Sobolev spaces

comments on L^p

good: dual spaces well understood: $(L^p)^* \cong L^q$

bad: fns not diff-able

contrast: $\frac{\partial}{\partial x_j} : C_b^1(\mathbb{R}^n) \longrightarrow C_b^0(\mathbb{R}^n)$ is a bdd lin. op. (good!)

bad: $(C^0)^*$ is a space of measures, not fns!

$(C^1)^*$ is worse

defn (1st try): $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $m \geq 0$ integer, $1 \leq p \leq \infty$, defi. for $f \in C^\infty(\Omega)$,

$$\|f\|_{W^{m,p}} := \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p} \in [0, \infty]. \qquad (\text{note: } W^{0,p} = L^p)$$

$W^{m,p}(\Omega) :=$ closure w.r.t. the $W^{m,p}$-norm of $\{f \in C^\infty(\Omega) \mid \|f\|_{W^{m,p}} < \infty\}$

case $p = 2$, $\Omega = \mathbb{R}^n$

$$\|f\|_{W^{m,2}} \overset{\text{(Plancherel)}}{=} \sum_{|\alpha| \leq m} \|\widehat{\partial^\alpha f}\|_{L^2} = \sum_{|\alpha| \leq m} \|(2\pi i p)^\alpha \hat{f}\|_{L^2}$$

note: $|\alpha| \leq m \Rightarrow |p^\alpha| \leq c\,(1 + |p|^2)^{m/2}$ for some const $c > 0$

$\Rightarrow$ the $W^{m,2}$-norm on $\mathbb{R}^n$ is equivalent to

$$\boxed{\|f\|_{H^m} := \|(1 + |p|^2)^{m/2} \hat{f}\|_{L^2} = \left(\int_{\mathbb{R}^n} (1 + |p|^2)^m |\hat{f}(p)|^2 \, dp\right)^{1/2}} \in [0, \infty].$$

defn: For $m \geq 0$, $H^m(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) \mid \|f\|_{H^m} < \infty\}$.

rk: Defn does not require $m \geq 0$ to be an integer. "fractional order of diff-ability"

thm: $H^m(\mathbb{R}^n)$ is a Hilbert space w/ inner product

$$\langle f, g\rangle_{H^m} := \int_{\mathbb{R}^n} (1+|p|^2)^m \langle \hat{f}(p), \hat{g}(p)\rangle \, dp = \langle (1+|p|^2)^{m/2}\hat{f}, (1+|p|^2)^{m/2}\hat{g}\rangle_{L^2}.$$

pf: $H^m(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) : f \longmapsto (1+|p|^2)^{m/2}\hat{f}$ is a bijective isometry

$\Rightarrow$ completeness follows from compl. of L^2. $\boxtimes$

EX: For $|\alpha| = m$, $s \geq 0$, $\partial^\alpha : C_0^\infty(\mathbb{R}^n) \longrightarrow C_0^\infty(\mathbb{R}^n)$ extends to a bdd lin. op. $H^{s+m}(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n)$. (express via F.T. & mult. by polynomials)

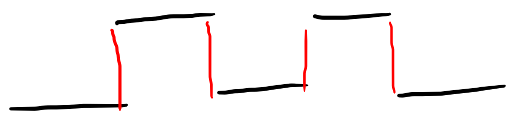
fully periodic fns: for $f \in L^2(\mathbb{T}^n)$, defn.

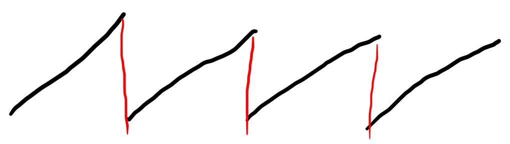
$$\|f\|_{H^m} := \|(1+|k|^2)^{m/2}\hat{f}\|_{\ell^2} = \left(\sum_{k\in\mathbb{Z}^n} (1+|k|^2)^m |\hat{f}_k|^2\right)^{1/2} \in [0,\infty]$$

$$H^m(\mathbb{T}^n) := \{ f \in L^2(\mathbb{T}^n) \mid \|f\|_{H^m} < \infty \}.$$

thm: $H^m(\mathbb{T}^n)$ is a Hilbert space w.r.t. $\langle f, g\rangle_{H^m} := \sum_{k\in\mathbb{Z}} (1+|k|^2)^m \langle \hat{f}_k, \hat{g}_k\rangle$ $\boxtimes$

EX: $\mathcal{S}(\mathbb{R}^n) \subseteq H^m(\mathbb{R}^n)$, $C^\infty(\mathbb{T}^n) \subseteq H^m(\mathbb{T}^n)$ $\forall\ m \geq 0$.

EX: square wave  both $\in H^m(S^1) := H^m(\mathbb{T}^1)$

sawtooth wave  iff $m < \frac{1}{2}$.

ex ($H^1 \not\Rightarrow C^0$ on $\mathbb{R}^2$): choose $\beta \in C_0^\infty(\mathbb{R}^2)$ s.t. $\beta = 1$ near $0 \in \mathbb{R}^2$,

let $f(x) := \beta(x)\,(-\log|x|^2)^{1/3}$. $f \notin C^0(\mathbb{R}^2)$, but $f \in H^1(\mathbb{R}^2)$.

<u>def</u>: We say $\exists$ a <u>continuous inclusion</u> $H^m \hookrightarrow C^k$ if $\exists$ a bdd lin. op. $H^m \to C^k : f \mapsto \tilde{f}$ s.t. $\tilde{f} = f$ a.e.

i.e. $\exists\ C > 0$ s.t. $\forall\ f \in H^m$, after changing f on a set of measure 0, $f \in C^k$ & $\|f\|_{C^k} \le C \cdot \|f\|_{H^m}$.

<u>thm</u> (Sobolev embedding thm, case $p=2$, $\Omega = \mathbb{R}^n$)

If $2s > n$, then $\forall$ integers $m \ge 0$, $\exists$ contin. inclusions

$H^{s+m}(\mathbb{R}^n) \hookrightarrow C^m(\mathbb{R}^n)$ & $H^{s+m}(\mathbb{T}^n) \hookrightarrow C^m(\mathbb{T}^n)$.

<u>ex</u>: For $s > \frac{1}{2}$, $H^s(S^1) \hookrightarrow C^0(S^1) \Rightarrow$ square/sawtooth $\notin H^s(S^1)$.

<u>Pf</u>: Consider $s > \frac{n}{2}$ & $f \in H^s(\mathbb{R}^n)$, show $H^s(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n)$.

<u>trick</u>: Show $\hat{f} \in L^1(\mathbb{R}^n)$, then $f \overset{a.e.}{=} \mathcal{F}^* \hat{f} \in C^0(\mathbb{R}^n)$ &

$\|\mathcal{F}^* \hat{f}\|_{C^0} \le \|\hat{f}\|_{L^1}$, $\therefore$ we need an estimate $\|\hat{f}\|_{L^1} \le c\, \|f\|_{H^s}$.

$$\|\hat{f}\|_{L^1} = \int_{\mathbb{R}^n} \frac{1}{(1+|p|^2)^{s/2}} \left|(1+|p|^2)^{s/2} \hat{f}\right| dp \overset{\text{Cauchy-Schwarz}}{\le} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|p|^2)^s} dp \right)^{1/2} \cdot \|f\|_{H^s}.$$

The integral on RHS converges iff (polar coords.) $\int_1^\infty \frac{r^{n-1}}{(1+r^2)^s} dr < \infty$

$$\Leftrightarrow \int_1^\infty r^{n-1-2s}\, dr < \infty \Leftrightarrow r^{n-2s}\Big|_1^\infty < \infty \Leftrightarrow n - 2s < 0 \Leftrightarrow 2s > n.$$

$\Rightarrow \|\hat{f}\|_{L^1} \le$ const $\cdot \|f\|_{H^s}$.

If $f \in H^{s+m}(\mathbb{R}^n)$, then $\forall\ |\alpha| \le m$, same trick gives $\|p^\alpha \hat{f}\|_{L^1} \le C \cdot \|f\|_{H^{s+m}}$

this bounds $\|\partial^\alpha f\|_{C^0} \Rightarrow \|f\|_{C^m} \le c \cdot \|f\|_{H^{s+m}}$.

<u>periodic</u> <u>case</u>: key point is, if $2s > n$, then $\displaystyle\sum_{k \in \mathbb{Z}^n} \frac{1}{(1+|k|^2)^s} < \infty$

(compare it to the integral). $\boxtimes$

(polar coords: for a bdd fn. $f : \mathbb{R}^n \to [0,\infty]$, depending only on $|x|$ for $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} f(x)\, dx < \infty \Leftrightarrow \int_1^\infty f(r)\, r^{n-1}\, dr < \infty.$$

<u>cor</u>: $\displaystyle\bigcap_{s \ge 0} H^s(\mathbb{R}^n) \subseteq C_b^\infty(\mathbb{R}^n)$, $\displaystyle\bigcap_{s \ge 0} H^s(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$.

compactness

defn: $A \in \mathcal{L}(X,Y)$ is *compact* if it maps bdd sets in X to precompact sets in Y, i.e. $\forall$ bdd seqs. $x_n \in X$, $Ax_n \in Y$ has a conv. subseq.

ex: If $\dim X = \infty$, $\mathrm{Id}: X \to X$ is *not* cpct.

ex: If $A : X \to Y$ has *finite rank*, then A is cpct.

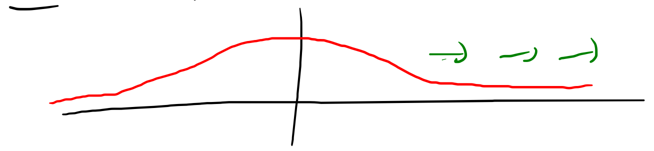
pf: $x_n \in X$ bdd, Ax_n is a bdd seq in the fin.-dim. space $\operatorname{im} A$
$\Rightarrow \exists$ conv. subseq.

ex: $C^1([0,1]) \hookrightarrow C^0([0,1])$ is cpct.

pf: $f_n \in C^1([0,1])$ bdd $\Rightarrow f_n$ is unif. bdd & so is f_n'
$\Rightarrow f_n$ is *equicontinuous*, Arzelà-Ascoli $\Rightarrow f_n$ has a C^0-conv. subseq.

EX: $C^m(\mathbb{T}^n) \hookrightarrow C^k(\mathbb{T}^n)$ for $m > k$ is cpct.

ex: $C^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ is *not* cpct, e.g. choose $f \in C^1$ s.t. $\lim_{x \to \pm\infty} f(x) = 0$



$f_n(x) := f(x+n)$. now $f_n \xrightarrow{p.w.} 0$,
but $\|f_n\|_{C^0} = \|f\|_{C^0} \not\to 0 \Rightarrow \nexists$ conv. subseq.

prop: The set $\{A \in \mathcal{L}(X,Y) \mid A \text{ cpct}\}$ is closed in $\mathcal{L}(X,Y)$ w.r.t. the operator norm.

pf: Suppose $A_n : X \to Y$ cpct, $A_n \to A \in \mathcal{L}(X,Y)$.
$x_k \in X$ bdd $\Rightarrow A_1 x_k$ conv. after replacing x_k with some subseq. $x_k^{(1)}$.
$A_2 x_k^{(2)}$ also conv. for some subseq. $x_k^{(2)}$ of $x_k^{(1)}$.
Keep going $\rightsquigarrow$ Cantor diagonal trick:
$x_k^{(\infty)} := x_k^{(k)}$ is a subseq. of x_k s.t. $A_n x_k^{(\infty)}$ conv. as $k \to \infty$ $\forall n$.

claim: $A x_k^{(\infty)}$ also converges.

pf:
$$\|Ax_n^{(\infty)} - Ax_m^{(\infty)}\| \le \underbrace{\|Ax_n^{(\infty)} - A_M x_n^{(\infty)}\|}_{\text{small if } M \text{ large}} + \overbrace{\|A_M x_n^{(\infty)} - A_M x_m^{(\infty)}\|}^{\text{small for } m,n \text{ large since } A_M x_k^{(\infty)} \text{ is Cauchy}} + \underbrace{\|A_M x_m^{(\infty)} - A x_m^{(\infty)}\|}$$

$\Rightarrow A x_k^{(\infty)}$ is Cauchy. $\square$

cor: Any limit of a seq. of finite-rank ops. is cpct.

thm (Rellich-Kondrachov cptness thm, case $p=2$)
$\forall\ t > s \ge 0$, the obvious inclusion $H^t(\mathbb{T}^n) \hookrightarrow H^s(\mathbb{T}^n)$ is cpct.