Equivariant transversality, super-rigidty and all that

Chris Wendl

Humboldt-Universität zu Berlin

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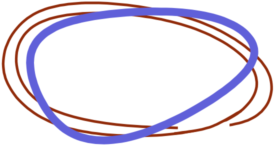


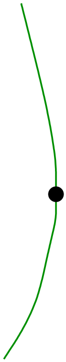
(slides available at `www.math.hu-berlin.de/~wendl/WesternHemisphere.pdf`)

Motivation

J-holomorphic curves are great!

Example (Gromov-McDuff, 1980's):

$u : (S^2, i) \to (M^4, J)$ with $[u] \cdot [u] = 1$
(M, ω) minimal



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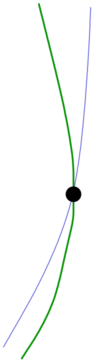

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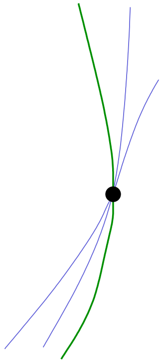

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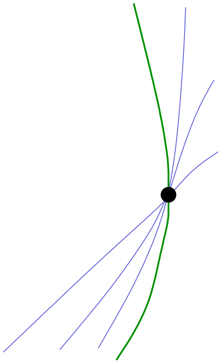

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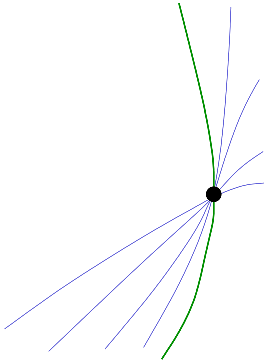

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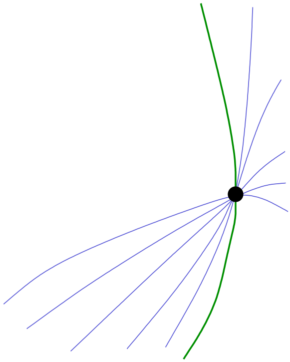

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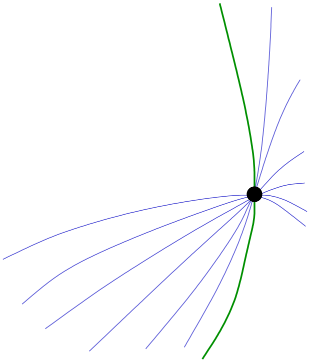

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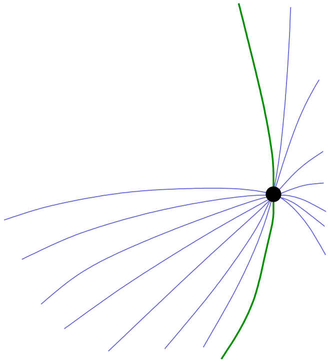

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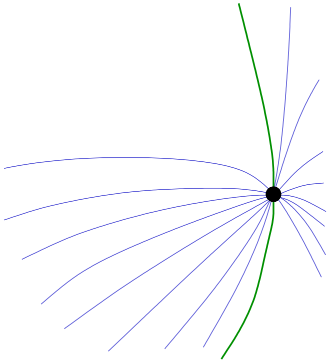

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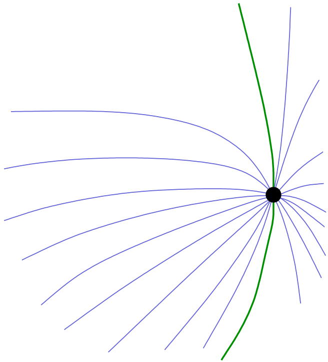

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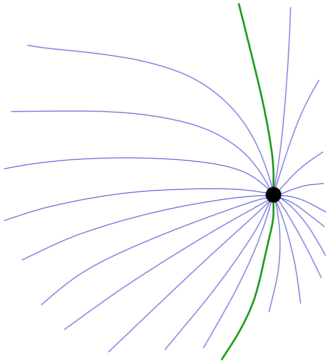

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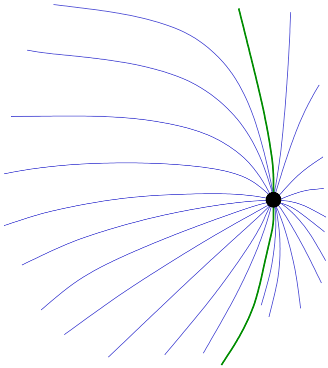

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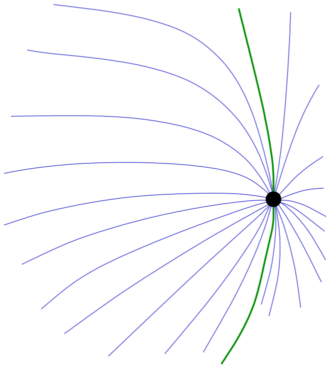
$\Rightarrow$ **Theorem** : $(M, \omega) \cong (\mathbb{C}P^2, c\,\omega_{\mathrm{FS}})$. $\square$



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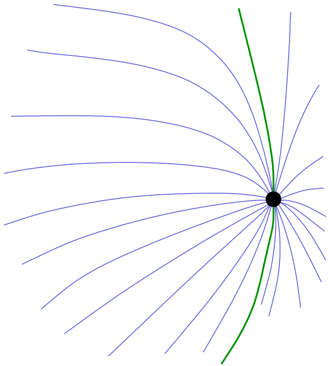

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Equivariant transversality is not possible... unless it is.

My aim in this talk is to address the following general questions:

- How do we recognize when equivariant transversality is possible?
  Claim: In many settings, if it is **possible**, *then it holds* **generically**.
- When it is not possible, why not, and what is true instead?
  (key words: **clean intersections**, **obstruction bundles**)
- If I want to apply these ideas to my favorite nonlinear elliptic PDE with symmetry, what do I need to prove?

We will consider three classes of problems as examples:

1. The zero-set of a section of a **finite-dimensional orbibundle***
2. The space of of an oriented line field*
3. The moduli space of

**no claim of originality*

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Problem 1: Finite dimensions

M a compact n-dimensional orbifold, $E \to M$ an orbibundle of rank m.

Question

For generic $\sigma \in \Gamma(E)$, is $\sigma^{-1}(0) \subset M$ a **suborbifold** of dimension $n - m$? Does $\sigma \pitchfork 0$ hold generically? **Answer**: Typically not.

Local example

Call $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ $\mathbb{Z}_2$-**equivariant** if $\sigma(x, -y) = -\sigma(x, y)$. Then $\sigma^{-1}(0)$ is **never** 0-dimensional, e.g. it contains $\mathbb{R} \times \{0\}$.

Next best thing ("Morse-Bott" condition):
Say $\sigma \in \Gamma(E)$ intersects zero **cleanly** if all components $\mathcal{M}_i \subset \sigma^{-1}(0)$ are **suborbifolds** (of dimensions $\geq n - m$) with $T_x\mathcal{M}_i = \ker D\sigma(x)$.

We can then compute the Euler number of E via **obstruction bundles**:

$$\langle e(E), [M] \rangle = \sum_i \langle e(\mathcal{O}b_i), \mathcal{M}_i \rangle, \qquad \mathcal{O}b_x := \operatorname{coker} D\sigma(x).$$

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Call $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ $\mathbb{Z}_2$-**equivariant** if $\sigma(x, -y) = -\sigma(x, y)$.
Then $\sigma^{-1}(0)$ is **never** 0-dimensional, e.g. it contains $\mathbb{R} \times \{0\}$.

Next best thing ("Morse-Bott" condition):
Say $\sigma \in \Gamma(E)$ intersects zero **cleanly** if all components $\mathcal{M}_i \subset \sigma^{-1}(0)$ are **suborbifolds** (of dimensions $\geq n - m$) with $T_x\mathcal{M}_i = \ker D\sigma(x)$.

We can then compute the Euler number of E via **obstruction bundles**:

$$\langle e(E), [M] \rangle = \sum_i \langle e(\mathcal{O}b_i), \mathcal{M}_i \rangle, \qquad \mathcal{O}b_x := \operatorname{coker} D\sigma(x).$$

Problem 1: Finite dimensions

M a compact n-dimensional orbifold, $E \to M$ an orbibundle of rank m.

Question

For generic $\sigma \in \Gamma(E)$, is $\sigma^{-1}(0) \subset M$ a **suborbifold** of dimension $n - m$?
Does $\sigma \pitchfork 0$ hold generically? **Answer**: Typically not.

Local example

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Sample theorem 1.A

If $\dim M = \operatorname{rank} E$ and isotropy groups satisfy $|G_x| \leq 3$ for all x, then generic sections of E intersect zero **cleanly**.

Key observation behind the proof (to be discussed):
$\mathbb{Z}_2$ and $\mathbb{Z}_3$ each have **only two real irreducible representations**.

Sample theorem 1.B (cf. Wasserman '69, Hepworth '09)

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For an **oriented line field** $\ell \subset TM$ generated by $R \in \mathfrak{X}(M)$, we consider the moduli space of **closed orbits**

$$\mathcal{M}(\ell) := \left\{ \gamma : S^1 \looparrowright M \,\middle|\, \dot{\gamma} \in \ell \right\} \Big/ \mathrm{Diff}(S^1) \cong \boldsymbol{\sigma}_R^{-1}(0) \Big/ S^1,$$

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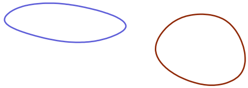

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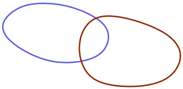

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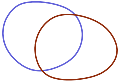

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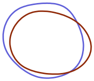

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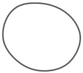

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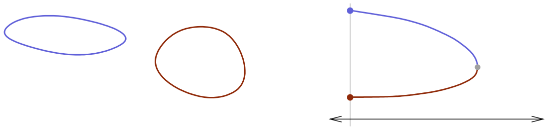

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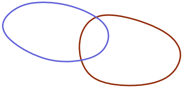

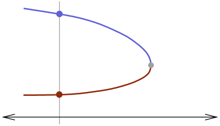

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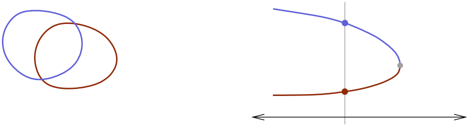

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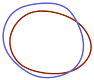

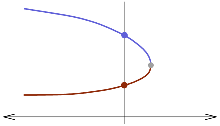

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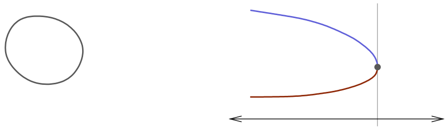

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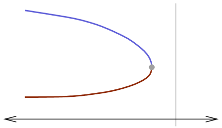

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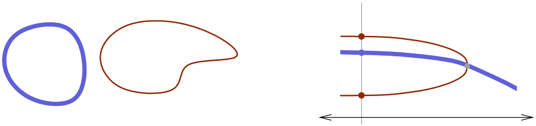

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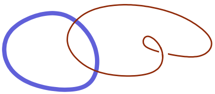

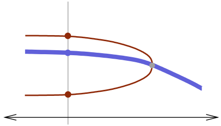

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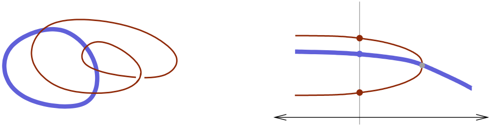

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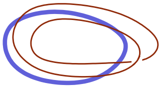

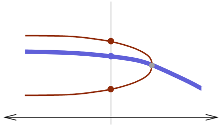

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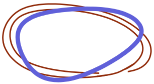

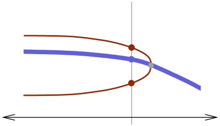

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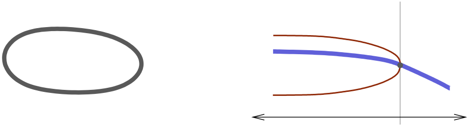

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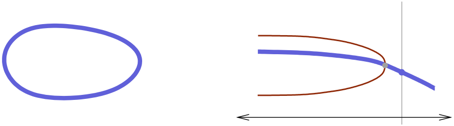

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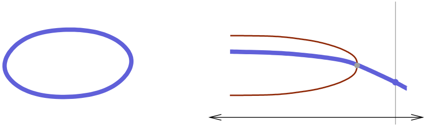

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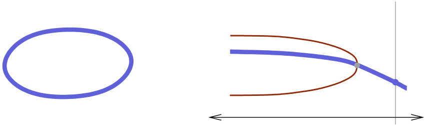

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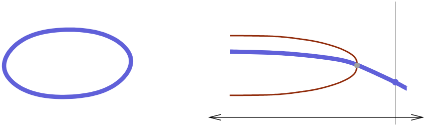

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There is only **birth-death** and **period-doubling** for generic $\{\ell_s\}_{s\in[0,1]}$.

Remark 1: If the ℓ_s are also **geodesible**, then components of $\mathcal{M}(\{\ell_s\})$ are **compact up to period-doubling**, i.e. **no blue sky catastrophes**.

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Fix a $2n$-dimensional symplectic manifold (M, ω) and consider compatible almost complex structures J.

Theorem 3.A (W. '16–'19)

If (M, ω) is a **symplectic Calabi-Yau 3-fold** $(\dim M = 6,\ c_1(M, \omega) = 0)$ and J is generic, then $\bar{\partial}_J$ intersects the zero-section **cleanly**, i.e. all simple curves are **super-rigid**.

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If $\dim M \geq 4$ and J is generic, all **unbranched covers** of simple J-holomorphic curves are **cut out transversely**.

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Paradigm

Each of our problems involves a **moduli space** $\mathcal{M}(\sigma)$ defined via geometric data σ, such that to every $x \in \mathcal{M}(\sigma)$ corresponds:

- A **finite symmetry group** G_x, which is **trivial** on a subset $\mathcal{M}^*(\sigma) \subset \mathcal{M}(\sigma)$ for which transversality holds generically.
- A **Fredholm operator** $\mathbf{D}_x$, which is **surjective** if and only if **transversality** holds at x.

Here is the general strategy

1. **Isosymmetric strata** (easy):
   Decompose $\mathcal{M}(\sigma)$ into subsets $\mathcal{M}^G(\sigma) \subset \mathcal{M}(\sigma)$ on which G_x is constant. For generic σ, these are **submanifolds**.
2. **Walls** (the technical part):
   Stratify each $\mathcal{M}^G(\sigma)$ further into submanifolds on which $\ker \mathbf{D}_x$ and $\operatorname{coker} \mathbf{D}_x$ vary smoothly (i.e. **constant dimensions**).
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Given $\sigma \in \Gamma(E)$, write $\mathcal{M}(\sigma) := \sigma^{-1}(0) \subset M$.
For each finite group G, define

$$M^G := \left\{ x \in M \;\middle|\; G_x \cong G \right\},$$

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Key observations:

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At each $x \in \mathcal{M}^G(\sigma)$, there is a **linearization**

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Let $\{\boldsymbol{\theta}_i : G \to \mathrm{Aut}_{\mathbb{R}}(W_i)\}_{i=1}^N$ denote the **real irreducible representations** of G, with $\boldsymbol{\theta}_1$ as the trivial representation.

Since $\mathbf{D}_x : T_xM \to E_x$ is G_x**-equivariant**, Schur's lemma implies that it splits with respect to the **isotypic decompositions** $T_xM = \bigoplus_{i=1}^N T_xM^i$ and $E_x = \bigoplus_{i=1}^N E_x^i$, giving

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Corollary (of stratification)

For generic σ, if $\mathcal{M}_i \subset \mathcal{M}^G(\sigma)$ is a component whose points $x \in \mathcal{M}_i$ satisfy $\operatorname{ind} \mathbf{D}_x^i \geq 0$ **for all** i, then $\sigma \pitchfork 0$ on an **open dense** subset of $\mathcal{M}_i$. Similarly for **clean intersections** if $\operatorname{ind} \mathbf{D}^i \leq 0$ for $i \geq 2$. □

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We consider $E := T^*M$ and $df \in \Gamma(E)$ and need to show $df \pitchfork 0$ for generic $f : M \to \mathbb{R}$. **Two new feaures**:

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where $\mathbb{K}_i := \operatorname{End}_G(W_i) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ has dimension $\in \{1, 2, 4\}$, $k_i := \dim_{\mathbb{K}_i} \ker \mathbf{D}_x^i$ and $c_i := \dim_{\mathbb{K}_i} \operatorname{coker} \mathbf{D}_x^i$.

Proof of Theorem 1.A (clean intersections), case $|G_x| \le 2$:
For $x \in \mathcal{M}^{\mathbb{Z}_2}(\sigma)$, there are two irreps $\theta_\pm : \mathbb{Z}_2 \to \mathrm{GL}(1, \mathbb{R})$, both with $\operatorname{End}_{\mathbb{Z}_2}(\mathbb{R}) = \mathbb{R}$. Write $\mathbf{D}_x = \mathbf{D}_x^+ \oplus \mathbf{D}_x^-$, where $\mathbf{D}_x^+$ is surjective and $\ker \mathbf{D}_x^+ = T_x \mathcal{M}^{\mathbb{Z}_2}(\sigma)$. We have $\operatorname{ind} \mathbf{D}_x = \dim M - \operatorname{rank} E = 0$, thus

$$\operatorname{ind} \mathbf{D}_x^- = -\operatorname{ind} \mathbf{D}_x^+ \le 0,$$

and need to show that $\mathbf{D}_x^-$ is injective. If not, then $x \in \mathcal{M}^{\mathbb{Z}_2}(\sigma\,;\,k,c)$ for $k := \dim \ker \mathbf{D}_x^- > 0$ and $c := k - \operatorname{ind} \mathbf{D}_x^- = k + \operatorname{ind} \mathbf{D}_x^+$. Then $\dim \mathcal{M}^{\mathbb{Z}_2}(\sigma\,;\,k,c) = \dim \mathcal{M}^{\mathbb{Z}_2}(\sigma) - kc = \operatorname{ind} \mathbf{D}_x^+ - k(k + \operatorname{ind} \mathbf{D}_x^+) < 0.$ $\square$

Problem 3 (holomorphic curves): Preparation

Linearizations

Each $u : (\Sigma, j) \to (M, J)$ has a **linearized Cauchy-Riemann operator**

$$\mathbf{D}_u := D\bar{\partial}_J(u) : \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$$

and a **normal Cauchy-Riemann operator**

$$\mathbf{D}_u^N := \pi_N \circ \mathbf{D}_u\big|_{N_u} : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u),$$

for the projection $u^*TM = T_u \oplus N_u \xrightarrow{\pi_N} N_u$ along the subbundle $T_u \subset u^*TM$ with $(T_u)_z = \operatorname{im} du(z)$ at all noncritical points z.

Lemma: (i) u is cut out **transversely** iff $\mathbf{D}_u^N$ is **surjective**.
(ii) For an immersed simple curve with index 0, u is **super-rigid** iff $\mathbf{D}_{u\circ\varphi}^N$ is **injective** for all branched covers $\varphi : (\Sigma', j') \to (\Sigma, j)$. □

This makes $\mathbf{D}_u^N$ the more convenient operator to work with. But we need it to vary continuously on isosymmetric strata...

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Define strata of the form

$$\mathcal{M}^d(J) = \{u = v \circ \varphi\} \subset \mathcal{M}_g(A, J)$$

such that:

- v varies among **simple** curves $v : (\Sigma, j) \to (M, J)$ with a prescribed number of **critical** points, each of **prescribed order**;
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For generic J, $\mathcal{M}^d(J)$ is a smooth manifold, and the operators $\mathbf{D}_u^N$ vary smoothly as u varies in $\mathcal{M}^d(J)$. □

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Consider $\mathbf{D} := \mathbf{D}_v^N : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ on $E := N_v$, and

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Then $G := \mathrm{Aut}(\varphi) = \mathbb{Z}_2$ and there is a unique nontrivial **deck transformation** $\psi : \Sigma' \to \Sigma'$. We define

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Idea

Replace $\Gamma(\varphi^* E)$ with $\Gamma(E \otimes_{\mathbb{R}} W)$ for some flat bundle W.

Lemma (via asymptotic regularity):
For a finite set $\Theta \subset \Sigma$, restricting $\mathbf{D}$ to the **punctured** domain $\dot{\Sigma} := \Sigma \setminus \Theta$ produces an operator on weighted Sobolev spaces (with small exponential growth at punctures) that has the **same index and kernel** as $\mathbf{D}$. □

Now **remove branch points** and consider $\varphi : \dot{\Sigma}' \to \dot{\Sigma}$ as a covering map of punctured Riemann surfaces.

Lemma (covering space theory):
There exists a **regular** cover $\pi : \dot{\Sigma}'' \to \dot{\Sigma}$ with **finite automorphism group** G and an injective homomorphism $\rho : G \to S_d$ to the symmetric group such that φ is equivalent to the cover

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Given a representation $\boldsymbol{\theta} : G \to \mathrm{Aut}_{\mathbb{R}}(W)$, define the **flat vector bundle**

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This gives a **twisted** bundle $E^{\boldsymbol{\theta}} := E \otimes_{\mathbb{R}} W^{\boldsymbol{\theta}} \to \dot{\Sigma}$ with Cauchy-Riemann operator $\mathbf{D}^{\boldsymbol{\theta}}$ defined by $\mathbf{D}^{\boldsymbol{\theta}}(\eta \otimes v) := (\mathbf{D}\eta) \otimes v$ for flat sections v.

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Corollary (the general splitting of $\mathbf{D}_u^N$)

If $\boldsymbol{\rho} \cong \bigoplus_{i=1}^N \boldsymbol{\theta}_i^{\oplus m_i}$, then $\varphi^* \mathbf{D} \cong \mathbf{D}^{\boldsymbol{\rho}} \cong \bigoplus_{i=1}^N (\mathbf{D}^{\boldsymbol{\theta}_i})^{\oplus m_i}$. $\square$

Remark: If $\mathrm{ind}\, \mathbf{D} = 0$, a computation via the punctured Riemann-Roch formula shows $\mathrm{ind}\, \mathbf{D}^{\boldsymbol{\theta}} \leq 0$ always. This is 45% of the reason why Theorem 3.A (super-rigidity) is true.

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Problem 3 (holomorphic curves): Walls

Walls in $\mathcal{M}^d(J)$ are defined by fixing the dimensions of the kernel and cokernel of $\mathbf{D}_u^N$ and its summands. Locally near u, this is the **zero-set** of a map to $\mathrm{Hom}_G(\ker \mathbf{D}_u^N, \operatorname{coker} \mathbf{D}_u^N)$ whose derivative with respect to a variation $\mathbf{T}$ in $\mathbf{D}_u^N$ is

$$\ker \mathbf{D}_u^N \xrightarrow{\mathbf{T}} \Omega^{0,1}(\Sigma, N_u) \xrightarrow{\text{proj}} \operatorname{coker} \mathbf{D}_u^N.$$

Why is this derivative surjective?

Perturbing J causes **zeroth-order** perturbations in $\mathbf{D}_u^N$, so $\mathbf{T}$ should be realized by a **bundle map** $A : N_u \to \Lambda^{0,1}T^*\Sigma \otimes N_u$. If not every map $\ker \mathbf{D}_u^N \to \operatorname{coker} \mathbf{D}_u^N$ arises this way, then given bases $(\eta_i) \in \ker \mathbf{D}_u^N$ and $(\xi_j) \in \ker(\mathbf{D}_u^N)^* \cong \operatorname{coker} \mathbf{D}_u^N$, there exist nontrivial coefficients $c_{ij} \in \mathbb{R}$ such that for **all** zeroth-order perturbations A,

$$\sum_{i,j} c_{ij} \langle A\eta_i, \xi_j \rangle_{L^2} = 0 = \int_\Sigma \langle\ ,\ \rangle \circ (A \otimes \mathbb{1}) \Big(\sum_{i,j} c_{ij} \eta_i \otimes \xi_j \Big)\, d\,\mathrm{vol}\,.$$

In other words:

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Crucial technical lemma

For each $\ell \in \mathbb{N}$, there exists an integer $k \geq \ell$ and a **Baire set** of compatible almost complex structures J such that for every **simple curve** $u : (\Sigma, j) \to (M, J)$ and point $z \in \Sigma$, if η_i, ξ_j are **local solutions** to $\mathbf{D}_u^N \eta_i = 0$ and $(\mathbf{D}_u^N)^* \xi_j = 0$ near z such that the tensor product

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vanishes to order ℓ at z, then $\Pi(t)$ **does not vanish to order** k at z.

Corollary (via unique continuation): Generically all $\mathbf{D}_u^N$ satisfy Petri.

"**Proof**": Sard-Smale theorem + dimension counting in jet spaces at z...

Remark: The proof requires u to be simple for the usual (Sard-Smale) reasons, but the result is **local**, so it **carries over to all multiple covers**.

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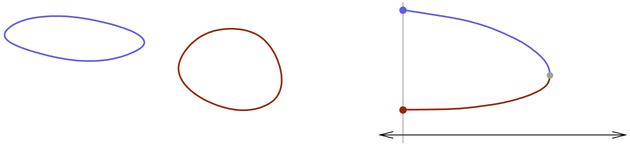

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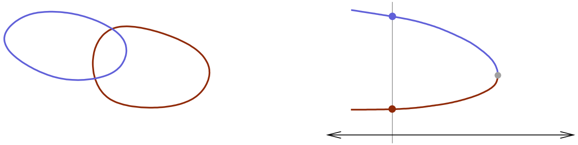

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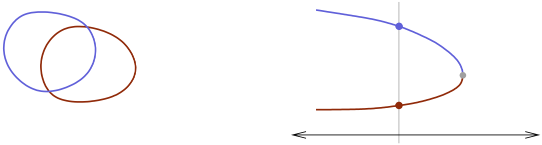

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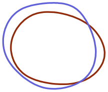

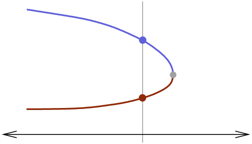

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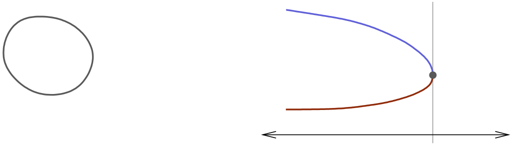

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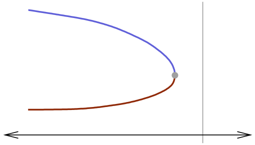

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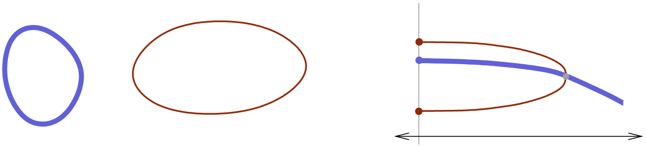

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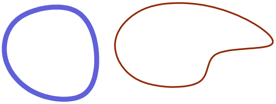

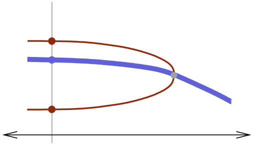

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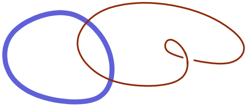

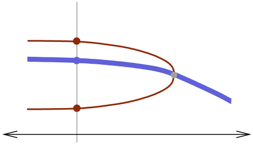

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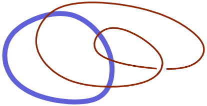

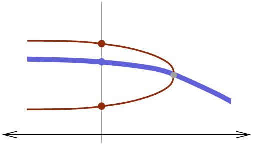

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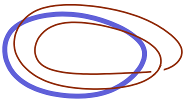

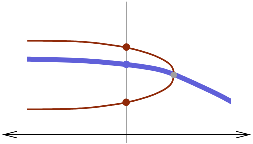

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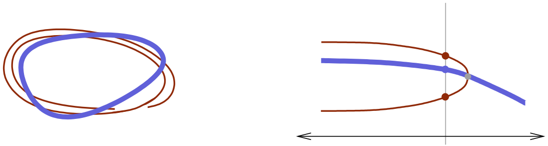

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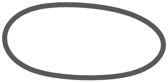

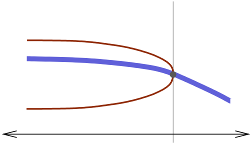

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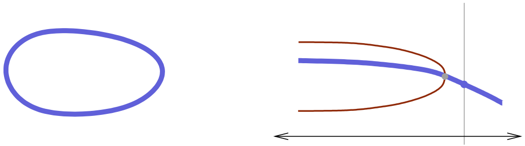

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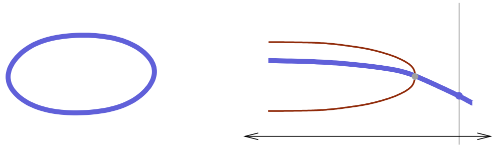

Sample theorem 2.B

For generic deformations $\{\ell_s\}_{s\in[0,1]}$ of an oriented line field, if lengths of orbits are bounded, nothing else goes wrong.

Back to Problem 2 (closed orbits)

Why not?

Isosymmetric strata: For $d = 1, 2, 3, \ldots$,

$$\mathcal{M}^d(\{\ell_s\}) := \left\{(s, \gamma) \in \mathcal{M}(\{\ell_s\}) \;\middle|\; \operatorname{cov}(\gamma) = d\right\}$$

is a smooth 1-manifold for generic $\{\ell_s\}$.

Splitting: For $(s, \gamma) \in \mathcal{M}^d(\{\ell_s\})$,

$$\mathbf{D}_\gamma = \bigoplus_{i=1}^{N} \mathbf{D}_\gamma^{\boldsymbol{\theta}_i}$$

with $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ the **irreps of $\mathbb{Z}_d$**. All summands have **index** 0.

Bifurcations = crossing walls of codimension 1:

$$\operatorname{codim} \mathcal{M}^d(\{\ell_s\}; k, c) = \sum_{i=1}^{N} t_i k_i c_i$$

with t_i = dimension of the equivariant endomorphism algebra of $\boldsymbol{\theta}_i$.

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Real irreps of $\mathbb{Z}_d$ come in two types:

- *Real type:* $\boldsymbol{\theta}_\pm : \mathbb{Z}_d \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R})$ with

$$\boldsymbol{\theta}_+(m) = 1, \qquad \boldsymbol{\theta}_-(m) = (-1)^m \text{ (if } d \text{ even).}$$

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$$\dim\ker \mathbf{D}_\gamma^{\boldsymbol{\theta}_+} = \dim\operatorname{coker} \mathbf{D}_\gamma^{\boldsymbol{\theta}_+} = 1 \quad \Rightarrow \quad \textbf{birth-death}.$$
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All other walls have codimension ≥ 2. □

Final remark:

In the Hamiltonian case, orbits are critical points of an **action functional** $\Rightarrow$ linearizations are **self-adjoint**. This changes $\operatorname{codim} \mathcal{M}^d(\{\ell_s\}; k, c)$ so that **complex-type representations** also play a role.

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