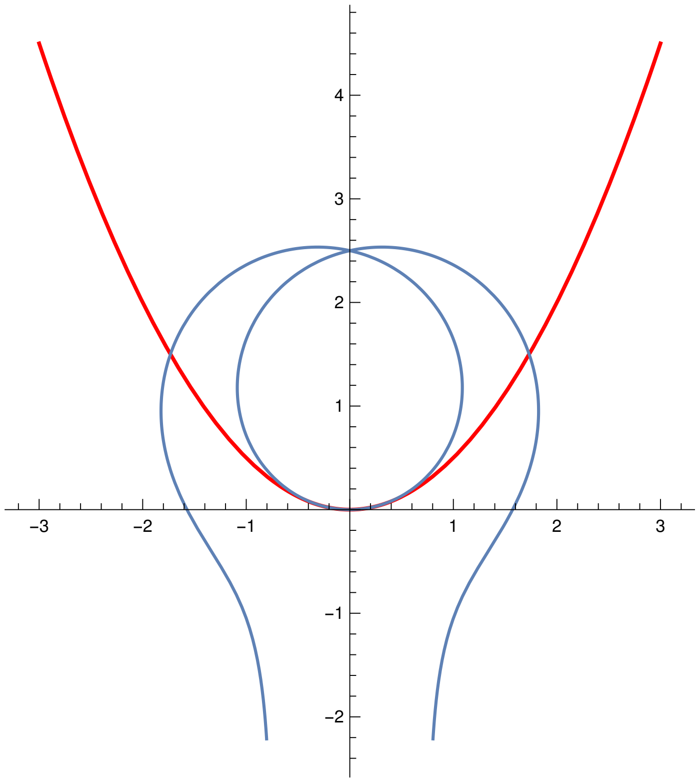
par

4.1. Example. The Parabola

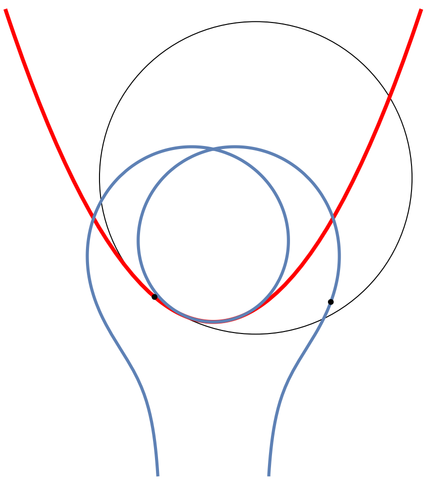
In this subsection we consider a parabola and find its natural parameter and its Möbius curvature. Furthermore we define anticurves, osculating circles and normal vectors for generally curved plane curves and test these concepts with the considered parabola.

4.1.1. Curvatures and Natural Parameters, epsilon2D

4.1.2.Anticurves, Anti-Parabola



4.1.3 Osculating Circles

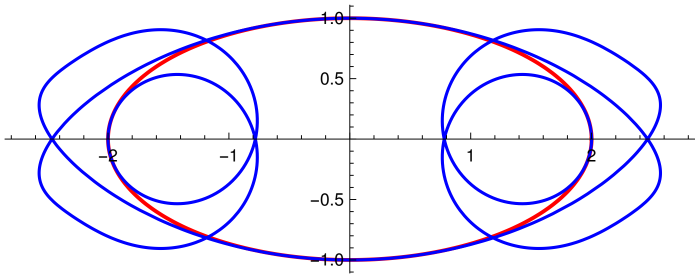


ell

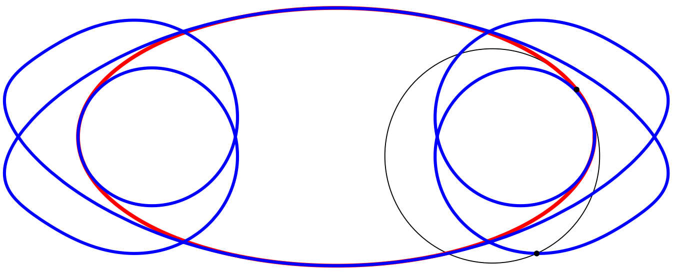
4.2. Example. The Ellipse

3.4.1. The Curvatures

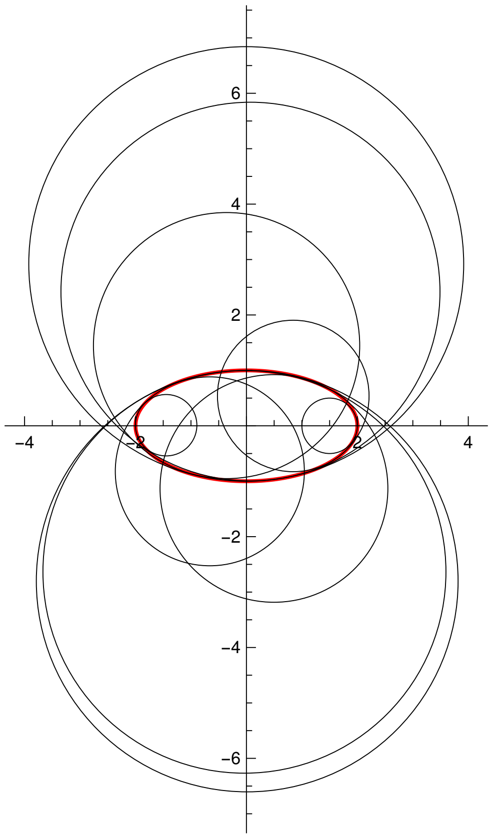
3.4.2. The Anti-Ellipse



3.4.3. Osculating Circles



3.4.4. A Family of Osculating Circles



4.3. Example. The Logarithmic Spiral

The logarithmic spirals are treated also in my notebook Loxodromes [lox}.

3.5.1. Curvatures and natural parameter

3.5.2. The Anti-Spiral

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Show[{spir, antispir}, PlotRange → All]
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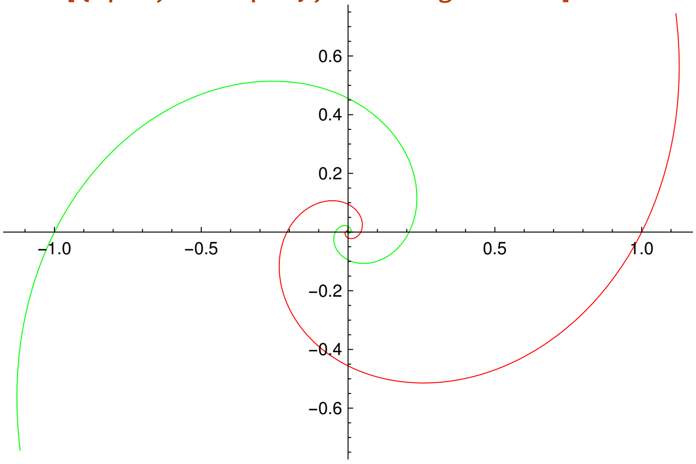
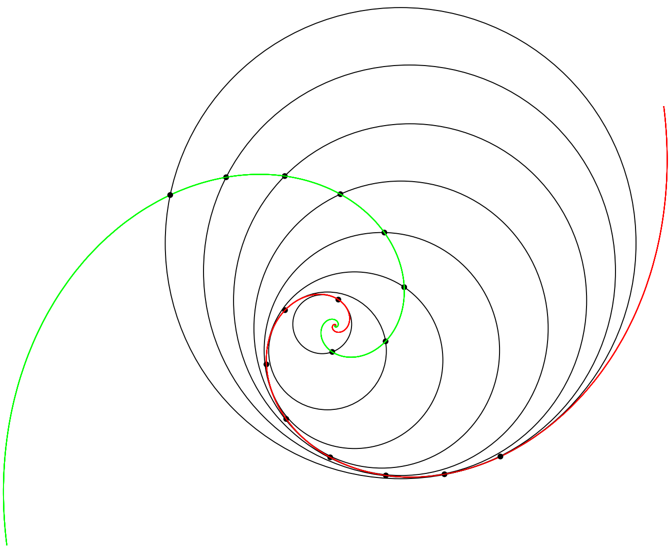


3.5.3. Osculating Circles



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Gray

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ECG

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OS3

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MDG

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lox

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Homepage

Home

http://www-irm.mathematik.hu-berlin.de/~sulanke