(1) In Lecture 3, we saw the theorem of Hofer-Wysocki-Zehnder that for any Reeb orbit \( \gamma : S^1 \to M \) in a contact 3-manifold \((M, \xi = \ker \alpha)\), with asymptotic operator \( A_\gamma \) and trivialisation \( \tau \) of \( \gamma^* \xi \to S^1 \), the function
\[
\sigma(A_\gamma) : \lambda \mapsto \text{wind}^\tau(\lambda) := \text{wind}^\tau(f) \text{ for any nontrivial } f \in \ker(A_\gamma - \lambda)
\]
is well defined, monotone increasing, and attains every value in \( \mathbb{Z} \) exactly twice (counting multiplicity of eigenvalues).

(a) Verify that the above theorem holds for the \( L^2 \)-symmetric operator
\[
A := -J_0 \frac{d}{dt} - c : C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R}^2),
\]
where \( J_0 \) denotes the standard complex structure on \( \mathbb{R}^2 = \mathbb{C} \) and \( c \in \mathbb{R} \) is any constant. (The general case can be derived from this using a deformation argument.)

(b) If \( \gamma(t) = \gamma_0(kt) \) for another Reeb orbit \( \gamma_0 : S^1 \to M \), then the \( k \)-fold cover of each eigenfunction of \( A_{\gamma_0} \) is an eigenfunction of \( A_\gamma \). Assuming \( \tau \) is the pullback under \( S^1 \to S^1 : t \mapsto kt \) of a trivialisation of \( \gamma_0^* \xi \to S^1 \), show that a nontrivial eigenfunction \( f \) of \( A_\gamma \) is a \( k \)-fold cover if and only if \( \text{wind}^\tau(f) \) is divisible by \( k \).

(c) Assume \( \gamma_0 \) is an embedded orbit that is \( k \)-fold covered by \( \gamma \), and \( \tau \) is defined by pulling back a trivialisation of \( \gamma_0^* \xi \to S^1 \). Show that for any nontrivial eigenfunction \( f \) of \( A_\gamma \),
\[
\text{cov}(f) := \max\{k \in \mathbb{N} \mid f \text{ is a } k \text{-fold cover}\} = \gcd(k, \text{wind}^\tau(f)).
\]

(d) Show that if \( \gamma \) is a Reeb orbit that has even Conley-Zehnder index, then so does every multiple cover \( \gamma^k \) of \( \gamma \).

(2) Assume \( \gamma : S^1 \to M \) is a nondegenerate Reeb orbit in a contact 3-manifold \((M, \xi = \ker \alpha)\), with covering multiplicity
\[
\text{cov}(\gamma) = \max \{k \in \mathbb{N} \mid \gamma(t + 1/k) = \gamma(t) \text{ for all } t \in S^1\}.
\]
Given \( J \in \mathcal{J}(\alpha) \), let \( u_\gamma : \mathbb{R} \times S^1 \to \mathbb{R} \times M \) denote the associated \( J \)-holomorphic orbit cylinder.

(a) Show that \( c_N(u_\gamma) = -p(\gamma) \), where \( p(\gamma) \in \{0, 1\} \) is the parity of the Conley-Zehnder index of \( \gamma \).

(b) Show that \( u_\gamma \ast u_\gamma = -\text{cov}(\gamma) \cdot p(\gamma) \).

(c) Deduce from part (b) that if \( u^k \) denotes a \( k \)-fold cover of a given asymptotically cylindrical \( J \)-holomorphic curve \( u \), it is \textit{not} generally true that \( u^k \ast v^\ell = k\ell(u \ast v) \).

\textbf{Remark:} One can show however that in general,
\[
u^k \ast v^\ell \geq k\ell(u \ast v).
\]

(d) \((*)\) Use the adjunction formula to show the following: if \( \gamma \) is a multiple cover of a Reeb orbit with even Conley-Zehnder index, and \( J' \) is an arbitrary almost complex structure on \( \mathbb{R} \times M \) that is compatible with \( d(e^t \alpha) \) and belongs to \( \mathcal{J}(\alpha) \) outside a compact subset, then there is no simple \( J' \)-holomorphic curve homotopic to \( u_\gamma \) through asymptotically cylindrical maps.