introduction

- linear
- nonlinear

} Func. Anal. is { linear alg. in ∞-dim. vec. spaces . . . ; differential calculus/geometry in ∞-dim. manifolds . . .

. . . of functions — with applications to diff. eqns / PDEs

defn: A Banach space is a complete normed vector space $(X, \|\cdot\|)$ (i.e. Cauchy seqs. converges)

exs: (1) $\mathbb{R}^n$, (2) $\mathbb{C}^n$ (w/ Euclidean norm), (3) all fin.-dim. vec. spaces over $\mathbb{R}$ or $\mathbb{C}$ with any norm.

(4) $C^0([0,1]) := \{ f : [0,1] \to \mathbb{R} \mid f \text{ continuous} \}$, with norm

$$\|f\|_{C^0} := \sup_{t\in[0,1]} |f(t)| \quad \left(= \max_{t\in[0,1]} |f(t)| \text{ since } [0,1] \text{ is compact} \right)$$

a seq. $f_n \to f$ in $C^0([0,1])$ iff $\|f - f_n\|_{C^0} \to 0$ iff $f_n \to f$ uniformly.

thm from Anal. I: Unif. Cauchy seqs. converge uniformly $\Rightarrow C^0([0,1])$ is complete.

non-ex: (5) $C^\infty([0,1]) := \{ f : [0,1] \to \mathbb{R} \mid f \text{ smooth } (\infty\text{-differentiable}) \}$, with $\|\cdot\|_{C^0}$ as norm. A unif. Cauchy seq of C^∞-fns. can converge to a non-smooth (but contin.!) fn.

remark: Many "obvious" facts in L.A. are *false* in func. ana.

(1) If $\dim X < \infty$, all linear subspaces $V \subseteq X$ are *closed* subsets.

counterex. for $\dim X = \infty$: $C^\infty([0,1])$ is a *dense* linear subspace of $C^0([0,1])$.

Weierstrass: approx. of contin. fns. by polynomials (which are smooth).

(2) If X, Y are fin-dim vec. spaces, all linear maps $X \to Y$ are continuous.

defn: a linear map $A : X \to Y$ btwn normed vec. spaces is bounded if $\exists$ a const. $c > 0$ s.t. $\|Ax\| \leq c\|x\| \ \forall x \in X$, i.e.

$$\|A\| := \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|}{\|x\|} \text{ is finite.}$$

thm: $A : X \to Y$ is bdd $\iff$ continuous.

pf: If A bdd, then for a seq. $x_n \to x$ in X,

$$\|Ax - Ax_n\| = \|A(x - x_n)\| \leq \|A\| \cdot \|x - x_n\| \to 0 \implies Ax_n \to Ax$$

$\implies A$ is contin.

If A not bdd, then $\exists$ seq. $x_n \in X$ s.t. $\frac{\|Ax_n\|}{\|x_n\|} \to \infty$,

then $y_n := \frac{x_n}{\|Ax_n\|}$ satisfies $y_n \to 0$, but $\|Ay_n\| = 1 \ \forall n$

$\implies Ay_n \not\to 0 = A(0)$, $\implies A$ not contin. at 0. ∎

defn: $\mathcal{L}(X, Y) := \{$contin. linear maps ("operators") $A : X \to Y\}$.

check: $\mathcal{L}(X, Y)$ with $\|A\|$ def'd as above (the "operator norm") is a normed vec. sp.

pf of Δ-ineq: $A, B \in \mathcal{L}(X, Y)$, then $\|Ax\| \leq \|A\| \cdot \|x\|$, $\|Bx\| \leq \|B\| \cdot \|x\|$ $\forall x \in X$, $\implies \|(A+B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\|$

$$\leq (\|A\| + \|B\|) \cdot \|x\| \implies \|A+B\| \leq \|A\| + \|B\|.$$

∎

Thm: If Y is complete, then so is $\mathcal{L}(X,Y)$.

Pf: Assume $A_n \in \mathcal{L}(X,Y)$ Cauchy. Then $\forall\ x \in X$, $\|A_n x - A_m x\| =$

$\|(A_n - A_m)x\| \leq \|A_n - A_m\| \cdot \|x\|$ small for m,n large $\Rightarrow$

$A_n x$ is a Cauchy seq in $Y \Rightarrow$ converges.

Let $A : X \to Y$ by $\displaystyle Ax := \lim_{n\to\infty} A_n x \quad \forall\ x \in X$.

check: A is linear.

still to show: (1) $A \in \mathcal{L}(X,Y)$, (i.e. A is bdd) (2) $\|A - A_n\| \to 0$ as $n \to \infty$.

(1) claim: $\|A_n\|$ is a Cauchy seq in $\mathbb{R}$.

$\big|\, \|A_n\| - \|A_m\| \,\big| = \big|\, \|A_n - A_m + A_m\| - \|A_m\| \,\big| \leq \|A_n - A_m\|$ small for m,n large.

Let $\displaystyle C := \lim_{n\to\infty} \|A_n\| \geq 0$. Then given $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow \|A_n\| \leq C + \varepsilon$, i.e. $\forall\ x \in X$, $\|A_n x\| \leq (C+\varepsilon)\|x\|$

$\Rightarrow \|Ax\| \leq (C+\varepsilon)\|x\| \Rightarrow \|A\| \leq C + \varepsilon$. (in fact: ε arbitrary $\Rightarrow$ $\|A\| \leq C$.)

$\Rightarrow A \in \mathcal{L}(X,Y)$.

(2) Given $\varepsilon > 0$, fix $N \in \mathbb{N}$ s.t. $m,n \geq N \Rightarrow \|A_m - A_n\| \leq \varepsilon$

$\Rightarrow \forall\ x \in X$, $\|A_m x - A_n x\| \leq \varepsilon \|x\| \Rightarrow \|Ax - A_n x\| \leq \varepsilon \|x\|$

$\downarrow m \to \infty$ Ax $\qquad \forall\ n \geq N \Rightarrow \|A - A_n\| \leq \varepsilon$

$\Rightarrow A_n \to A$ in $\mathcal{L}(X,Y)$. $\square$

Recall: A series $\sum_{n=1}^{\infty} x_n$ in a normed vec. sp. converges absolutely if $\sum_n \|x_n\| < \infty$. thm (Ana. I): If X is complete, abs. conv. $\Rightarrow$ conv.

cor: Assume X is a Banach space, thus so is $\mathcal{L}(X) := \mathcal{L}(X,X)$.
Then $\forall\, A \in \mathcal{L}(X)$ with $\|A\| < 1$, $I + A$ ($I :=$ identity $X \to X$)
has an inverse $(I+A)^{-1} \in \mathcal{L}(X)$.

Pf: Let $B := I - A + A^2 - A^3 + \dots$ check: $\|A^k\| \le \|A\|^k \quad \forall\, k \in \mathbb{N}$,
$\Rightarrow \sum_{n=0}^{\infty} \|(-1)^n A^n\| \le \sum_{n \ge 0} \|A\|^n < \infty$ since $\|A\| < 1$, $\Rightarrow$ series converges
& $B \in \mathcal{L}(X)$. check (PSET 1): ~~$BA = AB = I$~~. $B(I+A) = (I+A)B = I$

cor of cor: X, Y Banach, if $A \in \mathcal{L}(X,Y)$ has an inverse $A^{-1} \in \mathcal{L}(Y,X)$,
then so does $A + B$ for any $B \in \mathcal{L}(X,Y)$ with $\|B\|$ suff small.
Pf: $A + B = A(I + A^{-1}B) \Rightarrow (A+B)^{-1} = (I + A^{-1}B)^{-1} A^{-1}$
if $\|A^{-1}B\| < 1$. $\boxtimes$

APPLICATION: Consider the 2nd-order boundary value problem

$$(\text{BVP}) \quad \begin{cases} \ddot{x}(t) + P(t)\,x(t) = f(t) \\ x(0) = x(1) = 0 \end{cases}$$

for fns. $x : [0,1] \to \mathbb{R}$, given fns. $P, f : [0,1] \to \mathbb{R}$ contin.

funct.-ana. approach:
Let $X := \{ x : [0,1] \to \mathbb{R} \mid x$ of class C^2 with $x(0) = x(1) = 0 \}$ & defn.
norm $\|x\| := \|x\|_{C^2} := \|x\|_{C^0} + \|\dot{x}\|_{C^0} + \|\ddot{x}\|_{C^0}$.
$Y := C^0([0,1])$ with usual C^0-norm.

$\rightsquigarrow$ linear map $T_P : X \to Y$, $T_P x := \ddot{x} + Px$

EX (PSET 1): X & Y are Banach spaces, $T_P \in \mathcal{L}(X,Y)$ for every $P \in C^0([0,1])$,
T_0 has an inverse $T_0^{-1} \in \mathcal{L}(Y,X)$, & $\|T_P - T_0\| \le \|P\|_{C^0}$.

$\Rightarrow$ thm: $\exists$ const. $c > 0$ s.t. $\forall$ fns. $f, P \in C^0([0,1])$ s.t. $\|P\|_{C^0} < c$,
(BVP) has a unique sol. $x : [0,1] \to \mathbb{R}$ of class C^2.