TOPICS IN TOPOLOGY ("TOPOLOGIE III'), SOMMERSEMESTER 2024, HU BERLIN

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This is not a set of lecture notes, but merely a brief summary of the contents of each lecture, with reading suggestions and a compendium of exercises. The suggested reading will usually not correspond precisely to what was covered in the lectures, but there will often be a heavy overlap.

Prologue: notation

Before getting into the content of the course, here is a glossary of important notation that is used in the lectures, including some comparison with other sources such as [tD08, DK01, Wen23] where different notation is sometimes used. This glossary will be updated during the semester as needed, and it is not in alphabetical order, but there is some kind of ordering principle... maybe you can figure out what it is.

Categories.

- General shorthand: For any category $\mathscr{C}$, I often abuse notation by writing $X \in \mathscr{C}$ to mean "X is an **object** in $\mathscr{C}$"; many other authors denote this by "$X \in \mathrm{Ob}(\mathscr{C})$" or something similar. For two objects $X, Y \in \mathscr{C}$, I write
$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \qquad \text{or sometimes just} \qquad \mathrm{Hom}(X, Y)$$
for the set of **morphisms** $X \to Y$. The notation $\mathrm{Mor}(X, Y)$ is also frequently used in many sources, and would make more sense linguistically, but it seems to be less popular. Given two functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$, the notation
$$T : \mathcal{F} \to \mathcal{G}$$
means that T is a **natural transformation** from $\mathcal{F}$ to $\mathcal{G}$.
- Top: the category of topological spaces and continuous maps
- Top_*: the category of **pointed spaces** and **pointed maps**, i.e. an object (X, x) is a topological space X equipped with a base point $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are continuous maps $X \to Y$ that send x to y. This notation is common but not universal, e.g. [tD08] uses a superscript 0 to indicate base points, so Top_* is called TOP^0.
- Set: the category of sets and maps (with no continuity requirement)
- Set_*: the category of **pointed sets** and (not necessarily continuous) **pointed maps**, i.e. an object (X, x) is a set X with a base point $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are arbitrary maps $X \to Y$ that send x to y.
- $\mathsf{Top}^{\mathrm{rel}}$: the category of **pairs of spaces** (X, A) and **maps of pairs**, i.e. an object (X, A) is a topological space X equipped with a subset $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $X \to Y$ that send A into B. Despite the uniquity of this category, there doesn't seem to be any common standard notation for it; [tD08] calls it $\mathsf{TOP}(2)$, and similarly writes $\mathsf{TOP}(3)$ for the category of **triples** (X, A, B) with $B \subset A \subset X$, and so forth. In [Wen23] I used a subscript instead of a superscript, but I'm changing it so that I can also define the next item on this list.

- $\mathsf{Top}_*^{\mathrm{rel}}$: the category of **pointed pairs of spaces**, i.e. an object (X, A, x) is a topological space X equipped with a subset $A \subset X$ and a base point $x \in A$, and morphisms $f : (X, A, x) \to (Y, B, y)$ are maps of pairs $(X, A) \to (Y, B)$ that also send x to y. I have no idea what anyone else calls this, but it's a subcategory of what [tD08] calls $\mathsf{TOP}(3)$, and is in any case clearly important since e.g. it is the domain of the relative homotopy functors π_n.
- hTop, hTop_*, $\mathsf{hTop}^{\mathrm{rel}}$, $\mathsf{hTop}_*^{\mathrm{rel}}$: the **homotopy categories** associated to Top, Top_*, $\mathsf{Top}^{\mathrm{rel}}$ and $\mathsf{Top}_*^{\mathrm{rel}}$ respectively, meaning we define categories with the same objects, but instead of taking morphisms to be actual maps, we define them to be *homotopy classes* of maps (respecting subsets and/or base points where appropriate, so e.g. **pointed homotopy** for hTop_*, and homotopy of maps of pairs for $\mathsf{hTop}^{\mathrm{rel}}$). This notation (or similar) for homotopy categories is very common, but different from my Topology I–II notes [Wen23], which wrote e.g. Top_*^h instead of hTop_*.
- Diff: the category of smooth finite-dimensional **manifolds** without boundary, and **smooth maps**
- Grp: the category of **groups** and group homomorphisms
- Ab: the category of **abelian groups** and homomorphisms, which is a subcategory of Grp
- $\mathsf{Ring} \supset \mathsf{CRing} \supset \mathsf{Fld}$: the category of **rings with unit** and its subcategories of **commutative rings** and **fields** respectively, with ring homomorphisms (preserving the unit)
- R-Mod: the category of **modules** over a given commutative ring R and **R-module homomorphisms**. In [Wen23] I called this Mod^R, and other variations such as Mod-R are also common.
- $\mathbb{K}$-Vect: the category of **vector spaces** over a given field $\mathbb{K}$ and **$\mathbb{K}$-linear maps**, i.e. this is R-Mod in the special case where R is a field $\mathbb{K}$. In [Wen23] I called this $\mathsf{Vec}_{\mathbb{K}}$.
- Categories of **(co-)chain complexes**: given any *additive* category $\mathscr{A}$ such as Ab or R-Mod,

  $$\mathsf{Ch}(\mathscr{A}) \qquad \text{or sometimes simply} \qquad \mathsf{Ch}$$

  denotes the category of chain complexes $\ldots \to A_{n+1} \to A_n \to A_{n-1} \to \ldots$ formed out of objects and morphisms in $\mathscr{A}$, with the morphisms of $\mathsf{Ch}(\mathscr{A})$ defined to be **chain maps**. There is a similar category $\mathsf{CoCh}(\mathscr{A})$ of cochain complexes $\ldots \to A_{n-1} \to A_n \to A_{n+1} \to \ldots$, though I am not really happy with this notation and I doubt that anyone else is either. In [Wen23] I denoted $\mathsf{Ch}(\mathsf{Ab})$, $\mathsf{CoCh}(\mathsf{Ab})$, $\mathsf{Ch}(R\text{-}\mathsf{Mod})$ and $\mathsf{CoCh}(R\text{-}\mathsf{Mod})$ by Chain, $\mathsf{Cochain}$, Chain^R and $\mathsf{Cochain}^R$ respectively. One sometimes sees a meaningless subscript such as $\mathsf{Ch}_\bullet(\mathscr{A})$ added, but there are also meaningful subscripts that define important subcategories such as e.g. $\mathsf{Ch}_{\geqslant 0}(\mathscr{A})$, the chain complexes that are trivial in all negative degrees.
- **Homotopy categories of chain complexes**: analogously to the homotopy categories of spaces, one can take the objects in $\mathsf{Ch}(\mathscr{A})$ and define morphisms to be chain homotopy classes of chain maps instead of actual chain maps. The internet seems quite insistent that I should call the resulting category

  $$\mathsf{K}(\mathscr{A}) := \text{the (naive) homotopy category associated to } \mathsf{Ch}(\mathscr{A}),$$

  even though I'd rather call it $\mathsf{hCh}(\mathscr{A})$, and in [Wen23] I wrote e.g. Chain^h instead of $\mathsf{K}(\mathsf{Ab})$; on occasion I have even seen $\mathsf{Ho}(\mathscr{A})$ in place of $\mathsf{K}(\mathscr{A})$. I have no idea what notation to use for the homotopy category of cochain complexes. People who like derived categories will tell you that there are other things more deserving of the name "homotopy category of chain complexes," and I added the word "naive" above in order to avoid getting into conversations about it with those people, which would be completely unnecessary for the purposes of the present course.

Topological constructions.

- $X \amalg Y$: This is how I write the **disjoint union** of two topological spaces (and similarly for pairs of spaces), and most sensible people use either this notation or $X \sqcup Y$, but [tD08] instead writes $X + Y$ and calls it the **topological sum** of X and Y, presumably because—like the direct sum of abelian groups and many other constructions that use the word "sum"—it is a coproduct. The book by tom Tieck becomes significantly easier to read once you realize this.
- $X \coprod Y$: the **coproduct** of X and Y, whatever that means in whichever category X and Y happen to live in, so e.g. in Top, it means the same thing as $X \amalg Y$, though in Top_* it means $X \vee Y$.
- $[X, Y]$: If X and Y are just topological spaces (i.e. objects in Top), then this denotes the **set of homotopy classes** of maps $X \to Y$, i.e.
$$[X, Y] := \mathrm{Hom}_{\mathsf{hTop}}(X, Y).$$
If X and Y are equipped with additional data (which may be suppressed in the notation) and are thus objects in Top_*, $\mathsf{Top}^{\mathrm{rel}}$ or $\mathsf{Top}_*^{\mathrm{rel}}$, then I use the same notation $[X, Y]$ to mean the corresponding notion of homotopy classes in each category, so e.g. in the context of pointed spaces, I would write
$$[X, Y] := \mathrm{Hom}_{\mathsf{hTop}_*}(X, Y),$$
and similarly for (pointed or unpointed) pairs of spaces. This convention is popular but not universal, e.g. [tD08] writes $[X, Y]^0$ for the set of pointed homotopy classes and uses $[X, Y]$ only to mean unpointed homotopy classes; [DK01] does the same but writes $[X, Y]_0$ instead of $[X, Y]^0$.
- $X \vee Y$ and $X \wedge Y$: these are the **wedge sum** and **smash product** respectively of pointed spaces, and mercifully, everyone seems to agree on what they mean and how to write them.
- **Implied base points**: for a pair of spaces (X, A), the **quotient space** X/A is often interpreted as a pointed space, with the collapsed subset A as base point. Similarly, for two pointed spaces X, Y, the set of **pointed homotopy classes** $[X, Y]$ is viewed as a pointed set (i.e. an object in Set_*) whose base point is the homotopy class of the constant map to the base point of Y.
- **One-point spaces**: the symbol $*$ is often used to mean either a one-point space, the unique point in that space, or sometimes a previously unnamed base point of a given pointed space. It should usually be clear from context which is meant.
- I: this usually denotes the **unit interval**
$$I := [0, 1],$$
as appears in domains of paths, homotopies etc.
- **Homotopy relations**: Given maps $f, g : X \to Y$, I write
$$f \underset{h}{\sim} g$$
to means that f and g are homotopic ([tD08] writes "$f \simeq g$"), and
$$f \overset{H}{\rightsquigarrow} g$$
to mean that H is a homotopy from f to g, thought of as a *path* in the space of maps, hence $H : I \times X \to Y$ with $H(0, \cdot) = f$ and $H(1, \cdot) = g$. This can also mean e.g. pointed homotopy or homotopy of maps of pairs if working in Top_* or $\mathsf{Top}^{\mathrm{rel}}$ respectively. Where I write $f \overset{H}{\rightsquigarrow} g$, [tD08] writes $H : f \simeq g$.

- **Homotopy commutative diagrams**: I use a diagram of the form

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle g} & \sim & \big\downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{\psi} & Q \end{array}$$

  to mean that $\varphi \circ f$ and $\psi \circ g$ need not be identical but are homotopic, whatever that means in whichever category the objects of the diagram live in, e.g. if they are pointed spaces it means pointed homotopic, for spaces without base points it just means homotopic—it may also mean *chain* homotopic if the objects are chain complexes. If I write the variant

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle g} & \underset{\alpha}{\sim} & \big\downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{\psi} & Q \end{array},$$

  then it means that α is a homotopy (or chain homotopy as the case may be) from $\varphi \circ f$ to $\psi \circ g$. It wasn't easy to figure out how to render this in LaTeX, so maybe that's why most textbooks don't do it.
- $Z(f)$, $Z(f, g)$, $\mathrm{cone}(f)$: **mapping cylinders**, **double mapping cylinders** and **mapping cones** (see Week 2, Lecture 3)
- CX, SX: the **cone** and **suspension** respectively of a space X. In the context of pointed spaces the same notation may instead mean the *reduced* cone/suspension.

1. Week 1

Lecture 1 (15.04.2024): Motivation and colimits.

- Motivational theorem on exotic spheres (Milnor 1956): There exists a smooth manifold Y that is homeomorphic but not diffeomorphic to S^7. (In fact, Kervaire and Milnor proved shortly afterwards that there are exactly 28 such manifolds up to diffeomorphism.)
- Outline of a proof (slightly ahistorical), with notions that will be major topics in this course written in red:
  (1) Pontryagin classes: Associate topological invariants $p_k(E) \in H^{4k}(X; \mathbb{Z})$ for each $k \in \mathbb{N}$ to every isomorphism class of vector bundles E over a given space X. Since every smooth manifold M has a tangent bundle TM, we can define $p_k(M) := p_k(TM) \in H^{4k}(M; \mathbb{Z})$ as an invariant of smooth (but not topological) manifolds.
  (2) Intersection form and signature: For a compact oriented $4k$-manifold M (possibly with boundary), the intersection form is the quadratic form Q_M on $H^{2k}(M, \partial M; \mathbb{Z})$ defined by
  $$Q_M(\alpha) := \langle \alpha \cup \alpha, [M] \rangle \in \mathbb{Z},$$
  and it's called the "intersection form" because it can be interpreted as a signed count of intersections between two generic closed oriented submanifolds representing the class in $H_{2k}(M; \mathbb{Z})$ Poincaré dual to α. The signature $\sigma(M) \in \mathbb{Z}$ is essentially the number of positive eigenvalues minus the number of negative eigenvalues[1] of this quadratic form.

[1] What I really mean is: first rewrite Q_M as a quadratic form on $H^{2k}(M, \partial M; \mathbb{Q})$ or $H^{2k}(M, \partial M; \mathbb{R})$, which is a vector space, so that by standard linear algebra, you can present it in terms of a symmetric linear transformation and look at the eigenvalues of that transformation. One can define this in a more obviously invariant way by talking about maximal subspaces on which Q_M is positive/negative definite.

(3) Hirzebruch signature theorem (8-dimensional case): For M a closed oriented 8-manifold,

$$\sigma(M) = \frac{1}{45}\langle 7p_2(M) - p_1(M) \cup p_1(M), [M]\rangle.$$

(4) (the clever bit) Construct a compact oriented smooth 8-manifold X with simply connected boundary $Y := \partial X$ such that $\sigma(X) = 8$, $H_2(Y)$ and $H_3(Y)$ both vanish, and the tangent bundle TX is stably trivial, which implies its Pontryagin classes vanish. The construction can be described (key words: "plumbing of spheres"), and the computations carried out, using only methods from Topology 2.

(5) Deduce via Poincaré duality, the Hurewicz theorem and Whitehead's theorem[2] that Y is homotopy equivalent to S^7. By Smale's solution to the higher-dimensional Poincaré conjecture,[3] it follows that Y is homeomorphic to S^7.

(6) Argue by contradiction: If Y is diffeomorphic to S^7, then one can construct a closed smooth 8-manifold M by gluing X to an 8-disk along a diffeomorphism $\partial X = Y \cong S^7 = \partial\mathbb{D}^8$,

$$M := X \cup_{S^7} \mathbb{D}^8.$$

Methods from Topology 2 (e.g. Mayer-Vietoris) now imply $p_1(M) = 0$ and $\sigma(M) = 8$, so Hirzebruch says

$$45\sigma(M) = 45 \cdot 8 = 7\langle p_2(M), [M]\rangle.$$

But the right hand side of this relation is a multiple of 7, and the left hand side is not.

- Interpretation of a functor $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ as a **diagram** in $\mathscr{C}$ over $\mathscr{J}$, constant functors $\mathcal{X} : \mathscr{J} \to \mathscr{C}$ as **targets**, the **universal property** and definition of the **colimit** $\mathrm{colim}(\mathcal{F})$
- Interpreting direct systems as diagrams and direct limits as colimits
- Defining the quotient space X/A as colimit of the diagram

$$\begin{array}{ccc} A & \longrightarrow & * \\ \big\downarrow\!\!\!{}^{\cap} & & \\ X & & \end{array}$$

understood as a functor $\mathscr{J} \to \mathsf{Top}$, where $\mathscr{J}$ is a category with three objects and only two nontrivial morphisms.

Lecture 2 (18.04.2024): From coproducts to pullbacks and pushouts.

- The **limit** $\lim(\mathcal{F})$ of a diagram $\mathcal{F} : \mathscr{J} \to \mathscr{C}$
- Inverse limits as limits of diagrams
- Important special cases of limits and colimits:
  - **Coproducts** $\coprod$, and examples in the categories Top (disjoint union), Top_* (wedge sum), Ab (direct sum) and Grp (free product)
  - **Products** $\times$ (or $\prod$), and examples in Top
  - **Equalizers** and **co-equalizers**, realization in Top as subspaces or quotient spaces respectively

[2] A 3-dimensional version of this same argument is described in [Wen23, Lecture 57], using the theorems of Hurewicz and Whitehead as black boxes.

[3] This is the one major black box in this proof that I do not intend to fill in, because that would be a whole course in itself.

- Word of caution: limits and colimits are not guaranteed to exist, e.g. in the category Diff of smooth finite-dimensional manifolds without boundary, finite or countable coproducts exist (and are the same thing as in Top), but uncountable disjoint unions are not second countable and are thus not objects in Diff. Similarly, finite products exist in Diff but infinite products typically do not.
- Theorem: In any category $\mathscr{C}$, all (co-)limits can be presented in terms of (co-)products and (co-)equalizers, if they exist.
- Proof sketch (co-limit case): Given $\mathcal{F} : \mathscr{J} \to \mathscr{C} : \alpha \mapsto X_\alpha$, construct $\operatorname{colim}(\mathcal{F})$ as the equalizer of two morphisms $Y \xrightarrow{f,g} Z$ defined as follows. Write the set of all morphisms in $\mathscr{J}$ as $\operatorname{Hom}(\mathscr{J}, \mathscr{J})$; we then take Y to be the coproduct

$$Y := \coprod_{\phi \in \operatorname{Hom}(\mathscr{J},\mathscr{J})} X_\phi, \qquad \text{where} \qquad \text{for } \phi \in \operatorname{Hom}(\alpha,\beta),\ X_\phi := X_\alpha,$$

while Z is the slightly simpler coproduct

$$Z := \coprod_{\beta \in \mathscr{J}} X_\beta.$$

For each $\alpha, \beta \in \mathscr{J}$ and $\phi \in \operatorname{Hom}(\alpha,\beta)$, let $f_\phi : X_\phi \to Z$ denote the composition of the morphism $\phi_* : X_\phi = X_\alpha \to X_\beta$ with the canonical morphism $X_\beta \to \coprod_{\gamma \in \mathscr{J}} X_\gamma$ of the coproduct; the universal property of the coproduct then dictates that the collection of morphisms $f_\phi : X_\phi \to Z$ determines a morphism $f : Y \to Z$. Similarly, $g : Y \to Z$ is determined by the collection of morphisms $g_\phi : X_\phi \to Z$ defined for each $\phi \in \operatorname{Hom}(\alpha,\beta)$ as the compositions of $\operatorname{Id}_{X_\alpha} : X_\phi = X_\alpha \to X_\alpha$ with the canonical morphism $X_\alpha \to \coprod_{\gamma \in \mathscr{J}} X_\gamma$. Now check that the universal property is satisfied (exercise).
- Upshot: In Top, colimits are quotients of disjoint unions, limits are subspaces of products.
- **Fiber products**: presenting the fiber product of two maps $f : X \to Z$ and $g : Y \to Z$ in Top as the "intersection locus"

$$X\ {}_f\!\times_g Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

with the obvious projections to X and Y.
- Interpreting fiber products as **pullbacks**
- **Pushouts**: presenting the pushout of two maps $f : Z \to X$ and $g : Z \to Y$ in Top as "gluing spaces together" along a map:

$$X\ {}_f\!\cup_g Y := (X \amalg Y) \Big/ f(z) \sim g(z) \text{ for all } z \in Z.$$

- Question for thought: How many of these constructions of limits or colimits work in the homotopy categories hTop or hTop_*? (Hint: Do not try too hard to make sense of equalizers and co-equalizers.)

Suggested reading. The main definitions involving direct systems and direct limits can all be found in [Wen23, Lecture 39], with the generalization to colimits explained in Exercise 39.24. If you're really serious about this stuff, you can also try reading [Mac71].

If you want to read more about exotic spheres, there's a nice collection of relevant literature assembled at https://www.maths.ed.ac.uk/~v1ranick/exotic.htm.

Exercises (for the Übung on 25.04.2024). Since the Übung on 25.04 was cancelled due to illness, most of the exercises for Week 1 have now been supplemented with written answers and/or some discussion.

Exercise 1.1. In what sense precisely are the limit and colimit of a diagram $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ unique, if they exist?

Answer: If the limit or colimit exists (of which there is no guarantee, cf. Exercise 1.7), then it is unique up to canonical isomorphisms. Precisely: Suppose $X, Y \in \mathscr{C}$ are two objects, together with collections of morphisms $\mathcal{F}(\alpha) \xrightarrow{\varphi_\alpha} X$ and $\mathcal{F}(\alpha) \xrightarrow{\psi_\alpha} Y$ for all $\alpha \in \mathscr{J}$, such that both satisfy the universal property for $\operatorname{colim}(\mathcal{F})$. Then there is a uniquely determined isomorphism

$$f : X \xrightarrow{\cong} Y \qquad \text{such that} \qquad \psi_\alpha = f \circ \varphi_\alpha \text{ for all } \alpha \in \mathscr{J}.$$

The existence and uniqueness of a morphism f satisfying this condition follows from the universal property of X, and the fact that it is an isomorphism follows by reversing the roles of X and Y, since Y also satisfies the universal property. For $\lim(\mathcal{F})$ there is a similar uniqueness statement, proved in a similar way.

Note that in most categories, uniqueness "up to canonical isomorphisms" is the best that one could hope to get from universal properties, as one will always have the freedom to replace a given object playing the role of $\operatorname{colim}(\mathcal{F})$ or $\lim(\mathcal{F})$ with a different object that is isomorphic to it. In practice, our favorite categories often come with canonical constructions that lead to specific objects, e.g. the disjoint union (also known as the coproduct) of a given collection of topological spaces is a specific space, not just an equivalence class of spaces up to homeomorphism. But in various situations, limits or colimits can also arise from something other than the canonical construction, and finding an isomorphism with that canonical construction may be harder than explicitly verifying the universal property.

Exercise 1.2 (morphisms between (co-)products)**.** Assume J is a set, and $\{X_\alpha\}_{\alpha \in J}$ and $\{Y_\alpha\}_{\alpha \in J}$ are collections of objects in some category $\mathscr{C}$ such that the products

$$\left\{ \prod_{\alpha \in J} X_\alpha \xrightarrow{\pi^X_\beta} X_\beta \right\}_{\beta \in J}, \qquad \left\{ \prod_{\alpha \in J} Y_\alpha \xrightarrow{\pi^Y_\beta} Y_\beta \right\}_{\beta \in J},$$

and coproducts

$$\left\{ X_\beta \xrightarrow{i^X_\beta} \coprod_{\alpha \in J} X_\alpha \right\}_{\beta \in J}, \qquad \left\{ Y_\beta \xrightarrow{i^Y_\beta} \coprod_{\alpha \in J} Y_\alpha \right\}_{\beta \in J}$$

exist. In what sense does an arbitrary collection of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha \in J}$ uniquely determine morphisms

$$\coprod_{\alpha \in J} f_\alpha : \coprod_{\alpha \in J} X_\alpha \to \coprod_{\alpha \in J} Y_\alpha, \qquad \text{and} \qquad \prod_{\alpha \in J} f_\alpha : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha?$$

Argue in terms of universal properties, without using your knowledge of how to represent products and coproducts in any specific categories.

Answer: The morphisms $\coprod_\alpha f_\alpha$ and $\prod_\alpha f_\alpha$ are uniquely determined by the condition that the diagrams

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_\beta} & Y_\beta \\ \downarrow{\scriptstyle i^X_\beta} & & \downarrow{\scriptstyle i^Y_\beta} \\ \coprod_\alpha X_\alpha & \xrightarrow{\coprod_\alpha f_\alpha} & \coprod_\alpha Y_\alpha \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \prod_\alpha X_\alpha & \xrightarrow{\prod_\alpha f_\alpha} & \prod_\alpha Y_\alpha \\ \downarrow{\scriptstyle \pi^X_\beta} & & \downarrow{\scriptstyle \pi^Y_\beta} \\ X_\beta & \xrightarrow{f_\beta} & Y_\beta \end{array}$$

commute for every $\beta \in J$. One gets the existence and uniqueness of $\coprod_\alpha f_\alpha$ from the universal property of the coproduct $\coprod_\alpha X_\alpha$, because the morphisms $i^Y_\beta \circ f_\beta : X_\beta \to \coprod_\alpha Y_\alpha$ make $\coprod_\alpha Y_\alpha$ a target of the diagram whose colimit is $\coprod_\alpha X_\alpha$. Similarly, the existence and uniqueness of $\prod_\alpha f_\alpha$ follows from the universal property of the product $\prod_\alpha Y_\alpha$, using the collection morphisms $f_\beta \circ \pi^X_\beta : \prod_\alpha X_\alpha \to Y_\beta$.

Exercise 1.3 (finite limits and colimits)**.** Show that in any category $\mathscr{C}$, finite colimits always exist if and only if all pushouts exist and $\mathscr{C}$ has an initial object (see Exercise 1.5). Dually, finite limits always exist if and only if all pullbacks (also known as fiber products) exist and $\mathscr{C}$ has a terminal object.[4]
Hint: By a theorem stated in the lecture, it suffices if you can express arbitrary (co-)equalizers and finite (co-)products in terms of pushouts or pullbacks.

Solution: Note that the statement of this exercise has been revised; the original version had two errors, one being its failure to mention initial and terminal objects, and the other an oversimplification of what it means for a limit or colimit to be *finite*—we need the category $\mathscr{I}$ underlying the diagram to have finitely-many morphisms, not just finitely-many objects.

With that understood, let's assume all pushouts exist and that $\mathscr{C}$ also has an initial object $0 \in \mathscr{C}$. If we can show that all finite coproducts and all coqualizers exist, then the theorem from lecture uses these to construct a colimit for any diagram $\mathcal{F} : \mathscr{I} \to \mathscr{C}$ such that $\mathscr{I}$ has only finitely many objects and morphisms. (Regarding the errors in the original version: note that if $\mathscr{I}$ has finitely-many objects but infinitely-many morphisms, then one of the coproducts needed for the theorem from lecture is not finite.)

You should be able to convince yourself via an inductive argument that if the coproduct of two objects $X, Y \in \mathscr{C}$ always exists, then all finite coproducts exist. So let's show first that $X \coprod Y$ exists for arbitrary $X, Y \in \mathscr{C}$. At this point I find it helpful to think about how coproducts and pushouts are constructed concretely in the example $\mathscr{C} = \mathsf{Top}$: the coproduct of X and Y is their disjoint union, and the pushout of a pair of maps $f : Z \to X$ and $g : Z \to Y$ is a quotient of that disjoint union by the equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$. If we want to make that equivalence relation trivial so that the pushout turns out to be the same thing as the coproduct, the solution is to choose the empty set for Z; the maps f, g are uniquely determined by this choice, because the empty set is an initial object in Top (see Exercise 1.5). This suggests that in our given category $\mathscr{C}$ with initial object $0 \in \mathscr{C}$, the pushout of the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \\ Y & & \end{array}$$

should be the coproduct of X and Y; note that only one diagram of this form is possible since 0 being initial means that the morphisms $0 \to X$ and $0 \to Y$ are unique. Now suppose P is the pushout of this diagram, equipped with morphisms $\varphi : X \to P$ and $\psi : Y \to P$, and suppose we are given another object Z with morphisms $f : X \to Z$ and $g : Y \to Z$. The diagram

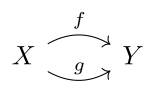
$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

then trivially commutes, since there is only one morphism $0 \to Z$, and the universal property of the pushout gives rise to a unique morphism $u : P \to Z$ such that $f = u \circ \varphi$ and $g = u \circ \psi$, which

[4] The word "finite" in this context refers to limits or colimits of diagrams $\mathcal{F} : \mathscr{I} \to \mathscr{C}$ such that $\mathscr{I}$ has only finitely many objects and morphisms.

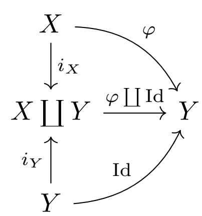
amounts to the statement that P with its morphisms φ and ψ also satisfies the universal property of the coproduct $X \coprod Y$.

We show next that the coequalizer of an arbitrary pair of morphisms

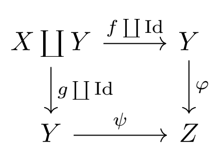


in $\mathscr{C}$ can also be constructed as a pushout. Think again about how it works in the case $\mathscr{C} = \mathsf{Top}$: the coequalizer here is the quotient of Y by the equivalence relation such that $f(x) \sim g(x)$ for all $x \in X$. If we instead take the pushout of f and g, the resulting space is too large: it is a quotient of $Y \amalg Y$ instead of Y, meaning that we glue together *two* copies of Y by identifying $f(x)$ in one copy with $g(x)$ in the *other* copy for each $x \in X$. But the correct space can be obtained from this by making the equivalence relation larger, so that for every $y \in Y$, y in the first copy gets identified with y in the second copy. The way to realize this is by enlarging the domain of the pair of maps used in defining the pushout: instead of the two maps $f, g : X \to Y$, we consider the pushout of the two maps $f \amalg \mathrm{Id}, g \amalg \mathrm{Id} : X \amalg Y \to Y$.

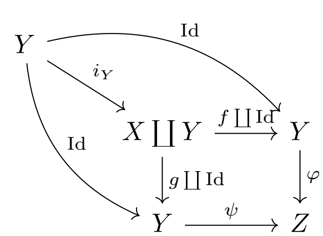
Let's say that again without assuming $\mathscr{C} = \mathsf{Top}$. We've already shown that the coproduct $X \coprod Y$ of two objects in $\mathscr{C}$ can be constructed, and if we write $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ for the canonical morphisms that coproducts come equipped with, then by the universal property of the coproduct, every morphism $\varphi : X \to Y$ determines a unique morphism $\varphi \coprod \mathrm{Id} : X \coprod Y \to Y$ for which the diagram



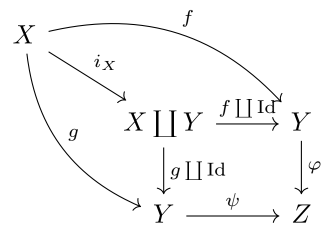
commutes. Claim: Given two morhisms $f, g : X \to Y$, a diagram of the form



commutes if and only if $\varphi = \psi$ and $\varphi \circ f = \varphi \circ g$. To see this, we can enhance the diagram in two ways using the universal property of the coproduct: first,



shows that if the given diagram commutes, then $\varphi = \varphi \circ \mathrm{Id} = \psi \circ \mathrm{Id} = \psi$. Assuming this, the second enhanced diagram



then proves $\varphi \circ f = \psi \circ g = \varphi \circ g$. Conversely, if one assumes $\varphi = \psi$ and $\varphi \circ f = \varphi \circ g$, then $\varphi \circ (f \coprod \mathrm{Id})$ and $\psi \circ (g \coprod \mathrm{Id})$ are two morphisms $X \coprod Y \to Z$ whose compositions with i_X and i_Y are identical, so the uniqueness in the universal property of the coproduct requires them to be the same.

The result of the claim is that pushout diagrams for the two morphisms $f \coprod \mathrm{Id} : X \coprod Y \to Y$ and $g \coprod \mathrm{Id} : X \coprod Y \to Y$ are equivalent to coequalizer diagrams for $f, g : X \to Y$. It is a short step from there to the conclusion that an object Z with morphism $Y \to Z$ satisfies the universal property of the coequalizer if and only if Z with two copies of that same morphism $Y \to Z$ satisfies the universal property of the pushout.

For the dual case of this whole story, I will just say this: if $1 \in \mathscr{C}$ is a terminal object, then the uniqueness of morphisms to 1 implies that the pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & 1 \end{array}$$

satisfies the universal property of the product $X \times Y$. Having shown that finite products exist, one then obtains the equalizer of any pair of morphisms $f, g : X \to Y$ as the pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \mathrm{Id} \times f} \\ X & \xrightarrow{\mathrm{Id} \times g} & X \times Y \end{array} .$$

If finite products and equalizers always exist, then all finite limits can be constructed out of them.

Exercise 1.4. Let's talk about some coproducts and products in algebraic settings.

(a) What is a coproduct of two objects in the category Ring of rings with unit? Try to describe it explicitly.
(b) Same question about products in Ring. (This one is perhaps easier.)
(c) Show that two fields of different characteristic can have neither a product nor a coproduct in the category Fld of fields.

Answers: The coproduct of two rings A, B is their tensor product $A \otimes B$, equipped with the ring homomorphisms

$$A \xrightarrow{i_A} A \otimes B : a \mapsto a \otimes 1, \qquad B \xrightarrow{i_B} A \otimes B : b \mapsto 1 \otimes b.$$

As a set, $A \otimes B$ is the same thing as the tensor product of A and B as abelian groups; one then gives it a ring structure by defining

$$(a \otimes b)(a' \otimes b') := (aa') \otimes (bb').$$

It is easy to check that the required universal property is satisfied. Perhaps more interesting is to observe that in the more familiar categories Ab and R-Mod in which we are used to talking about tensor products, they do *not* arise as colimits, and there is an obvious reason why they shouldn't: the only obviously canonical homomorphisms I can think of from a pair of abelian groups A and B to their tensor product $A \otimes B$ are the trivial ones. The big difference in Ring as that rings have multiplicative units, and these give rise to canonical *nontrivial* morphisms from A and B to $A \otimes B$ as described above. (For similar reasons, you also should not try to think of tensor products as categorical products—for a more useful categorical perspective on tensor products, see Exercise 1.9.)

The product in Ring is exactly what you'd expect: the product of rings.

For fields, the problem is that there are in fact no field homomorphisms *at all* between a pair of fields with different characteristics. So for any fields A and B, the need to have morphisms $A, B \to A \coprod B$ and $A \times B \to A, B$ means that neither the coproduct nor the product can exist unless A and B have the same characteristic (which their product and coproduct must then also have). For example, $\mathbb{Z}_2$ and $\mathbb{Q}$ have no coproduct in Fld, though they do have a coproduct in Ring, namely $\mathbb{Z}_2 \otimes \mathbb{Q}$, which is an extremely indirect way of writing the trivial ring. (Amusing exercise: show that $1 = 0$ in $\mathbb{Z}_2 \otimes \mathbb{Q}$. The elements 1 and 0 are never equal in a field.)

Exercise 1.5 (initial and terminal objects)**.** In defining limits and colimits of diagrams $\mathcal{F} : \mathscr{J} \to \mathscr{C}$, the set of objects in $\mathscr{J}$ is not required to be nonempty. When it is empty, we can think of $\operatorname{colim}(\mathcal{F})$ is a coproduct of an empty collection of objects in $\mathscr{C}$, and $\operatorname{colim}(\mathcal{F})$ is then called an **initial object** in $\mathscr{C}$. Similarly, the product $\lim(\mathcal{F})$ of an empty collection of objects is called a **terminal** (or **final**) object in $\mathscr{C}$.

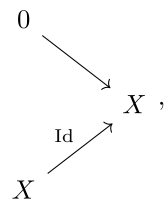
(a) Reformulate the definitions given above for the terms "initial object" and "terminal object" in a way that makes no reference to limits or colimits, and using this reformulation, give a short proof that both are unique up to canonical isomorphisms, if they exist.
(b) Show that for any initial object $0 \in \mathscr{C}$, the coproducts $0 \coprod X$ and $X \coprod 0$ exist and the canonical morphisms of X to each are isomorphisms. Similarly, for any terminal object $1 \in \mathscr{C}$, the products $1 \times X$ and $X \times 1$ exist and their canonical morphisms to X are isomorphisms.
(c) Describe what initial and terminal objects are in each of the following categories, if they exist: Top, Top_*, Ab, Ring, and Fld.
Hint: You might guess the last two from Exercise 1.4.

Answers: If $\mathscr{J}$ is the empty category, then there is a unique diagram $\mathcal{F} : \mathscr{J} \to \mathscr{C}$, but it carries no information. If we want to define a colimit of this diagram, then any object $X \in \mathscr{C}$ can be considered a target; there is no need to specify any morphisms since $\mathscr{J}$ has no objects. The condition of X being a *universal* target is, however, nontrivial: it means that for any other target Y, there is a unique morphism $u : X \to Y$ such that... well, at this point we would normally say that certain morphisms admit factorizations through the morphism u, but since $\mathscr{J}$ has no objects, there are no morphisms to be factored and thus no further conditions to impose. We are left only with this: *$X \in \mathscr{C}$ is an initial object if and only if for every object $Y \in \mathscr{C}$, there is a unique morphism $X \to Y$.* That's the usual definition—we stated it in a much more roundabout way by talking about coproducts over the empty category.

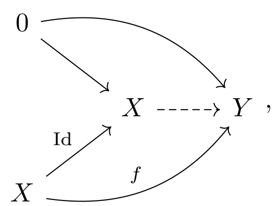
Here's the dual version: *$X \in \mathscr{C}$ is a terminal object if and only if for every object $Y \in \mathscr{C}$, there is a unique morphism $Y \to X$.*

With these definitions understood: if $0, 0' \in \mathscr{C}$ are two initial objects, then there is a unique morphism $0 \to 0'$, and there is also a unique morphism $0' \to 0$. Moreover, there are unique morphisms $0 \to 0$ and $0' \to 0'$, and both of those have to be identity morphisms, since identity morphisms must always exist. It follows that the unique morphisms $0 \to 0'$ and $0' \to 0$ are inverse to each other, and are thus isomorphisms. The uniqueness of terminal objects up to unique isomorphisms is proved similarly; there is only a slightly different reason for the uniqueness of the morphisms $1 \to 1'$ and so forth.

Let's consider the coproduct of an initial object $0 \in \mathscr{C}$ with an arbitrary $X \in \mathscr{C}$. We claim that X itself plays the role of the coproduct, together with the two morphisms



the first of which is determined by the condition that 0 is an initial object. Indeed, suppose Y is given, along with a morphism $f : X \to Y$ and the unique morphism $0 \to Y$ (for which there is no freedom of choice). The dashed arrow in the following diagram is then uniquely determined,



and this establishes the universal property of the coproduct. In this way of representing $0 \coprod X$, the canonical morphism $X \to 0 \coprod X$ is imply the identity morphism $X \to X$, and thus an isomorphism. Similar arguments prove the analogous statements about $X \coprod 0$, $1 \times X$ and $X \times 1$.

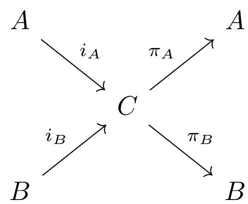
Here is an inventory of initial and terminal objects in specific categories:

- Top: the empty set $\varnothing$ is initial, and every one-point space $*$ is terminal. Note that the initial object in this case is not just unique *up to isomorphism*, but is actually unique, i.e. there really is only one object in Top called $\varnothing$. By contrast, the unique point in a one-point space can be anything, and the collection of all possible one-point spaces is therefore too large to qualify as a set; it is a proper class. Nonetheless, there is indeed a unique homeomorphism between any two of them.
- Top_*: every one-point space is both an initial and a terminal object.
- Ab: every trivial group is both initial and terminal. The answer in R-Mod is the same, in case you'd wondered.
- Ring: this one's more interesting. According to Exercise 1.4, tensor products are coproducts in Ring, so an initial object $R \in \mathsf{Ring}$ should be a ring with the property that $R \otimes A \cong A \cong A \otimes R$ for all rings $A \in \mathsf{Ring}$; plugging in $A := \mathbb{Z}$ as a special case, one deduces $R \cong \mathbb{Z}$. And indeed, for any other ring B, a ring homomorphism $\mathbb{Z} \to B$ is uniquely determined by the condition that it preserve the 0 and 1 elements. Terminal objects are trivial rings, i.e. those in which $1 = 0$.
- Fld: there are no initial or terminal objects in Fld, because as discussed in the answer to Exercise 1.4(c), there do not exist any fields that admit homomorphisms either to or from every other field (of arbitrary characteristic).

Exercise 1.6 (biproducts)**.** Assume $\mathscr{A}$ is a category in which the sets $\operatorname{Hom}(A, B)$ of morphisms $A \to B$ for each $A, B \in \mathscr{A}$ are equipped with the structure of abelian groups such that composition $\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) : (f, g) \mapsto g \circ f$ is always a bilinear map. (Popular examples are the categories Ab of abelian groups and R-Mod of modules over a commutative ring R.) A **biproduct**

of two objects $A, B \in \mathscr{A}$ is an object $C \in \mathscr{A}$ equipped with four morphisms

(1.1)



that satisfy the five relations

$$\pi_A i_A = \mathbb{1}_A, \quad \pi_B i_B = \mathbb{1}_B, \quad \pi_A i_B = 0, \quad \pi_B i_A = 0, \quad i_A \pi_A + i_B \pi_B = \mathbb{1}_C. \tag{1.2}$$

In the categories Ab or R-Mod, an example of a biproduct of A and B is the direct sum $A \oplus B$ with its canonical inclusion and projection maps. The category $\mathscr{A}$ is called **additive** if every pair of objects has a biproduct.

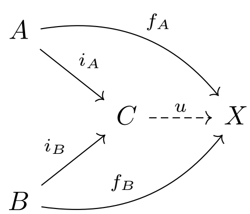
(a) Show that for any biproduct as in the diagram (1.1), C with the morphisms i_A, i_B is a coproduct of A and B, and with the morphisms π_A, π_B it is also a product of A and B.

(b) Show that in the categories Ab and R-Mod, every biproduct of two objects A, B admits an isomorphism to $A \oplus B$ that identifies the four maps in (1.1) with the obvious inclusions and projections.

(c) A (covariant or contravariant) functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ between two additive categories is called an **additive functor** if the map defined by $\mathcal{F}$ from $\mathrm{Hom}(A, B)$ to $\mathrm{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ or (in the contravariant case) $\mathrm{Hom}(\mathcal{F}(B), \mathcal{F}(A))$ is a group homomorphism for all $A, B \in \mathscr{A}$. Show that additive functors send all biproducts in $\mathscr{A}$ to biproducts in $\mathscr{B}$.

Remark: Popular examples of additive functors Ab $\to$ Ab *or* R-Mod $\to$ R-Mod *are* $\otimes G$, $G\otimes$, $\mathrm{Hom}(\cdot, G)$ *and* $\mathrm{Hom}(G, \cdot)$ *for any fixed module* G*, as these arise in the universal coefficient theorems for homology and cohomology.*

Answers: Let's show first that (1.1) and (1.2) make C with the morphisms $i_A : A \to C$ and $i_B : A \to B$ into a coproduct of A and B. We need to show that the dashed morphism u in the diagram



exists and is unique for any given object $X \in \mathscr{A}$ with morphisms f_A, f_B from A and B respectively. Start with uniqueness: if u is a morphism for which this diagram commutes, then using (1.2) and the assumption that composition is bilinear, we have

$$u = u(i_A \pi_A + i_B \pi_B) = (u i_A)\pi_A + (u i_B)\pi_B = f_A \pi_A + f_B \pi_B.$$

For existence, we then just need to define u by this formula and show that it satisfies $u i_A = f_A$ and $u i_B = f_B$, which also follows easily from the relations (1.2). The proof that C with the morphisms π_A, π_B is a product of A and B is similar.

For part (b), we already know that $A \oplus B$ defines a biproduct of R-modules A and B, so what we really need is a general result about uniqueness of biproducts up to isomorphism. We already have such results for products and coproducts separately, but we cannot directly apply them here, even though we know that biproducts are both; the trouble is that doing so will produce *two* isomorphisms between any two biproducts of A and B, one that arises by viewing them as products, and another by viewing them as coproducts. We want to see that those two isomorphisms are *the same one*.

Concretely, let's suppose that (1.1) and (1.2) are given, and that we also have a second object C' and set of morphisms $i'_A, i'_B, \pi'_A, \pi'_B$ satisfying the same set of relations. We do not need to assume $\mathscr{A}$ is Ab or R-Mod for this discussion, as it will make sense in any category for which biproducts can be defined, but some intuition about direct sums may nonetheless be helpful for writing down suitable morphisms between C and C'. Explicitly, define

$$f := i'_A \pi_A + i'_B \pi_B : C \to C', \qquad \text{and} \qquad g := i_A \pi'_A + i_B \pi'_B : C' \to C.$$

Using (1.2), we then have

$$\begin{aligned} gf &= (i_A \pi'_A + i_B \pi'_B)(i'_A \pi_A + i'_B \pi_B) = i_A(\pi'_A i'_A)\pi_A + i_A(\pi'_A i'_B)\pi_B + i_B(\pi'_B i'_A)\pi_A + i_B(\pi'_B i'_B)\pi_B \\ &= i_A \pi_A + i_B \pi_B = \mathbb{1}_C, \end{aligned}$$

and by a similar calculation, $fg = \mathbb{1}_{C'}$, so f is an isomorphism with $g = f^{-1}$. Using f to identify C with C' now transforms the morphism $i_A : A \to C$ into

$$f i_A = (i'_A \pi_A + i'_B \pi_B) i_A = i'_A(\pi_A i_A) + i'_B(\pi_B i_A) = i'_A : A \to C',$$

and it transforms the morphism $\pi_A : C \to A$ into

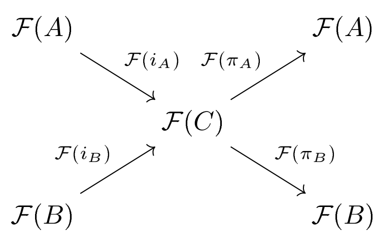
$$\pi_A f^{-1} = \pi_A(i_A \pi'_A + i_B \pi'_B) = (\pi_A i_A)\pi'_A + (\pi_A i_B)\pi'_B = \pi'_A : C' \to A,$$

and by similar calculations,

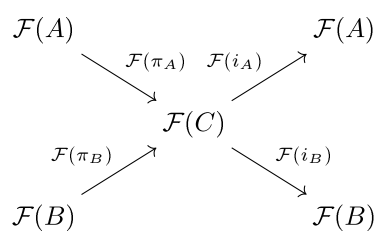
$$f i_B = i'_B, \qquad \pi_B f^{-1} = \pi'_B.$$

One can now appeal to abstract principles (i.e. the universal properties of products and coproducts) to deduce that f is indeed the *only* isomorphism $C \to C'$ that relates the morphisms i_A, i'_A and so forth in this way.

For a covariant additive functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, it is easy to check that $\mathcal{F}$ sends the four morphisms of (1.1) to morphisms



in $\mathscr{B}$ that satisfy the five relations (1.2), making $\mathcal{F}(C)$ a biproduct of $\mathcal{F}(A)$ and $\mathcal{F}(B)$. The amusing detail is what happens if $\mathcal{F}$ is *contra*variant: it still works, but the reversal of arrows means that some roles need to be switched, e.g. the diagram in $\mathscr{B}$ arising from (1.1) must be written as



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With $\mathcal{F}(\pi_A), \mathcal{F}(\pi_B)$ now playing the roles formerly played by i_A, i_B and $\mathcal{F}(i_A), \mathcal{F}(i_B)$ playing the roles of π_A, π_B, one easily checks that the five relations (1.2) are satisfied, so $\mathcal{F}(C)$ is again a biproduct of $\mathcal{F}(A)$ and $\mathcal{F}(B)$, with contravariance having transformed inclusions into projections and vice versa.

Exercise 1.7 (fiber products in Diff)**.** As mentioned in lecture, the category Diff of smooth manifolds is one in which many limits and colimits do not exist. An important example is the fiber product of two smooth maps $f : M \to Q$ and $g : N \to Q$, which matches the usual topological fiber product

$$M \; {}_f\!\times_g N := \left\{(x, y) \in M \times N \;\middle|\; f(x) = g(y)\right\} \subset M \times N$$

if the maps f and g are **transverse** to each other (written $f \pitchfork g$), because the implicit function theorem then gives $M \; {}_f\!\times_g N$ a natural smooth manifold structure for which the obvious projections to M and N are smooth.[5] If, on the other hand, f and g are not transverse, then the examples below show that all bets are off.

(a) Suppose $F : P \to M$ and $G : F \to N$ are smooth maps that define a target in Diff for the fiber product diagram defined by f and g; in other words, the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & M \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{g} & Q \end{array}$$

commutes and consists entirely of smooth manifolds and smooth maps. Interpret this diagram as defining a smooth map

$$u : P \to M \times N$$

whose image lies in the *topological* fiber product $M \; {}_f\!\times_g N \subset M \times N$, and show that if F and G satisfy the universal property for a fiber product in Diff, then u is a continuous bijection of P onto $M \; {}_f\!\times_g N \subset M \times N$.

(b) Deduce that if $M \; {}_f\!\times_g N \subset M \times N$ is a smooth submanifold of $M \times N$, then $M \; {}_f\!\times_g N$ with its projection maps to M and N does in fact define a fiber product in Diff. (Note that this may sometimes hold even if f and g are not transverse.)

(c) Consider the example $M = N = Q := \mathbb{R}$ with $f(x) := x^2$ and $g(y) := y^2$, thus

$$M \; {}_f\!\times_g N = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 = y^2\right\}.$$

You will easily convince yourself that this topological fiber product is not a manifold. Show that the pair of maps f, g does not admit any fiber product in Diff. Note that this is a stronger statement than just the observation that $\{x^2 = y^2\} \subset \mathbb{R}^2$ is not an object of Diff. *Hint: You can use parts (a) and (b) to show that if P is a smooth fiber product, then it contains a special point $p \in P$ such that $P \backslash \{p\}$ is diffeomorphic to $\{x^2 = y^2\} \backslash \{(0, 0)\}$.*

(d) Here's a weirder example: Let $M = Q := \mathbb{R}$, define $N := *$ as a manifold of one point with $g : N \to Q = \mathbb{R}$ mapping to 0, and choose $f : M = \mathbb{R} \to \mathbb{R} = Q$ to be any smooth function with

$$f^{-1}(0) = \{-1, -1/2, -1/3, \dots\} \cup \{0\} \cup \{\dots, 1/3, 1/2, 1\}.$$

(If you have doubts about the existence of such a function, try making minor modifications to the function e^{-1/x^2}, or something similar.) Show that in this case, a fiber product in Diff does exist, but is not homeomorphic to the topological fiber product.

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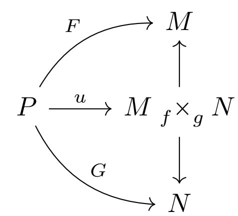
[5] Transversality is a condition on the derivatives of f and g at all points $x \in M$ and $y \in N$ such that $f(x) = g(y) =: p$; writing the derivatives at these points as linear maps $df(x) : T_xM \to T_pQ$ and $dg(y) : T_yM \to T_pQ$ between the appropriate tangent spaces, it means that the subspaces $\operatorname{im} df(x)$ and $\operatorname{im} dg(y)$ span all of T_pQ. Choosing suitable local coordinates near each point $(x, y) \in M \; {}_f\!\times_g N$, one can identify $M \; {}_f\!\times_g N$ locally with the zero-set of a smooth map whose derivative at (x, y) is surjective if and only if the transversality condition holds, so that the implicit function theorem makes $M \; {}_f\!\times_g N$ a smooth submanifold of $M \times N$.

Hint: What can you say about continuous maps from locally path-connected spaces to $f^{-1}(0) \subset \mathbb{R}$*?*

Answers: For part (a), note first that a fiber product diagram in Diff can always also be interpreted as a fiber product diagram in Top, so applying the universal property of the topological fiber product $M\ {}_f\!\times_g N$ immediately gives us a unique *continuous* map $u : P \to M\ {}_f\!\times_g N$ such that the diagram

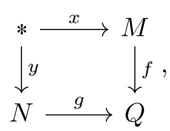


commutes, where the vertical arrows are the obvious projections. This diagram also gives us an explicit formula for u: its composition with the inclusion $M\ {}_f\!\times_g N \hookrightarrow M \times N$ is just

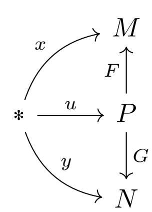
$$(F, G) : P \to M \times N,$$

which is a smooth map since F and G are smooth, though we cannot sensibly call it a smooth map to $M\ {}_f\!\times_g N$ unless the latter is known to be a smooth submanifold of $M \times N$.

We want to show that if P with the maps F and G satisfies the universal property for a fiber product in Diff, then the map $u : P \to M\ {}_f\!\times_g N$ described above is a bijection. Indeed, pick any point $(x, y) \in M\ {}_f\!\times_g N$ and consider the pullback diagram



where the labels "x" and "y" on arrows are used to indicate the images of maps from a one-point space labelled $*$. The latter is (trivially) a smooth 0-manifold, and the maps defined on it are (trivially) smooth, so this diagram lives in Diff, and the universal property of the fiber product P therefore produces a unique map $u : * \to P$ for which the diagram



commutes. The image of $u : * \to P$ is thus the unique point $p \in P$ satisfying $u(p) = (F(p), G(p)) = (x, y)$.

Part (b) follows almost immediately from what was said above: if $M\ {}_f\!\times_g N$ is a smooth submanifold of $M \times N$, then the map $u : P \to M\ {}_f\!\times_g N$ obtained from any smooth fiber product diagram by applying the universal property in Top is automatically also smooth, with the consequence that $M\ {}_f\!\times_g N$ also satisfies the universal property in Diff.

For the example in part (c), $M\ {}_f\!\times_g N \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the union of the two lines $\{y = x\}$ and $\{y = -x\}$, so it is not globally a manifold, though it becomes a smooth 1-manifold if one deletes the singular point $(0, 0)$. Suppose there exists a smooth manifold P and smooth functions

$F, G : P \to \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & \mathbb{R} \\ \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle f} \\ \mathbb{R} & \xrightarrow{g} & \mathbb{R} \end{array}$$

defines a fiber product in Diff. By part (a), the smooth map $(F, G) : P \to \mathbb{R}^2$ is then a bijection onto the set $\{y = \pm x\}$, so that there is a unique point $p \in P$ with $F(p) = G(p) = 0$. The manifold P must be path-connected, because any point in $\{y = \pm x\}$ can be joined to $(0, 0)$ by a smooth path lying in one of the smooth submanifolds $\{y = x\}$ or $\{y = -x\}$, and the universal property will then produce a smooth map from this submanifold to P, whose image thus contains a path from any given point to p. Now let $\Sigma := \{y = \pm x\} \backslash \{(0, 0)\} \subset \mathbb{R}^2$, defining a smooth 1-dimensional submanifold of $\mathbb{R}^2$, and observe that the restrictions to Σ of the two projections $\mathbb{R}^2 \to \mathbb{R}$ define a smooth fiber product diagram, and thus (since P satisfies the universal property) give rise to a smooth map $u : \Sigma \to P$, which is inverse to the bijection $P \backslash \{p\} \to \Sigma$ defined by (F, G). This shows that $P \backslash \{p\}$ and Σ are diffeomorphic, thus P is a connected smooth manifold that can be turned into a 1-manifold with four connected components by deleting one point. There is no such manifold, so this is a contradiction.

For the example in part (d), we can identify $M \times N = \mathbb{R} \times *$ with $\mathbb{R}$ and thus identify the topological fiber product with the set

$$M \;{}_f\!\times_g N = f^{-1}(0) \subset \mathbb{R},$$

carrying the subspace topology it inherits as a subset of $\mathbb{R}$. It is not a manifold, because the point $0 \in f^{-1}(0)$ does not have any connected neighborhood. However, for any given smooth fiber product diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & \mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle f} \\ * & \xrightarrow{0} & \mathbb{R} \end{array},$$

P is a smooth manifold with a smooth function $F : P \to \mathbb{R}$ whose image is contained in $f^{-1}(0)$, and there is very little freedom in finding functions F with this property: since P is locally path-connected, F must be locally constant. It follows that F does factor through a smooth manifold with an obvious smooth bijection onto $f^{-1}(0)$: the manifold in question is $f^{-1}(0)$ itself, but with the *discrete* topology instead of the subspace topology. Conclusion: the fiber product in Diff for our given pair of maps is given by

$$\begin{array}{ccc} f^{-1}(0) & \hookrightarrow & \mathbb{R} \\ \downarrow & & \downarrow{\scriptstyle f} \\ * & \xrightarrow{0} & \mathbb{R} \end{array},$$

where $f^{-1}(0)$ in the corner is understood to carry the discrete topology and is thus a smooth 0-manifold. Its obvious bijection to the topological fiber product ($f^{-1}(0)$ with the subspace topology) is continuous, but not a homeomorphism.

Exercise 1.8. The following bit of abstract nonsense provides a useful tool for proving that objects are isomorphic in various categories, e.g. one can apply it in hTop to establish homotopy equivalences, or (as in Exercise 1.9 below) to deduce properties of tensor products from a universal property.

In any category $\mathscr{C}$, each object $X \in \mathscr{C}$ determines a covariant functor

$$\mathrm{Hom}(X, \cdot) : \mathscr{C} \to \mathsf{Set},$$

which associates to each object $Y \in \mathscr{C}$ the set $\mathrm{Hom}(X,Y)$ of morphisms and to each morphism $f : Y \to Z$ in $\mathscr{C}$ the map

$$\mathrm{Hom}(X,Y) \xrightarrow{f_*} \mathrm{Hom}(X,Z) : g \mapsto f \circ g.$$

There is similarly a contravariant functor $\mathrm{Hom}(\cdot,X) : \mathscr{C} \to \mathsf{Set}$ for which morphisms $f : Y \to Z$ induce maps

$$\mathrm{Hom}(Z,X) \xrightarrow{f^*} \mathrm{Hom}(Y,X) : g \mapsto g \circ f.$$

(a) Show that for any two objects $X, Y \in \mathscr{C}$, each morphism $f : X \to Y$ determines a natural transformation $T_f : \mathrm{Hom}(Y,\cdot) \to \mathrm{Hom}(X,\cdot)$ associating to each object $Z \in \mathscr{C}$ the set map $f^* : \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$, and that if f is an isomorphism, then the map f^* is bijective for every $Z \in \mathscr{C}$, i.e. T_f is then a *natural isomorphism*.[6]
(b) Show conversely that every natural transformation $T : \mathrm{Hom}(Y,\cdot) \to \mathrm{Hom}(X,\cdot)$ is T_f for a unique morphism $f : X \to Y$, which is an isomorphism of $\mathscr{C}$ if and only if T_f is a natural isomorphism. It follows that X and Y are isomorphic whenever the sets of morphisms $\mathrm{Hom}(X,Z)$ and $\mathrm{Hom}(Y,Z)$ are in bijective correspondence for every third object Z, in a way that is natural with respect to Z.
(c) Prove contravariant analogues of parts (a) and (b) involving the functors $\mathrm{Hom}(\cdot,X)$ and $\mathrm{Hom}(\cdot,Y)$.

Solution: The interesting step is part (b), so let's just talk about that. (One could give a quick answer to part (a) more or less by mumbling the word "functor".) Suppose a natural transformation $T : \mathrm{Hom}(Y,\cdot) \to \mathrm{Hom}(X,\cdot)$ is given, so for every object $Z \in \mathscr{C}$, T defines a set map

$$T_Z : \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$$

which is required to fit into certain commutative diagrams as dictated by the word "natural". In particular, choosing $Z := Y$, we observe that T determines a distinguished morphism $f : X \to Y$ by

$$f := T_Y(\mathrm{Id}_Y) \in \mathrm{Hom}(X,Y).$$

We claim now that, in fact, $T = T_f$. Indeed, given any $Z \in \mathscr{C}$ and $g \in \mathrm{Hom}(Y,Z)$, naturality implies that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xrightarrow{T_Y} & \mathrm{Hom}(X,Y) \\ \downarrow g_* & & \downarrow g_* \\ \mathrm{Hom}(Y,Z) & \xrightarrow{T_Z} & \mathrm{Hom}(X,Z) \end{array}$$

commutes, hence

$$T_Z(g) = T_Z(g \circ \mathrm{Id}_Y) = (T_Z \circ g_*)(\mathrm{Id}_Y) = (g_* \circ T_Y)(\mathrm{Id}_Y) = g_* f = g \circ f = f^* g = T_f(g).$$

Now that we know all natural transformations arise in this way, and after verifying the formula $T_{f \circ g} = T_g \circ T_f$, it follows easily that the morphism $f : X \to Y$ has an inverse if and only if the corresponding natural transformation T_f has an inverse.

One way to apply this result in homotopy theory is as follows. Suppose we are given a map $f : X \to Y$ for which we can verify that for all spaces Z, the induced maps

$$f^* : [Y,Z] \to [X,Z] : g \mapsto g \circ f$$

are bijective. This means that the natural transformation on Hom-functors corresponding to f is a natural isomorphism, therefore implying that f itself is an isomorphism, i.e. the conclusion in this

[6]A **natural isomorphism** $T : \mathcal{F} \to \mathcal{G}$ between two functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ is a natural transformation such that the morphism $T(\alpha) : \mathcal{F}(\alpha) \to \mathcal{G}(\alpha)$ in $\mathscr{B}$ associated to each object $\alpha \in \mathscr{A}$ is an isomorphism. It follows that T has an inverse, which is also a natural transformation $T^{-1} : \mathcal{G} \to \mathcal{F}$.

setting is that f is a homotopy equivalence. The variant in part (c) would imply similarly that if the maps

$$f_* : [Z, X] \to [Z, Y] : g \mapsto f \circ g$$

are known to be bijective for all spaces Z, then f is a homotopy equivalence.

Exercise 1.9 (tensor products)**.** On the category R-Mod of modules over a commutative ring R, the tensor product satisfies the following universal property: for any three R-modules A, B, C, the natural map

$$\mathrm{Hom}(A \otimes B, C) \xrightarrow{\alpha} \mathrm{Hom}(A, \mathrm{Hom}(B, C)), \qquad \alpha(\Phi)(a)(b) := \Phi(a \otimes b)$$

is a bijection. Indeed,

$$\mathrm{Hom}_2(A, B; C) := \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

can be interpreted as the set of R-bilinear maps $A \times B \to C$, so the fact that α is bijective means that every such bilinear map factors through the canonical R-bilinear map $A \times B \to A \otimes B$ and a uniquely determined R-module homomorphism $A \otimes B \to C$. In fact, α is not just a bijection; it is also an R-module isomorphism, though we will not make use of this fact in the following. The important observation for now is that α defines a natural isomorphism between the two functors $\mathrm{Hom}(\cdot \otimes \cdot, \cdot)$ and Hom_2 from R-$\mathsf{Mod} \times R$-$\mathsf{Mod} \times R$-Mod to Set, which are contravariant in the first two variables and covariant in the third.

More generally, suppose $\mathscr{C}$ is any category for which the sets $\mathrm{Hom}(X, Y)$ can be regarded as objects in $\mathscr{C}$ for every $X, Y \in \mathscr{C}$, and suppose $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a functor such that the functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathsf{Set}$ defined by $\mathrm{Hom}(\cdot \otimes \cdot, \cdot)$ and $\mathrm{Hom}_2 := \mathrm{Hom}(\cdot, \mathrm{Hom}(\cdot, \cdot))$ are naturally isomorphic, so in particular, for every triple of objects $X, Y, Z \in \mathscr{C}$, there is a bijection of sets

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

that is natural with respect to all three.

(a) Prove that there is a natural isomorphism relating any two functors $\otimes, \otimes' : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ that satisfy the condition described above. In other words: tensor products are uniquely determined (up to natural isomorphism) by the universal property.

(b) Prove that $\otimes$ is associative in the sense that the functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ defined by $(X, Y, Z) \mapsto X \otimes (Y \otimes Z)$ and $(X, Y, Z) \mapsto (X \otimes Y) \otimes Z$ are naturally isomorphic. Prove it using only the universal property, i.e. do not use any knowledge of how $\otimes$ is actually defined in any specific categories.

Solutions: Both parts are applications of Exercise 1.8, which is the right tool for the job because the universal property of $\otimes$ does not tell us what $X \otimes Y$ is, but instead tells us what other functor $\mathrm{Hom}(X \otimes Y, \cdot)$ is naturally isomorphic to, namely $\mathrm{Hom}_2(X, Y; \cdot) := \mathrm{Hom}(X, \mathrm{Hom}(Y, \cdot))$. If we are given two versions $\otimes$ and $\otimes'$ that both satisfy the universal property, we obtain from this a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, \cdot) \cong \mathrm{Hom}(X \otimes' Y, \cdot)$$

for every pair of objects $X, Y \in \mathscr{C}$, and therefore (via Exercise 1.8) an isomorphism $X \otimes Y \cong X \otimes' Y$.

Associativity follows similarly because one can follow two chains of natural bijections that both end at the same destination: for any spaces X, Y, Z, V we have:

$$\mathrm{Hom}(X \otimes (Y \otimes Z), V) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y \otimes Z, V)) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, V))),$$

and also

$$\mathrm{Hom}((X \otimes Y) \otimes Z, V) \cong \mathrm{Hom}(X \otimes Y, \mathrm{Hom}(Z, V)) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, V))).$$

Exercise 1.10 (tensor products of pairs)**.** Let $\mathsf{Top}^{\mathrm{rel}}$ denote the category of pairs of spaces and maps of pairs. When defining the cross and cup products on relative homology and cohomology, one often sees the product of two pairs defined as

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B).$$

(a) Why is this definition of $\times$ not actually a *product* (in the sense of category theory) on the category $\mathsf{Top}^{\mathrm{rel}}$? What do categorical products in $\mathsf{Top}^{\mathrm{rel}}$ actually look like?

(b) In the spirit of Exercise 1.9, I would like to argue that $\times$ as defined above should be interpreted as a *tensor product* on $\mathsf{Top}^{\mathrm{rel}}$. Due to some subtle point-set topological issues that I'd rather not get into until next week, it's best for now to dispense with topologies and work instead in the category $\mathsf{Set}^{\mathrm{rel}}$, whose objects are pairs (X, A) of sets with $A \subset X$, and whose morphisms $(X, A) \to (Y, B)$ are arbitrary (not necessarily continuous) maps $X \to Y$ that send A into B. In this setting, how can you regard each of the sets $\mathrm{Hom}((X, A), (Y, B))$ as an object of $\mathsf{Set}^{\mathrm{rel}}$ such that there are natural bijections

$$\mathrm{Hom}((X, A) \times (Y, B), (Z, C)) \cong \mathrm{Hom}\big((X, A), \mathrm{Hom}((Y, B), (Z, C))\big)$$

for all choices of pairs?

Answers: Categorical products require projection morphisms, but e.g. the projection map $X \times Y \to X$ does not generally send $A \times Y \cup X \times B$ into A, and thus does not define a map of pairs $(X, A) \times (Y, B) \to (X, Y)$. For a categorical product on $\mathsf{Top}^{\mathrm{rel}}$, the correct definition would be the obvious one,

$$(X, A) \times (Y, B) := (X \times Y, A \times B).$$

If (X, A) and (Y, B) are objects in $\mathsf{Set}^{\mathrm{rel}}$, then $\mathrm{Hom}((X, A), (Y, B))$ also becomes an object in $\mathsf{Set}^{\mathrm{rel}}$ after singling out the subset

$$\big\{\phi \in \mathrm{Hom}((X, A), (Y, B)) \bigm| \phi(X) \subset B\big\} \subset \mathrm{Hom}((X, A), (Y, B)).$$

It is then straightforward to check that set maps of pairs from (X, A) to $\mathrm{Hom}((Y, B), (Z, C))$ are in natural bijective correspondence with set maps of pairs from $(X, A) \times (Y, B)$ to (Z, C).

The case of this with $A = B = C = \varnothing$ is often written in a more appealing way by using the notation

$$X^Y := \mathrm{Hom}(Y, X) \qquad \text{in } \mathsf{Set},$$

so that $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ becomes the so-called **exponential law**

$$Z^{X \times Y} \cong (Z^Y)^X.$$

Note that this is one of the few situations in which the categorical product can also sensibly be called a tensor product; they are not the same thing in $\mathsf{Set}^{\mathrm{rel}}$, but in Set they are.

The reason we removed topologies from the picture before starting this discussion was that one needs to be very careful about defining the right topology on the set $C(X, Y)$ of continuous maps $X \to Y$ between two spaces if one wants to have a natural bijection

$$C(X \times Y, Z) \cong C(X, C(Y, Z)).$$

In fact, there is *no* right way to define the topology on $C(X, Y)$ so that this works for *all* spaces; one must first restrict the category of spaces under consideration, and then make slight modifications to the definitions of both $C(X, Y)$ and $X \times Y$ as topological spaces. We will go into a little bit of detail about this when it becomes necessary, as without it, one would miss out on some very clever tools coming from stable homotopy theory.

2. Week 2

The lecture on 22.04.2024 was cancelled due to illness, so this week contains only one lecture.

Lecture 3 (25.04.2024): The homotopy category and mapping cylinders.

- The homotopy categories hTop (without base points) and hTop_* (with base points)
- Notation for diagrams that commute up to homotopy (see the notational glossary above)
- The **double mapping cylinder** of two maps $f : Z \to X$ and $g : Z \to Y$,

$$Z(f,g) := \big(X \amalg (I \times Z) \amalg Y\big)\Big/\!\sim, \qquad \text{where } (0,z) \sim f(z) \text{ and } (1,z) \sim g(z) \text{ for all } z \in Z.$$

- Role of $Z(f,g)$ as a weak form of pushout in hTop (it is called a **homotopy pushout**): the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle g}\downarrow & \sim & \downarrow{\scriptstyle i_X} \\ Y & \xrightarrow[i_Y]{} & Z(f,g) \end{array}$$

commutes up to an obvious homotopy, though not on the nose (the obvious inclusions i_X and i_Y have disjoint images). Diagrams

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle g}\downarrow & \underset{H}{\sim} & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow[\psi]{} & Q \end{array}$$

determine maps $Z(f,g) \xrightarrow{u} Q$, constructed in an obvious way out of φ, ψ and the homotopy $\varphi \circ f \overset{H}{\rightsquigarrow} \psi \circ g$, so that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & \\ {\scriptstyle i_X}\downarrow & & \searrow \\ Z(f,g) & \xrightarrow{\ u\ } & Q \\ {\scriptstyle i_Y}\uparrow & & \nearrow \\ Y & \xrightarrow{\ \psi\ } & \end{array}$$

commutes (on the nose, i.e. not just up to homotopy).
- Special cases:
  (1) **Mapping cylinder** of $f : X \to Y$:

$$Z(f) := Z(\mathrm{Id}_X, f) = (I \times Z) \cup_f Y,$$

  where the gluing occurs along $\{1\} \times Z$. Convenient feature: $Z(f)$ deformation retracts to Y, so $i_Y : Y \hookrightarrow Z(f)$ is a homotopy equivalence. We can therefore view *every* map $X \to Y$ "up to homotopy equivalence" as inclusion of a subspace, namely $i_X : X \hookrightarrow Z(f)$. (This trick was used once at the end of *Topologie II*, cf. the last two pages of [Wen23].)
  (2) **Mapping cone** of $f : X \to Y$: using the unique map $\epsilon : X \to *$, we define

$$\mathrm{cone}(f) := Z(\epsilon, f) = CX \cup_f Y,$$

  where $CX := (I \times X)\big/(\{0\} \times X)$ is the usual cone of X.
  (3) **Suspension** (unreduced): Not the most direct way to define it, but the familiar suspension SX of a space X is also the double mapping cylinder of a pair of maps

from X to one-point spaces:

$$\begin{array}{ccc} X & \longrightarrow & * \\ \big\downarrow & \sim & \big\downarrow \\ * & \hookrightarrow & SX \end{array} .$$

Here the two maps from $*$ to SX have images at the opposite poles, which are points obtained by collapsing $I \times X$ at $\{0\} \times X$ and $\{1\} \times X$ separately.

- Variant for hTop_*: If X, Y, Z are pointed spaces and f, g are pointed maps, defining a base point on $Z(f, g)$ requires modifying its definition by

$$Z(f,g) := \left(X \vee \frac{I \times Z}{I \times *} \vee Y \right) \Big/ \sim, \qquad \text{where } (0, z) \sim f(z) \text{ and } (1, z) \sim g(z) \text{ for all } z \in Z.$$

Note: Quotienting $I \times Z$ is necessary because $I \times Z$ on its own has no natural base point, but whenever Z, Z' are two pointed spaces,

$$\text{pointed homotopies } I \times Z \to Z' \qquad \Leftrightarrow \qquad \text{pointed maps } \frac{I \times Z}{I \times *} \to Z'.$$

Everything discussed above has analogues in which all maps are base-point preserving. The pointed version is sometimes called the **reduced** double mapping cylinder, and one can also derive from it special cases such as the **reduced mapping cone** and **reduced suspension**, which we'll have much more to say about later.

- Why is $Z(f, g)$ not really a pushout in hTop?
  (1) Our construction of the map $u : Z(f, g) \to Q$ uses more information than a diagram in hTop: it uses the actual maps in the diagram (not just their homotopy classes), plus a choice of homotopy. This doesn't mean it cannot work, but is a hint that we may be cheating.
  (2) (The real reason): The diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \\ {\scriptstyle i_X}\big\downarrow & \sim & \searrow \\ Z(f,g) & \xrightarrow{u} & Q \\ {\scriptstyle i_Y}\big\uparrow & \sim & \nearrow \\ Y & \xrightarrow{\psi} & \end{array}$$

does not always uniquely determine $[u] \in [Z(f, g), Q]$. Example: The mapping cone $\text{cone}(\alpha)$ of a degree 2 map $\alpha : S^1 \to S^1$, say $\alpha(e^{i\theta}) := e^{2i\theta}$ if we think of S^1 as the unit circle in $\mathbb{C}$. Now $\text{cone}(\alpha) \cong \mathbb{RP}^2$ and the natural inclusion $S^1 \hookrightarrow \text{cone}(\alpha)$ defines the nontrivial element of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. A homotopy pushout diagram

$$\begin{array}{ccc} S^1 & \longrightarrow & * \\ {\scriptstyle \alpha}\big\downarrow & \sim & \big\downarrow \\ S^1 & \xrightarrow[\beta]{} & Q \end{array}$$

now means a choice of space Q and homotopy class $\beta \in [S^1, Q]$ such that $\beta \cdot \beta$ is homotopic to a constant loop. The latter always holds if Q is simply connected, so take $Q := S^2$, and then observe that the diagram

$$\begin{array}{ccc} * & & \\ \big\downarrow & \sim & \searrow \\ \mathrm{cone}(\alpha) & \xrightarrow{u} & S^2 \\ \big\uparrow & \sim & \nearrow \\ S^1 & & \end{array}$$

always commutes up to homotopy, since $[S^1, S^2] \cong * \cong [*, S^2]$. But $[\mathbb{RP}^2, S^2]$ has more than one element, because there exist maps $\mathbb{RP}^2 \to S^2$ having either possible value of the mod-2 mapping degree (cf. Exercise 2.1).

- Theorem: There exists a category $\mathscr{P}$ whose objects are pushout diagrams (in Top)

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\big\downarrow & & \\ Y & & \end{array}$$

such that
  (1) Changing the maps f and g by homotopies produces isomorphic objects of $\mathscr{P}$;
  (2) There is a functor $\mathscr{P} \to \mathsf{hTop}$ sending each pushout diagram to its mapping cylinder $Z(f, g)$.
- Proof sketch: Morphisms in $\mathscr{P}$ are diagrams

$$\begin{array}{ccccc} Z & & \xrightarrow{\quad f \quad} & & X \\ & \searrow^{\gamma} & \underset{\phi}{\sim} & & & \searrow^{\alpha} \\ {\scriptstyle g}\big\downarrow & & Z' & \xrightarrow{\quad f' \quad} & & X' \\ & \underset{\psi}{\sim} & {\scriptstyle g'}\big\downarrow & & & \\ Y & & & & & \\ & \searrow^{\beta} & & & & \\ & & Y' & & & \end{array},$$

including choices of homotopies ϕ and ψ as part of the data. The notion of composition of such morphisms arises naturally by composing maps and concatenating homotopies.[7] Such a morphism determines a homotopy pushout diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\big\downarrow & \underset{H}{\sim} & \big\downarrow{\scriptstyle i_{X'} \circ \alpha} \\ Y & \xrightarrow[i_{Y'} \circ \beta]{} & Z(f', g') \end{array}$$

and therefore also an induced map $Z(f, g) \xrightarrow{u} Z(f', g')$. It is a bit tedious but straightforward to check:

[7] It seems likely that I'm oversimplifying this and ought to talk about "homotopy classes of homotopies" if I really want the composition in $\mathscr{P}$ to be associative, but I do not want to give these details more attention than they deserve. I am attempting to present a slightly more highbrow perspective on a sequence of lemmas in [tD08, §4.1–4.2] that seem rather technical and tedious.

(1) The map induced by a composition of two morphisms in $\mathscr{P}$ is homotopic to the composition of the two induced maps.
(2) If the maps α, β, γ all have homotopy inverses, one can use them to construct an inverse morphism in $\mathscr{P}$.

Both only require the same ideas that are needed for proving e.g. that multiplication in the fundamental group is associative. The second point implies, in particular, that the map $Z(f,g) \to Z(f',g')$ is a homotopy equivalence whenever α, β, γ are.

- Corollary: If $f \underset{h}{\sim} f'$ and $g \underset{h}{\sim} g'$, then $Z(f,g)$ and $Z(f',g')$ are homotopy equivalent.
- Theorem: Pushouts in hTop and hTop_* do not always exist.[8]
- Proof sketch in hTop_*: Fix the obvious base point in S^1 so that our previous degree 2 map $\alpha : S^1 \to S^1$ preserves base points. A pushout diagram in hTop_* of the form

$$\begin{array}{ccc} S^1 & \longrightarrow & * \\ {\scriptstyle\alpha}\big\downarrow & \sim & \big\downarrow \\ S^1 & \underset{\beta}{\longrightarrow} & P \end{array}$$

then means a pointed space P together with an element in the 2-torsion subgroup of its fundamental group

$$\beta \in \pi_1(P)_{(2)} := \left\{ \gamma \in \pi_1(P) \;\middle|\; \gamma^2 = 0 \right\}.$$

Then P and β satisfy the universal property for a pushout in hTop_* if and only if for every space Q and $\gamma \in \pi_1(Q)_{(2)}$, the map

$$[P, Q] \to \pi_1(Q)_{(2)} : u \mapsto u_* \beta$$

is a bijection. Assume this is true, and then consider the surjective map

$$\mathrm{SO}(3) \xrightarrow{p} S^2 : A \mapsto Ae_1,$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $e_1, e_2, e_3 \in \mathbb{R}^3$ denotes the standard basis. Taking e_1 as a base point in S^2, we have

$$p^{-1}(e_1) \cong \mathrm{SO}(2) \cong S^1,$$

giving rise to an exact sequence of pointed spaces

$$S^1 \overset{i}{\hookrightarrow} \mathrm{SO}(3) \overset{p}{\to} S^2.$$

We will see next week that the map $p : \mathrm{SO}(3) \to S^2$ has a special property: it is a *fibration*, with the consequence that for every space P, the induced sequence of pointed sets

$$[P, S^1] \overset{i_*}{\to} [P, \mathrm{SO}(3)] \overset{p_*}{\to} [P, S^2]$$

is also exact, meaning the preimage of the base point under p_* matches the image of i_*. (Here $[X, Y]$ means the set of homotopy classes of pointed maps $X \to Y$, so it is a set with an obvious base point.) Combining this with the bijection that we deduced above from the universal property of the pushout, we obtain an exact sequence

$$\pi_1(S^1)_{(2)} \to \pi_1(\mathrm{SO}(3))_{(2)} \to \pi_1(S^2)_{(2)},$$

in which the first and last terms both vanish. But $\mathrm{SO}(3) \cong \mathbb{RP}^3$ and thus $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, so the middle term does not vanish, and this is a contradiction.

- To do next week: Define what a fibration is and explain why the sequence of sets of homotopy classes in that proof was exact.

[8]. . . which is why we need to use *homotopy* pushouts instead.

Suggested reading. A more comprehensive treatment of mapping cylinders (including details that I left out of the proof of the theorem about the functor $\mathscr{P} \to \mathsf{hTop}$) can be found in [tD08, §4.1–4.2]. This does not include the proof that pushouts in hTop_* don't exist; I found that in the materials for a course on homotopy theory by Tyrone Cutler, available at https://www.math.uni-bielefeld.de/~tcutler/ (see the first set of exercises on homotopy pushouts).

Exercises (for the Übung on 2.05.2024).

Exercise 2.1. Review the notions of the $\mathbb{Z}_2$-valued and $\mathbb{Z}$-valued mapping degrees for maps between closed and connected topological manifolds of the same dimension, as covered e.g. in [Wen23, Lecture 35]. Then:

(a) Show that for every closed and connected topological manifold M of dimension $n \in \mathbb{N}$, the set $[M, S^n]$ contains at least two elements, and infinitely many if M is orientable.
(b) Does the set $[S^n, M]$ also always have more than one element?

Exercise 2.2. Deduce from the properties of double mapping cylinders the standard fact that there is a functor $S : \mathsf{Top} \to \mathsf{Top}$ assigning to every space $X \in \mathsf{Top}$ its (unreduced) suspension SX. *Note: This is just intended as a sanity check. There is nothing especially nontrivial to be done here, and there are also more direct ways to show that suspensions define a functor.*

Exercise 2.3. Show that the mapping cone $\operatorname{cone}(f)$ of any homotopy equivalence $f : X \to Y$ is a contractible space.
Hint: Find a useful morphism in the category $\mathscr{P}$ of pushout diagrams.

Exercise 2.4. Show that for any two maps $f : Z \to X$ and $g : Z \to Y$, the singular homologies (with arbitrary coefficients) of the spaces X, Y, Z and $Z(f,g)$ are related by a long exact sequence of the form

$$\ldots \to H_{n+1}(Z(f,g)) \to H_n(Z) \to H_n(X) \oplus H_n(Y) \to H_n(Z(f,g)) \to H_{n-1}(Z) \to \ldots,$$

and describe explicitly what the two homomorphisms in the middle of this sequence look like. Show that it also works with all homology groups replaced by their reduced counterparts, then write down the special case of a mapping cone and check that what you have is consistent with Exercise 2.3.
Hint: There is a relatively straightforward way to apply the Mayer-Vietoris sequence here, but you could also deduce this as a special case of the exact sequence of the generalized mapping torus derived in [Wen23, Lecture 34].

Exercise 2.5. Prove that pushouts in hTop do not always exist.
Hint: The proof carried out in lecture for hTop_* requires only minor modifications. Note that even if X and Y are spaces without base points, the set of homotopy classes $[X, Y]$ still has a natural base point whenever Y is path-connected. (Why?)

Exercise 2.6. Give explicit examples of homotopic maps

$$f \underset{h}{\sim} f' : Z \to X \qquad \text{and} \qquad g \underset{h}{\sim} g' : Z \to Y$$

such that the mapping cylinders $Z(f,g)$ and $Z(f',g')$ are not homeomorphic. (They will of course be homotopy equivalent!)

Exercise 2.7. The **join** $X * Y$ of two spaces X and Y is the double mapping cylinder $Z(\pi_X, \pi_Y)$ defined via the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Prove that the join of two spheres is always homeomorphic to a sphere: concretely, for every $m, n \in \mathbb{N}$,

$$S^m * S^n \cong S^{m+n+1}.$$

Hint: Split the double mapping cylinder in half so that you see $S^m * S^n$ as the union of two pieces glued along boundaries that both look like $S^m \times S^n$. Can you think of two compact manifolds that both have $S^m \times S^n$ as boundary? Stare closely at the two pieces, you might recognize them! Now glue them together and ask: what is $S^m * S^n$ the boundary of?

Exercise 2.8. Many constructions in homotopy theory have analogues in homological algebra, and one of these is the mapping cone. For two chain complexes (A_*, ∂_A) and (B_*, ∂_B) with a chain map $f : A_* \to B_*$, the **mapping cone of** f is the chain complex $(\mathrm{cone}(f)_*, \partial)$ with

$$\mathrm{cone}(f)_n := A_{n-1} \oplus B_n \qquad \text{and} \qquad \partial := \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}.$$

The analogy to the mapping cone in Top goes through cellular homology: if X, Y are two CW-complexes and $f : X \to Y$ is a cellular map, then the cone of f inherits a natural cell decomposition whose augmented cellular chain complex $\widetilde{C}_*^{\mathrm{CW}}(\mathrm{cone}(f))$ is the cone of the chain map $f_* : \widetilde{C}_*^{\mathrm{CW}}(X) \to \widetilde{C}_*^{\mathrm{CW}}(Y)$.[9]

Show that the mapping cone $\mathrm{cone}(f)_*$ of a chain map $f : A_* \to B_*$ similarly plays the role of a *homotopy pushout* in the category Ch of chain complexes and chain maps, with the role of a one-point space played by the trivial chain complex $0_* \in \mathsf{Ch}$. Specifically:

(a) There is a natural chain map $i_B : B_* \to \mathrm{cone}(f)_*$ such that the diagram

$$\begin{array}{ccc} A_* & \longrightarrow & 0_* \\ {\scriptstyle f}\big\downarrow & \sim & \big\downarrow \\ B_* & \xrightarrow[i_B]{} & \mathrm{cone}(f)_* \end{array}$$

commutes up to chain homotopy.

(b) Any homotopy-commutative diagram in Ch of the form

$$\begin{array}{ccc} A_* & \longrightarrow & 0_* \\ {\scriptstyle f}\big\downarrow & \underset{H}{\sim} & \big\downarrow \\ B_* & \xrightarrow[\psi]{} & D_* \end{array}$$

naturally determines a chain map $u : \mathrm{cone}(f)_* \to D_*$ such that $u \circ i_B$ is chain homotopic to ψ.

(c) If we were being strict about the analogy via cellular homology, then the trivial complex 0_* in the diagrams above ought to be replaced by $\widetilde{C}_*^{\mathrm{CW}}(*)$, the augmented cellular chain complex of a one-point space, which is not trivial: it has nontrivial entries in degrees 0 and -1, with the boundary operator giving an isomorphism between them. Explain why this discrepancy does not matter, and nothing in the discussion above would change if we used $\widetilde{C}_*^{\mathrm{CW}}(*)$ in place of 0_*.

Hint: None of this is hard... the quickest approach may be by guessing.

References

[DK01] J. F. Davis and P. Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2001.

[Mac71] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. Vol. 5, Springer-Verlag, New York-Berlin, 1971.

[9] This was Problem 2(b) on the take-home midterm for last semester's *Topologie II* course, but for Exercise 2.8, you do not need to know about it.

[tD08] T. tom Dieck, *Algebraic topology*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[Wen23] C. Wendl, *Topology I and II* (2023). Notes from the course at HU Berlin, available at https://www.mathematik.hu-berlin.de/~wendl/Winter2023/Topologie2/lecturenotes.pdf.