agenda:

(1) pf of Rellich-Kondrachov

(2) $C^\infty \subseteq H^s$ is dense

(3) Hölder spaces $C^{m,\alpha}$

(4) application to PDE

(1) Rellich-Kondrachov cptness thm: the inclusion $H^t(\mathbb{T}^n) \hookrightarrow H^s(\mathbb{T}^n)$ if $s<t$ is cpt, i.e. every H^t-bdd seq. of fns. on $\mathbb{T}^n$ has an H^s-conv. subseq.

main tool: any limit (in the op. norm) of a seq. of fin.-rank operators is cpt.

pf of thm: Let $A: H^t(\mathbb{T}^n) \hookrightarrow H^s(\mathbb{T}^n)$, for $N \in \mathbb{N}$, let

$$A_N : H^t(\mathbb{T}^n) \to H^s(\mathbb{T}^n) : f \mapsto \sum_{|k| \le N} e^{2\pi i k\cdot x} \hat{f}_k .$$

A_N has fin. rank.

claim: as $N \to \infty$, $A_N \to A \in \mathcal{L}(H^t(\mathbb{T}^n), H^s(\mathbb{T}^n))$.

$$\|(A - A_N) f\|_{H^s}^2 = \sum_{|k|>N} (1+|k|^2)^s |\hat{f}_k|^2 = \sum_{|k|>N} \frac{1}{(1+|k|^2)^{t-s}} (1+|k|^2)^t |\hat{f}_k|^2$$

$$< \frac{1}{(1+N^2)^{t-s}} \sum_{|k|>N} (1+|k|^2)^t |\hat{f}_k|^2 \le \frac{1}{(1+N^2)^{t-s}} \|f\|_{H^t}^2$$

$$\Rightarrow \|A - A_N\|^2 \le \frac{1}{(1+N^2)^{t-s}} \to 0 \text{ as } N \to \infty \quad (\text{since } t-s>0). \quad \square$$

rk: $H^t(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n)$ for $t>s$ is not cpt:

e.g. $f_k(x_1, \dots, x_n) = f(x_1 + k, x_2, \dots, x_n)$ for a fixed $f \neq 0 \in H^t(\mathbb{R}^n)$

$\|f_k\|_{H^t} = \|f\|_{H^t}$ $\forall k$, sim. $\|f_k\|_s \not\to 0 \Rightarrow \nexists$ H^s-conv. subseq.

(2) <u>thm</u>: $C^\infty(\mathbb{T}^n) \subseteq H^s(\mathbb{T}^n)$ & $\mathcal{S}(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n)$ are dense $\forall\ s \geq 0$.

Pf: Given $f \in H^s(\mathbb{T}^n)$, let $f_j(x) := \sum_{|k| \leq j} e^{2\pi i k \cdot x} \hat{f}_k$,

then $f_j \in C^\infty$; <u>check</u>: $f_j \xrightarrow{H^s} f$ as $j \to \infty$.

Given $f \in H^s(\mathbb{R}^n)$, $(1+|p|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n)$, $C_0^\infty \subseteq L^2$ dense

$\Rightarrow \exists$ seq. $h_j \in C_0^\infty(\mathbb{R}^n)$ s.t. $h_j \xrightarrow{L^2} (1+|p|^2)^{s/2} \hat{f}$,

let $g_j(p) := \dfrac{h_j(p)}{(1+|p|^2)^{s/2}}$, so $g_j \in C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \Rightarrow g_j = \hat{f}_j$ for

some $f_j \in \mathcal{S}(\mathbb{R}^n)$, $(1+|p|^2)^{s/2} \hat{f}_j \xrightarrow{L^2} (1+|p|^2)^{s/2} \hat{f} \Rightarrow f_j \xrightarrow{H^s} f$. $\boxed{\checkmark}$

<u>Rk</u>: Can also choose approx. id. ρ_j & prove $\rho_j * f \in H^s(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$

& $\rho_j * f \xrightarrow{H^s} f$. (PSET 8)

<u>cor</u>: The extension of $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ to a bdd lin. op.

$H^{s+|\alpha|}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is unique. $\boxed{\checkmark}$

(3) Hölder spaces

Recall: $s > \frac{n}{2} \Rightarrow H^s \hookrightarrow C^0$, $\quad H^{s+m} \hookrightarrow C^m \quad \forall\, m \in \mathbb{N}$

intuition: "fns in H^s have $s - \frac{n}{2}$ contin. derivs."

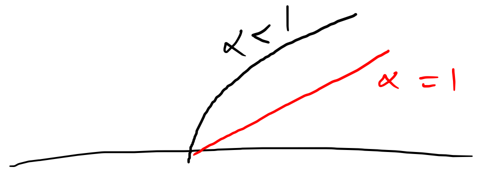
Q: Meaning of the non-integer part of $s - \frac{n}{2}$?

defn: A fn. f on $\Omega \subseteq \mathbb{R}^n$ or $\Omega \subseteq \mathbb{T}^n$ is Hölder continuous if $\exists\, \alpha \in (0,1]$ & const. $C > 0$ s.t. $\forall\, x \neq y$ in Ω,

$$|f(x) - f(y)| \leq C|x-y|^\alpha.$$

($\alpha :=$ "Hölder exponent")

$\alpha = 1 \iff$ Lipschitz



$\rightsquigarrow$ Hölder seminorm $\quad |f|_{C^{0,\alpha}} := \sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x-y|^\alpha}$

$$C^{0,\alpha}(\Omega) := \left\{ f \in C^0(\Omega) \;\middle|\; \|f\|_{C^{0,\alpha}} := \|f\|_{C^0} + |f|_{C^{0,\alpha}} < \infty \right\}$$

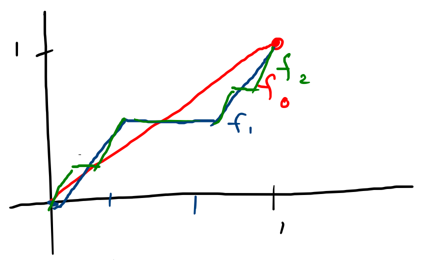
$m \in \mathbb{N}$, $\quad \|f\|_{C^{m,\alpha}} := \|f\|_{C^m} + \sum_{|\beta| = m} |\partial^\beta f|_{C^{0,\alpha}}$

$$C^{m,\alpha}(\Omega) := \left\{ f \in C^m(\Omega) \;\middle|\; \|f\|_{C^{m,\alpha}} < \infty \right\}.$$

thm (see notes for hints): These are all Banach spaces.

ex: $f(x) = |x|^\alpha$ is in $C^{0,\alpha}(\mathbb{R})$ but not Lipschitz if $\alpha < 1$.

ex: Cantor fn. $f = \lim_{n \to \infty} f_n$ not a.c. $\Rightarrow \notin C^{0,1}([0,1])$.



$\|f - f_n\|_{C^0} \leq \dfrac{C}{2^n}$ for some $C > 0$.

$|f_n|_{C^{0,1}} \leq C\left(\frac{3}{2}\right)^n$ for some $C > 0$.

Lemma (see notes for pf): If $f_k \xrightarrow{C^0} f$ s.t. for some consts $a > 1$, $b \geq 1$, $C > 0$ & $\beta \in (0,1]$ s.t. $\|f - f_k\|_{C^0} \leq \dfrac{C}{a^k}$ & $|f_k|_{C^{0,\beta}} \leq C b^k$, then $f \in C^{0,\alpha}$ for $\alpha := \dfrac{\beta}{1 + \log_a b}$.

cor: For Cantor fn. f, $a = 2$, $b = \frac{3}{2}$, $\beta = 1$, $\alpha = \dfrac{1}{1 + \log_2(3/2)}$

$$= \frac{1}{\log_2 3} = \log_3 2 \Rightarrow f \in C^{0,\alpha}([0,1]) \quad \forall\, \alpha \leq \log_3 2.$$

improvement of Sobolev emb. thm: Assume $n \in \mathbb{N}$, $s > 0$, $\alpha \in (0,1)$ satisfy $\alpha \leq s - \frac{n}{2}$. Then $\exists$ contin. inclusion $H^{s+m} \hookrightarrow C^{m,\alpha}$ $\forall\, m \in \mathbb{N}$ (on $\mathbb{R}^n$ or $\mathbb{T}^n$).

rk: Inclusions $C^{m,\alpha}(\mathbb{T}^n) \hookrightarrow C^m(\mathbb{T}^n)$ are cpct (Arzelà - Ascoli).

The Sobolev embedding $H^{s+m}(\mathbb{T}^n) \longrightarrow C^m(\mathbb{T}^n)$ factor for any $\alpha > 0$ suff. small

$$\begin{tikzcd} H^{s+m}(\mathbb{T}^n) \arrow[rr] \arrow[dr] & & C^m(\mathbb{T}^n) \\ & C^{m,\alpha}(\mathbb{T}^n) \arrow[ur, "\text{cpct}"'] & \end{tikzcd}$$

$\Rightarrow$ that embedding $H^{s+m} \hookrightarrow C^m$ is cpct.

pf sketch for $H^s(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$ for $0 < \alpha = s - \frac{n}{2} < 1$.

For $f \in H^s(\mathbb{R}^n)$ & $y \in \mathbb{R}^n$ suff. small, need to show $|f(x+y) - f(x)| \leq c|y|^\alpha$ $\forall\, x$, some const $c > 0$ which deps. linearly on $\|f\|_{H^s}$.

$$|f(x+y) - f(x)| = \left| \int_{\mathbb{R}^n} e^{2\pi i p \cdot (x+y)} \hat{f}(p)\, dp - \int_{\mathbb{R}^n} e^{2\pi i p \cdot x} \hat{f}(p)\, dp \right|$$

$$\leq \int_{\mathbb{R}^n} |e^{2\pi i p \cdot y} - 1|\, |\hat{f}(p)|\, dp = \int_{\mathbb{R}^n} \frac{|e^{2\pi i p \cdot y} - 1|}{(1+|p|^2)^{s/2}} (1+|p|^2)^{s/2} |\hat{f}(p)|\, dp$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \left(\int_{\mathbb{R}^n} \frac{|e^{2\pi i p \cdot y} - 1|^2}{(1+|p|^2)^s}\, dp \right)^{1/2} \cdot \|f\|_{H^s}$$

note: $|e^{2\pi i p \cdot y} - 1| \leq 2\pi |p \cdot y| \leq 2\pi |p||y|$ also $|e^{2\pi i p \cdot y} - 1| \leq 2$.

To estimate the integral on $|p| \leq 1/|y|$, use first estimate:

$$\leq C \int_{|p| \leq 1/|y|} \frac{|p|^2 |y|^2}{(1+|p|^2)^s}\, dp \overset{\text{(polar)}}{=} C'|y|^2 \int_0^{1/|y|} \frac{r^2}{(1+r^2)^s} r^{n-1}\, dr \leq C'|y|^2 \int_0^{1/|y|} \underbrace{r^{2-2s+n-1}}_{= r^{1-2s+n}}\, dr$$

$$= C''|y|^2 r^{2-2s+n} \Big|_0^{1/|y|} = \frac{C''|y|^2}{|y|^{2-2s+n}} = C''|y|^{2s-n} = C''|y|^{2\alpha}.$$

On $|p| \geq 1/|y|$, use other estimate ... same result.

$\Rightarrow$ whole thing $\leq C|y|^\alpha \cdot \|f\|_{H^s}$. $\square$

(4) application (elliptic regularity)

motivation: consider 2nd-order ODE $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$

for $F \in C^\infty$. Look for sols. $x \in C^2$.

$x \in C^2 \Rightarrow \dot{x} \in C^1$, $F \in C^1 \Rightarrow \ddot{x} \in C^1 \Rightarrow x \in C^3$. $\Rightarrow$ RHS $\in C^2$

$\Rightarrow \ddot{x} \in C^2 \Rightarrow x \in C^4$ ("bootstrapping argument") ... $\Rightarrow x \in C^\infty$.

Q: Which PDEs have this "regularizing" property.

ex 1: wave eqn for $u : \mathbb{R}^2 \to \mathbb{R}$, $\partial_t^2 u - \partial_t^2 x = 0$

(t, x)

this has sols. $u(t,x) = f(t \pm x)$ for any $f \in C^2$.

$\Rightarrow u \in C^2$, but maybe not better.

ex 2: Poisson's eqn: $\underbrace{\Delta u}_{\text{"Laplacian"}} := \sum_{j=1}^{n} \partial_j^2 u = g$ for $u : \mathbb{T}^n \to \mathbb{R}$,
$g : \mathbb{T}^n \to \mathbb{R}$ given

thm: If $u \in H^2(\mathbb{T}^n)$ satisfies $\Delta u = g \in H^m(\mathbb{T}^n)$ for some $m \in \mathbb{N}$,
then $u \in H^{m+2}(\mathbb{T}^n)$. (note: $\Delta : H^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is bdd lin. op.)

cor: If $u \in C^2(\mathbb{T}^n)$ satisfies $\Delta u = g \in C^\infty(\mathbb{T}^n)$, then $u \in C^\infty(\mathbb{T}^n)$.

pf: $u \in C^2 \Rightarrow u \in H^2$, $g \in C^\infty \Rightarrow g \in H^m \; \forall m \Rightarrow u \in H^{m+2} \; \forall m$
Sobolev emb thm $\Rightarrow u \in C^\infty$.

pf of thm: Use F.T. to convert $\Delta u = g$ into

$$\widehat{\sum_j \partial_j^2 u} = \hat{g} = \sum_j (2\pi i p_j)^2 \hat{u} = -4\pi^2 |p|^2 \hat{u}$$

$$\Rightarrow \hat{u}(p) = -\frac{\hat{g}(p)}{4\pi^2 |p|^2}. \quad \text{compute} \ldots$$

$\boxtimes$