We were proving (before we were rudely interrupted):

Assume $f \in L^p(\mathbb{R}^n)$ for some $p \in [1,\infty)$, $f_j := \rho_j * f$ for a sequence

$\rho_j : \mathbb{R}^n \xrightarrow{C^\infty} [0,\infty)$ s.t. $\operatorname{supp}(\rho_j) \subseteq B_{r_j}(0)$ & $\left| \int_{\mathbb{R}^n} \rho_j \, dm - 1 \right| < \varepsilon_j$,

$r_j, \varepsilon_j \longrightarrow 0$ as $j \to \infty$. Thm: $f_j \xrightarrow{L^p} f$.

pf under ASSUMPTION R: Assume $|f| \leq R$ & $\operatorname{supp}(f) \subseteq B_R(0)$ for some $R > 0$.

Then $f \in L^1(\mathbb{R}^n)$ & (easy EX.) $f_j \in L^1(\mathbb{R}^n)$. Claim: $f_j \xrightarrow{L^1} f$.

$$|f_j(x) - f(x)| = \left| \int f(x-y)\rho_j(y)\,dy - f(x) \right|$$

$$= \left| \int [f(x-y) - f(x)]\rho_j(y)\,dy + f(x)\left(\int \rho_j\,dm - 1\right) \right|$$

$$\leq \int |f(x-y) - f(x)| \cdot \rho_j(y)\,dy + |f(x)|\,\varepsilon_j.$$

$$\Rightarrow \|f_j - f\|_{L^1} \leq \underbrace{\int\left(\int |f(x-y) - f(x)|\,\rho_j(y)\,dy\right)dx}_{\text{// (Fubini)}} + \underbrace{\varepsilon_j \|f\|_{L^1}}_{\to 0 \text{ as } j \to \infty}$$

$$\int_{\mathbb{R}^n} \rho_j(y)\left(\int_{\mathbb{R}^n} |f(x-y) - f(x)|\,dx\right)dy = \int_{B_{r_j}(0)} \rho_j(y) \cdot \underbrace{\|\tau_{-y} f - f\|_{L^1}}_{\text{small for } |y| \leq r_j \text{ small}}\,dy$$

becomes arbitrarily small as $j \to \infty$ since $\int \rho_j\,dm$ is bdd.

Now $f_j \xrightarrow{L^1} f \Rightarrow$ for a subseq., $f_j \to f$ p.w. a.e., so $|f_j - f|^p \to 0$ p.w. a.e., but $\forall j$, $|f_j - f|^p$ is bdd by some fixed bdd fn. w/ cpt support ($\Rightarrow L^1$), so dominated convergence $\Rightarrow$

$$\int_{\mathbb{R}^n} |f_j - f|^p\,dm \to 0 \implies f_j \xrightarrow{L^p} f.$$

pf without Assumption R: Given $f \in L^p(\mathbb{R}^n)$, defn $f^R(x) := \begin{cases} f(x) & \text{if } |x| \leq R \text{ and } |f(x)| \leq R \\ 0 & \text{otherwise.} \end{cases}$

Then $f^R \xrightarrow{L^p} f$ as $R \to \infty$.

We know, $f_j^R := \rho_j * f^R \xrightarrow{L^p} f^R$ as $j \to \infty$.

$$\|f_j - f_j^R\|_{L^p} = \|\rho_j * f - \rho_j * f^R\|_{L^p} = \|\rho_j * (f - f^R)\|_{L^p} \overset{\text{(Young)}}{\leq} \underbrace{\|\rho_j\|_{L^1}}_{\text{bdd}} \cdot \|f - f^R\|_{L^p}.$$

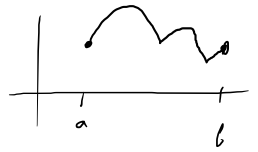
$$\text{Now } \|f - f_j\|_{L^p} \leq \underbrace{\|f - f^R\|_{L^p}}_{< \frac{\varepsilon}{3} \text{ for } R \gg 0} + \underbrace{\|f^R - f_j^R\|_{L^p}}_{< \frac{\varepsilon}{3} \text{ for } j \gg 0} + \underbrace{\|f_j^R - f_j\|_{L^p}}_{< \frac{\varepsilon}{3} \text{ for } R \gg 0} < \varepsilon \text{ for } j \gg 0.$$

$\square$

absolute continuity

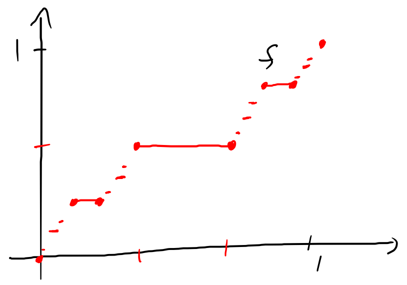
$\underline{Q}$: $f(x) = f(a) + \int_a^x f'(t)\, dt$ — what is the largest class of fns. $f : [a,b] \to V$ for which this holds?

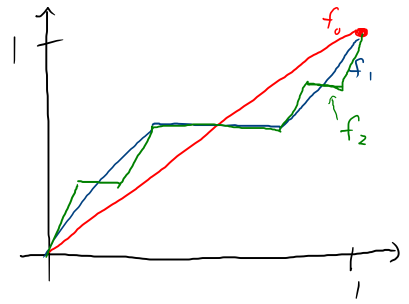
C^1?



C^0 + piecewise C^1? f differentiable a.e. & $f' \in L^1([a,b])$?

counterexample: "Cantor function" $f : [0,1] \to [0,1]$





This seq of fns. is uniformly Cauchy $\Rightarrow$ conv. to a continuous, surjective, increasing fn. $f : [0,1] \to [0,1]$ and $f' = 0$ on the complement of the Cantor set, i.e. $f' = 0$ a.e.

$f(x) \neq \int_0^x f'(t)\, dt = 0$

$\underline{Q}$: What kinds of fns $F : [a,b] \to V$ can be written as
$F(x) = c + \int_a^x f(t)\, dt$ for some $f \in L^1([a,b])$?

Lemma: For any measure space (X, μ) & $f \in L^1(X)$, given $\varepsilon > 0$,
$\exists\ \delta > 0$ s.t. $\forall\ A \subseteq X$, $\mu(A) < \delta \Rightarrow \int_A |f|\, d\mu < \varepsilon$.

Pf: If not, $\exists$ subsets $A_1, A_2, A_3, \ldots \subseteq X$ & $\varepsilon > 0$ s.t. $\int_{A_n} |f| \geq \varepsilon$ & $\mu(A_n) < \frac{1}{2^n}$. Defn $B_n := \bigcup_{k=n}^\infty A_k$.

$B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots \supseteq \bigcap_{n \in \mathbb{N}} B_n =: B$.

Now $\mu(B_n) \leq \sum_{k=n}^\infty \mu(A_k) < \sum_{k=n}^\infty \frac{1}{2^k} = \frac{1}{2^{n-1}} \to 0$ as $n \to \infty$ $\Rightarrow \mu(B) = 0$.

But $\int_{B_n} |f|\, d\mu \geq \int_{A_n} |f|\, d\mu \geq \varepsilon \Rightarrow \lim_{n\to\infty} \int_{B_n} |f|\, d\mu \neq \int_B |f|\, d\mu = 0$ contra! ∎

special case: $X = [a,b] \subseteq \mathbb{R}$, $\mu = m$, $A = \bigcup_{j=1}^N [a_j, b_j]$ for $a \leq a_1 \leq b_1 \leq \ldots \leq a_N \leq b_N \leq b$.

For $f \in L^1([a,b])$, let $F(x) = c + \int_a^x f(t)\, dt$ (c = const.),

$$\sum_{j=1}^N |F(b_j) - F(a_j)| = \sum_{j=1}^N \left| \int_{a_j}^{b_j} f(t)\, dt \right| \leq \sum_{j=1}^N \int_{[a_j, b_j]} |f|\, dm = \int_A |f|\, dm.$$

$\Rightarrow$ F satisfies the following:

defn: $I \subseteq \mathbb{R}$ an interval, a fn. F on I is absolutely continuous if $\forall\ \varepsilon > 0$,
$\exists\ \delta > 0$ s.t. $\forall$ finite seqs $a_1 \leq b_1 \leq \ldots \leq a_N \leq b_N$ of pts. in I,

$$\sum_{j=1}^N (b_j - a_j) < \delta \Rightarrow \sum_{j=1}^N |F(b_j) - F(a_j)| < \varepsilon.$$

rk: If we specified $N := 1$, this would be uniform continuity.

$\underline{EX}$: If $F : [a,b] \to \mathbb{R}$ is absolutely contin., then $\forall\ A \subseteq [a,b]$,
$m(A) = 0 \Rightarrow m(F(A)) = 0$.

ex: The Cantor fn $f : [0,1] \to [0,1]$ has $f(\underbrace{\text{Cantor set}}_{m=0}) = [0,1]$.
$\Rightarrow f$ is not abs. contin.

fundamental thm. of calculus for Lebesgue integral:

Following conditions on a fn. f on $[a,b]$ are equivalent:

(1) f is abs. contin.

(2) f is diff-able a.e., $f' \in L^1([a,b])$, & $f(x) = f(a) + \int_a^x f'(t)\, dt$.

EX: Lipschitz contin. $\Rightarrow$ abs. contin.

$\Rightarrow$ non-obvious cor: Lipschitz contin. fns. on an interval are diff-able a.e.

pf of FTC is in 2 steps:

(1) Given $f \in L^1([a,b])$, show that $F(x) := \int_a^x f(t)\, dt$ is diff-able a.e.

& $F' = f$.

(2) F A.C. $\Rightarrow$ $\exists$ $f \in L^1([a,b])$ s.t. $F(x) = c + \int_a^x f(t)\, dt$.

Lebesgue differentiation thm:

defn: For $f \in L^1_{loc}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ is a Lebesgue pt. for f if

average of $|f - f(x)|$ on the r-ball $B_r(x) \to 0$ as $r \to 0$, i.e.

$$\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0.$$

rk: x Lebesgue pt. $\Rightarrow$ avg. of f on $B_r(x) \to f(x)$ as $r \to 0$:

$$\left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f\, dm - f(x) \right| = \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} [f - f(x)]\, dm \right|$$

$$\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x)|\, dm \to 0 \text{ as } r \to 0.$$

rk: f contin. at x $\Rightarrow$ x is a Lebesgue pt.

thm: For $f \in L^1_{loc}(\mathbb{R}^n)$, almost every $x \in \mathbb{R}^n$ is a Lebesgue pt.

special case: $f \in L^1(\mathbb{R})$, $x \in (a,b) \subseteq \mathbb{R}$ a Lebesgue pt. of f.

Let $F(x) := \int_a^x f(t)\, dt$. Then at our Lebesgue pt., take $h > 0$ small,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} f(t)\, dt - f(x) \right| \leq$$

$$2 \cdot \frac{1}{2h} \int_x^{x+h} |f(t) - f(x)|\, dt \leq 2 \frac{1}{m(B_h(x))} \int_{B_h(x)} |f(t) - f(x)|\, dt \to 0 \text{ as } h \to 0.$$

Similar calculation for $h < 0$ $\Rightarrow$ $F'(x) = f(x)$.

cor: $\forall$ $f \in L^1([a,b])$, $F(x) := \int_a^x f(t)\, dt$ is diff-able a.e.

(i.e. at the Lebesgue pts of f) & $F' = f$. $\square$

rk: Leb. pt. condition is local $\Rightarrow$ suff. to consider $f \in L^1(\mathbb{R}^n)$.

For $r \geq 0$, let $f^r(x) := \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x)| \, dm$ for $r > 0$,

$$f^\circ(x) := \limsup_{r \to 0} f^r(x).$$

goal: $f^\circ = 0$ a.e.

For $N \in \mathbb{N}$, let $A_N := \{x \in \mathbb{R}^n \mid f^\circ(x) > \frac{1}{N}\}$.

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

Lemma: $\forall$ $N \in \mathbb{N}$, A_N is contained in a measurable set of measure $< \frac{1}{N}$.

$\Rightarrow$ all have measure 0 $\Rightarrow$ Leb. diff. thm.

Given $f \in L^1(\mathbb{R}^n)$, choose seq. f_j contin. s.t. $f_j \xrightarrow{L^1} f$. Now

$$f^r(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x)| \, dm$$

$$\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} \left(|f - f_k| + |f_k - f_k(x)| + |f_k(x) - f(x)| \right) dm$$

= sum of 3 integrals.

f_k contin. $\Rightarrow$ all x are Lebesgue pts. of f_k $\Rightarrow$ 2nd integral $\to 0$ as $r \to 0$ ($\forall$ k).

$|f_k(x) - f(x)|$ might not be small as $k \to \infty$ since $f_k \overset{C^0}{\not\to} f$, only $f_k \xrightarrow{L^1} f$.

Claim: 1st & 3rd integrals can be made arbitrarily small as $r \to 0$ for x outside some subset of arbitrarily small measure.