Background material for Lecture 5

Open book decompositions of 3-manifolds
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Consider $\pi : M \setminus B \to S^1$ such that:

- $B \subset M$ is an oriented link ("binding")
- $M \setminus B \xrightarrow{\pi} S^1$ is a fibration (fibres = "pages"),

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\text{nbhd}(B) \cong \bigsqcup (S^1 \times \mathbb{D}^2) \xrightarrow{\pi} S^1 \\
(\theta, (r, \phi)) \mapsto \phi
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Hence:

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\[S^3 = \mathbb{R}^3 \cup \{\infty\} \quad \text{and} \quad S^1 \times S^2\]
The Giroux correspondence

A contact structure $\xi$ is **supported** by an open book $\pi : M \setminus B \to S^1$ if $\xi = \ker \alpha$ for some contact form $\alpha$ such that

$\alpha|_{TB} > 0$ and $d\alpha|_{\text{pages}} > 0$. 
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$$\{\text{OBDs}\} \longrightarrow \{\text{ctct strs}\}/\text{isotopy}$$
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**Giroux:**\n
$$\{\text{ctct strs}\}/\text{isotopy} \overset{1:1}{\longleftrightarrow} \{\text{OBDs}\}/\text{stabilisation}$$
Bordered Lefschetz fibrations

Lefschetz fibration $W^4 \xrightarrow{\pi} \mathbb{D}^2$ with interior critical points and fibres with boundary:
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$W$ has boundary and corners: smooth faces

$$\partial W = \partial_v W \cup \partial_h W,$$

where

$$\partial_v W := \pi^{-1}(\partial \mathbb{D}^2) \xrightarrow{\text{fibration}} \partial \mathbb{D}^2 = S^1,$$

and

$$\partial_h W := \bigcup_{z \in \mathbb{D}^2} \partial \left( \pi^{-1}(z) \right) \cong \coprod (S^1 \times \mathbb{D}^2)$$
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$\implies \partial W$ inherits an open book.
Theorem
Any bordered Lefschetz fibration $W \xrightarrow{\pi} D^2$ admits (canonically up to deformation) a symplectic form $\omega$ such that fibres are symplectic and $(W, \omega)$ has convex boundary $(M, \xi)$ supported by the induced open book.
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If no irreducible components of singular fibres are closed (i.e. $W \xrightarrow{\pi} \mathbb{D}^2$ is “allowable”), then one can make $(W, \omega)$ a Stein filling of $(M, \xi)$. 

![Diagram](image-url)
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**Proposition**

The monodromy of the open book on $\partial W$ is a composition of positive Dehn twists, one for each critical point of $W \xrightarrow{\pi} \mathbb{D}^2$. 