(1) Recall that $H_2(\mathbb{C}P^2)$ is generated by an embedded sphere $\mathbb{C}P^1 \subset \mathbb{C}P^2$ with $[\mathbb{C}P^1] = 1$. A holomorphic curve $u : \Sigma \to \mathbb{C}P^2$ is said to have degree $d \in \mathbb{N}$ if $[u] = d[\mathbb{C}P^1]$. Show that all holomorphic spheres of degree 1 are embedded, and any other simple holomorphic sphere is embedded if and only if it has degree 2.

(2) Suppose $\Sigma$ and $\Sigma'$ are compact oriented surfaces with boundary, $M$ is a closed 4-manifold, and $u_s : \Sigma \to M$ and $v_s : \Sigma' \to M$ for $s \in [0, 1]$ are smooth homotopies such that for all $s$,

$$u_s(\partial \Sigma) \cap v_s(\Sigma') = u_s(\Sigma) \cap v_s(\partial \Sigma') = 0.$$ 

Show that if $u_s \cap v_s$ for $s = 0, 1$, then

$$\sum_{u_0(z) = v_0(\zeta)} \tau(u_0, z; v_0, \zeta) = \sum_{u_1(z) = v_1(\zeta)} \tau(u_1, z; v_1, \zeta),$$ 

where we denote by $\tau(u, z; v, \zeta) = \pm 1$ the sign of a transverse intersection $u(z) = v(\zeta)$.

(3) Given a compact surface $\Sigma$ with boundary, a complex line bundle $L \to \Sigma$, and a trivialisation $\tau$ of $L|_{\partial \Sigma}$, the relative first Chern number

$$c_1^L(L) \in \mathbb{Z}$$

can be defined as the signed count of zeroes of a generic section $\eta : \Sigma \to L$ such that $\eta|_{\partial \Sigma}$ is nonzero and constant with respect to $\tau$.

(a) Prove that $c_1^L(L)$ as described above does not depend on the choice of the section $\eta$.

(b) Prove that the relative first Chern number admits a unique and well-defined extension to higher rank complex vector bundles such that

$$(E, \tau) \cong (E', \tau') \Rightarrow c_1^L(E) = c_1^L(E')$$

and

$$c_1^{E_1 \oplus E_2}(E_1 \oplus E_2) = c_1^{E_1}(E_1) + c_1^{E_2}(E_2).$$

(4) Suppose $(W, \omega)$ is a symplectic cobordism with convex boundary $(M_+, \xi_+ = \ker \alpha_+)$ and concave boundary $(M_-, \xi_- = \ker \alpha_-)$, $(\hat{W}, \hat{\omega})$ is its completion, and $J$ is an almost complex structure on $\hat{W}$ that is compatible with $\omega$ on $W$, and on the cylindrical ends is translation-invariant and satisfies

$$J(\partial s) = R_{\alpha_\pm}, \quad J(\xi_\pm) = \xi_\pm \quad \text{and} \quad J|_{\xi_\pm} \text{ is compatible with } d\alpha_\pm|_{\xi_\pm},$$

where $R_{\alpha_\pm}$ denotes the Reeb vector field on $M_\pm$ determined by $\alpha_\pm$. For any smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi' > 0$ and $\varphi(s) = s$ near $s = 0$, consider the smooth 2-form $\omega_{\varphi} := \begin{cases} \omega & \text{on } W, \\
 \frac{d}{ds}(e^{\varphi(s)} \alpha_+) & \text{on } [0, \infty) \times M_+,
\frac{d}{ds}(e^{\varphi(s)} \alpha_-) & \text{on } (-\infty, 0] \times M_-.
\end{cases}$

Show that $\omega_{\varphi}$ is symplectic and $J$ is $\omega_{\varphi}$-compatible.