States: What is SFT a what is it good for?

Hamiltonian dynamics: $H \in C^\infty(\mathbb{R}^{2n})$

$(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$

"position" \rightarrow "momentum"

Hamilton's eqns: $\dot{q}_i = \frac{\partial H}{\partial p_i}(q, p), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(q, p)$

i.e. $x(t) = (q(t), p(t)) \in \mathbb{R}^{2n}$ satisfies $\dot{x} = X_H(x)$ for $X_H = \sum_{j=1}^{n} \left( \frac{2}{m_j} \frac{\partial}{\partial \dot{q}_j} - \frac{\partial V(q)}{\partial q_j} \right)$.

$\dot{q} = H(q, p) = \frac{1}{2} |\dot{p}|^2 + V(q) = $ total energy $\implies \dot{p} = -\nabla V(q)$

Neroton's 2nd law
Set \( \omega \in \Omega^2(\mathbb{R}^{2n}) \) ("standard symplectic form on \( \mathbb{R}^{2n} \))

\[ \omega \text{ is (i) closed: } d\omega = 0 \]

(ii) nondegenerate: \( \forall x \in \mathbb{R}^{2n}, \quad \exists ! X \neq 0 \in T_x \mathbb{R}^{2n} \text{ s.t. } \omega(x, y) = 0 \quad \forall y \in T_x \mathbb{R}^{2n} \)

(ii) \( \omega \) determines an iso \( T^*_x \mathbb{R}^{2n} \rightarrow T^*_x \mathbb{R}^{2n} : \quad X \mapsto \omega_x(\cdot, \cdot) \)

**EX:** The "Hamiltonian vec. field" \( X_H \) is uniquely characterized by

\[ \omega(X_H, \cdot) = -dH \]

defn: A symplectic form (symplectic structure) on a smooth 2n-mfld \( M \) is

(i) closed \& (ii) nondegenerate 2-form \( \omega \in \Omega^2(M) \), (\( M, \omega \)) is then a symplectic mfld.

Then any smooth fn \( H : M \rightarrow \mathbb{R} \) has vec. field \( X_H \) st.

\[ \omega(X_H, \cdot) = -dH \]
The flow \( \varphi_t : M \to M \) of \( X_\parallel \) preserves (1) \( x \) and (2) \( \omega \).

(1) \( d H \mid_{X_\parallel} = -\omega (X_\parallel, X_\parallel) = 0 \).

(2) Lie derivative \( \text{Lie}_v (\text{tensor}) \frac{1}{\parallel \omega \parallel} \omega + \frac{1}{\parallel X_\parallel \parallel} \omega = -d \omega \mid_{\parallel \omega \parallel} = 0 \).

\( \implies \varphi_t^* \omega = \omega \forall t \) (i.e. \( \varphi_t \) is a symplectomorphism \( (M, \omega) \to (M, \omega) \)).

\( \omega \in \Omega^2 (M^n) \) nondegen. \( \iff \) \( \omega^n : = \omega \wedge \cdots \wedge \omega \) is a volume form on \( M \).

\( \text{cor} : \text{Hamiltonian flows } \varphi_t \text{ preserves volume } \omega \text{ on } M \text{ w.r.t. } \omega^n \).

Q: For \( H : M \to \mathbb{R} \) \& \( c \in \mathbb{R} \) a regular value of \( H \), does \( X_\parallel \) admit a periodic orbit on \( H^{-1} (c) = : \Sigma \)?

\( \text{anw} : \text{answer depends on } \Sigma \text{ but not } H, \omega \) nondeg. \( \iff \forall x \in \Sigma, \exists ! 1 \text{-dim subspace } I_x \subseteq T_x \Sigma \text{ s.t. } \forall x \in I_x, \omega (X_\parallel, \cdot) \mid_{T_x \Sigma} = 0, \text{ i.e. } I_x = \ker (\omega \mid_{T_x \Sigma}) \).

\( \omega (X_\parallel, \cdot) \mid_{T_x \Sigma} = -dH \mid_{T_x \Sigma} = 0 \) \( \implies X_\parallel \) is a section of the subbundle \( I \subseteq T \Sigma \) (the characteristic line field). \( \therefore \) Up to parametrization, orbit depends on \( I \), not \( H \).
Then (Rellich/Kondrachov '58): in \((\mathbb{R}^n, \omega_{std})\), every star-shaped hypersurface has a closed orbit.

**Motivation:** Define vector field \(V(q,p) = \frac{i}{2} \sum_{j=1}^{n} (q_j \frac{\partial}{\partial q_j} + p_j \frac{\partial}{\partial p_j})\) so \(V \wedge \Sigma\). Then

\[
\omega_{std}(V, \cdot) = \sum_{j=1}^{n} (p_j dq_j - q_j dp_j) = \lambda_{std},
\]

\[
dV_{std} = \omega_{std}. \quad \text{Then } 2V \omega_{std} = dV \omega_{std} + \omega_{std} \frac{\partial}{\partial t}
\]

**Defn:** \(V \in \mathcal{X}(M)\) is a Liouville vector field on \((M, \omega)\) if

\[
2_V \omega = \omega; \quad \text{equivalently: } \int (\gamma_V)^* \omega = e^t \omega \quad \text{for the flow } \gamma_V^t \text{ of } V
\]

\[
\Lambda \colon V(\cdot, \cdot) \text{ is a primitive of } \omega \Leftrightarrow 2_V \Lambda = dV \Lambda + \omega \Lambda = V \omega = \Lambda, \quad \Rightarrow (\gamma_V^t)^* \Lambda = e^t \Lambda.
\]

**Defn:** A hypersurface \(\Sigma\) in \((M, \omega)\) is of contact type if a null of \(\Sigma\) admits a Liouville vector field \(V\) st. \(V \wedge \Sigma\).
Then \((\varepsilon, \varepsilon) \times \Sigma \xrightarrow{\varphi} M : (\varepsilon, x) \mapsto \varphi_V(x)\) presents a model of \(\Sigma = (\varphi_V^*)^1 \lambda = e^t \lambda \Rightarrow \) for \(\alpha := \lambda |_{\tau \Sigma} \) we have \(\varphi^* \lambda = e^t \alpha, \)

\(\Rightarrow \varphi^* \omega = d(e^t \alpha) = e^t (dt \wedge \alpha + d\lambda) \)

\(\Rightarrow \varphi^* \omega \) restricts to each hypersurface \(\varepsilon \tau \Sigma \times \Sigma \) as \(e^t d\lambda \)

\(\Rightarrow \) they all have the same characteristic field.

\(\Rightarrow \) if one of the hypers \(\varphi^* (\varepsilon \tau \Sigma \times \Sigma) \subseteq M\) has a closed orbit, they all do.

Weinstein conjecture: Closed \(\varepsilon \tau \Sigma\) type hypersurface always admit periodic orbits.

Status: For \(\dim \Sigma = 3\), proved in 2007 by Tonks (via Seiberg-Witten).

- Otherwise open except some special cases (e.g., residue, SFT).
(M, ξ) is called a contact manifold.

A diffeomorphism \( \varphi : M_1 \rightarrow M_2 \) is a contactomorphism \( (M_1, \xi_1) \rightarrow (M_2, \xi_2) \) if

\[ \varphi_* \xi_1 = \xi_2 \] and the co-orientation. If \( \xi_j = \ker \alpha_j \) for \( j = 1, 2 \), this means

\[ \varphi^* \alpha_2 = f \alpha_1 \] for some smooth function \( f : M_1 \rightarrow (0, \infty) \).
Gray's stability thm: If $M^{2n-1}$ is closed & $\{S_ \leq TM^\perp \}_{S \leq 0,1}$ a smooth 1-param. fam. of stlt stts, then $S_0 = (\psi_0)^* S_0$ for some smooth 1-param. fam. of diffes $\psi_S: M \to M$ w/ $\psi_0 = id$. (= all contactomorph.)

(ch: Not true for stlt forms!)

con: For a stlt hyp. $\Sigma \subseteq (M,\omega)$, the stlt stt. $\Sigma := \text{ker} (1_\omega|_\Sigma)$ in (up to isotopy) indp. of choice of Liouville vec. field.

defn: For closed stlt wfd $(M^{2n-1}_\pm, \Sigma_\pm)$, a symplectic cobordism from $(M_-, \Sigma_-)$ to $(M_+, \Sigma_+)$ is a cpct sig. wfd. $(W^{2n}, \omega)$ w/ oriented boundary $\partial W = M_+ \sqcup (-M_-)$ s.t. $M_\pm$ are stlt type hypersurfaces w/ induced stlt sets $\Sigma_\pm$ (up to isotopy).
Bordism theory $\Rightarrow$ A purely topological obstruction to

\[ \text{symp. cobordism between any 2 closed symplectic manifolds of same dim.} \]

\[ \text{defn: Cobordism from } M_- \to M_+ \begin{cases} \text{case } M_- = \emptyset: \text{ symplectic filling of } M_+ \text{ (soft)} \\ \text{case } M_+ = \emptyset: \text{ symplectic cap of } M_- \end{cases} \]

then ("soft"): All closed symplectic manifolds admit symplectic caps.

then ("hard" - via SFT): $\exists$ a seq. of non-fillable symplectic manifolds $\{M_k\}_{k=0}^\infty$ s.t. $\exists$ an exact symplectic cobd. $\bigcup_{k=1}^{\infty} M_k$ if $k \leq l$.