Topological K-theory

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Wednesdays, 13.30-15.00
Johnann von Neumann Haus, seminar room 4.007

The aim of algebraic topology is to understand a topological space X by associating to it groups, whose algebraic properties reflect topological properties of X. The motivation comes from the fact that we can usually distinguish groups more easily than topological spaces.
We will begin this lecture by introducing the formal definition of (extraordinary) cohomology theories, which associate to any topological space an infinite sequence of abelian groups. We sketch some examples and mention some facts about the classification of such theories without going deeply into details. Instead, we will quickly focus on a particular example of an extraordinary cohomology theory which we then study for the rest of the semester: topological K-theory. What sets K-theory apart from other cohomology theories is that it can be defined quite directly by vector bundles over X, or alternatively, in terms of continuous functions on X. We will discuss both constructions simultaneously, and show that they lead to the same cohomology theory. The lecture will be divided into five parts:

Part I: Cohomology theories - a brief survey

A few words on category theory; ordinary and extraordinary cohomology theories; examples; Brown's representability theorem.

Part II: Vector bundles and matrices of functions

Definition of vector bundles; constructing new vector bundles out of old ones; examples; matrices of continuous functions and their relation to vector bundles ("a bundle is a family of projections").

Part III: K-theory: definition and basic properties

The Grothendieck completion; a little bit homological algebra; definition of K-theory $K(X)$, relative K-theory $K(X, A)$, and odd K-theory $K^{-1}(X, A)$; the connecting homomorphism $\partial : K^{-1}(A) \to K(X, A)$; K-theory of locally compact spaces; verifying the axioms of an extraordinary cohomology theory; computation of some K groups.

Part IV: Bott periodicity

The famous Bott periodicity theorem states that there is a natural isomorphism $K(X) \to K^{-1}(X)$. This result is very important for K-theory...and we will need some time to prove it ;)

Part V: Additional topics

The content of this part will depend on the remaining time and the interests of the audience. Possible topics are: multiplicative structures, the Thom isomorphism theorem, the L-type definition of K-theory, operations, characteristic classes, the Atiyah-Jänich theorem.

We do not assume that the audience has a broad knowledge about algebraic topology. In particular, it should be possible to follow the first three parts for anyone who is acquainted with the basic notions of point set topology. Anything else can be provided and recalled during the lecture.

Litarature

- Atiyah, M. F. K-theory. Notes by D. W. Anderson. Second edition. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.
- Hatcher, A. Vector bundles and K-theory, available online.
- Lawson, H. Blaine, Jr. ; Michelsohn, Marie-Louise . Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.
- Park, Efton . Complex topological K-theory. Cambridge Studies in Advanced Mathematics, 111. Cambridge University Press, Cambridge, 2008.
- Switzer, Robert M. Algebraic topology-homotopy and homology. Reprint of the 1975 original, Classics in Mathematics. Springer-Verlag, Berlin, 2002.