X, Y Banach spaces, $V \subseteq X$ a closed subspace.

<u>defn</u>: The <u>direct sum</u> of X & Y is the Banach space $X \oplus Y := \{(x,y) \mid x \in X, y \in Y\}$

with the norm $\|(x,y)\| := \|x\| + \|y\|$.

<u>Easy</u> <u>EX</u>: $X \oplus Y$ is a Banach space, & $\|(x,y)\|_2 := \sqrt{\|x\|^2 + \|y\|^2}$

is an equivalent norm.

<u>defn</u>: The <u>quotient</u> of X by V is the Banach space $X/V = \left\{ [x] \,\middle|\, \begin{array}{l} x \in X \\ [x]=[y] \text{ iff} \\ x-y \in V \end{array} \right\}$

with norm $\|[x]\| := \inf_{v \in V} \|x+v\|$.

<u>prop</u>: X/V is a Banach space.

<u>pf sketch</u>: (1) $\|\cdot\|$ is a norm (not just a seminorm) since

$\|[x]\| = 0 \Rightarrow \exists$ seq $v_n \in V$ s.t. $\|x+v_n\| \to 0$, i.e. $x+v_n \to 0$

$\Rightarrow v_n \to -x \Rightarrow$ since V is closed, $-x \in V \Rightarrow x \in V \Rightarrow [x] = [0] \in X/V$.

(2) Completeness: Recall (ana. 3): in a normed vec. space, if every abs. conv. series converges, then the space is complete.

If $\sum_n \|[x_n]\| < \infty$, can choose $v_n \in V$ s.t. $\|x_n + v_n\| \leq 2\|[x_n]\| \ \forall n$,

$\Rightarrow \sum_n \|x_n + v_n\| < \infty \Rightarrow \sum_n (x_n + v_n)$ converges to some $y \in X$, then

$\left\| \sum_{n=1}^N [x_n] - [y] \right\| = \left\| \left[\sum_{n=1}^N (x_n + v_n) - y \right] \right\| \leq \left\| \sum_{n=1}^N (x_n+v_n) - y \right\| \to 0$ as $N \to \infty$

$\Rightarrow \sum_{n=1}^\infty [x_n] = [y]$. ☒

observe: quotient proj. $\pi : X \to X/V : x \mapsto [x]$ is a bdd lin. op.: $\|[x]\| \le \|x\|$.

defn: For subspaces $V, W \subseteq X$, write $X = V \oplus W$ if $V + W = X$ & $V \cap W = \{0\}$,
i.e. $\forall\ x \in X,\ \exists !\ v \in V,\ w \in W$ s.t. $x = v + w$.

defn: A closed subspace $V \subseteq X$ is complemented if $X = V \oplus W$ for some closed subspace $W \subseteq X$.

ex 1: For X a Hilbert space, then $X = V \oplus V^\perp$ & $V^\perp$ is always closed
$\Rightarrow$ all closed subspaces are complement.

ex 2: $\dim V < \infty \Rightarrow V$ is complemented (X any Banach sp.) — proved Tuesday via HB.
recall: construct complement W as $\ker \Pi$ for projection map
$\Pi \in \mathcal{L}(X)$ with $\operatorname{im} \Pi = V$.

ex 3: If $\operatorname{codim}(V) := \dim(X/V) < \infty$, then V is complemented.
pf: $W := \operatorname{Span}\{w_1, \dots, w_n\}$ for any basis $[w_1], \dots, [w_n]$ of X/V. $\boxtimes$

Q: Spse $V \subseteq X$ closed & $X = V \oplus W$, so $\pi|_W : W \to X/V$ is contin. & bijective. Is its inverse also contin?

Necessary cond: W is closed. pf: If $(\pi|_W)^{-1}$ contin. $X/V \to W$ & $w_n \in W$ s.t. $w_n \to w \in X$, then $w_n = (\pi|_W)^{-1} \circ \pi(w_n) \to (\pi|_W)^{-1} \circ \pi(w) \in W$
$\Rightarrow w \in W$.

thm: Closedness of W is also sufficient for $(\pi|_W)^{-1}$ to be contin. Follows from:

Inverse mapping thm (IMT): Every bdd lin. bijection btwn Banach spaces has a bdd inverse. (i.e. it is a "Banach space isomorphism") defn: If $X = V \oplus W$, the projection to V along W is the linear map

$\Pi : X \to X : v + w \mapsto v$ for $v \in V$, $w \in W$.

($\Leftrightarrow$ $\mathrm{im}\,\Pi = V$, $\ker \Pi = W$ & $\Pi^2 = \Pi$) ($\Leftrightarrow$ $\mathrm{Id} - \Pi$ is the proj. to W along V)

thm: Π is contin $\Leftrightarrow$ V & W are both closed.

cor: For a closed subspace $V \subseteq X$, $\exists$ a contin. lin. proj. to V

$\Leftrightarrow$ V is complemented.

pf of thm: $\Rightarrow$. Π contin $\Rightarrow$ $W = \ker \Pi$ is closed, $\mathrm{Id} - \Pi$ also contin.

$\Rightarrow$ $V = \ker(\mathrm{Id} - \Pi)$ also closed.

$\Leftarrow$: $V, W \subseteq X$ closed $\Rightarrow$ both are Banach spaces, consider their direct sum $V \oplus W$ with norm $\|(v,w)\| = \|v\| + \|w\|$ as a Banach space.

Then $\Phi : V \oplus W \to X : (v,w) \mapsto v + w$ is a contin. lin. bijection

$\overset{\text{IMT}}{\Longrightarrow}$ $\Phi^{-1} = (\Pi, \mathrm{Id} - \Pi)$ is bdd $\Rightarrow$ so are Π & $\mathrm{Id} - \Pi$. $\boxtimes$

Main goal: prove IMT.

defn: In a metric space M, a set $S \subseteq M$ is nowhere dense if its closure $\bar{S}$ has empty interior, i.e. no ball contained in $\bar{S}$.

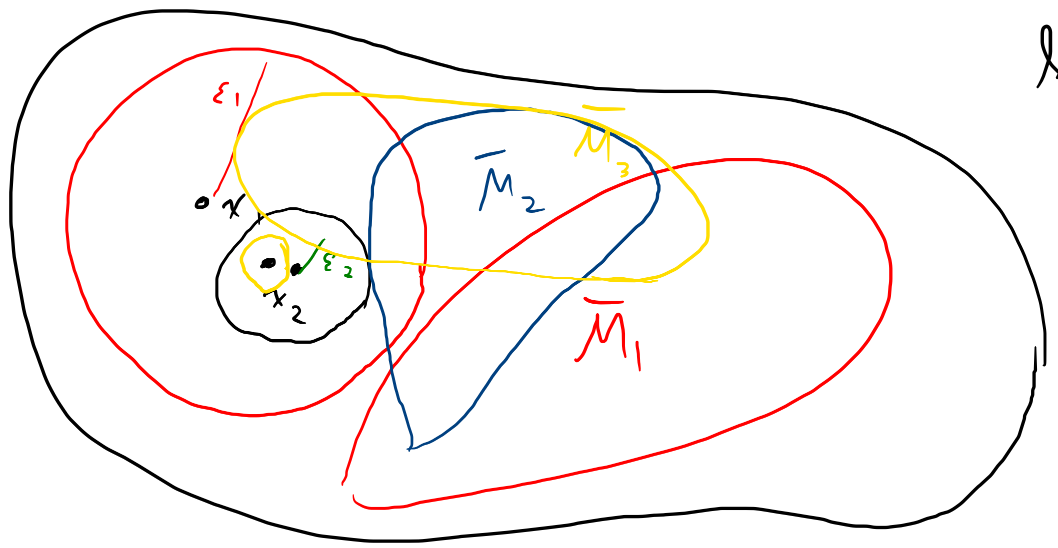
Baire category thm: No nonempty complete metric space is a $\underbrace{\text{countable union of nowhere dense sets}}_{\text{"meager"}}$.

(EX: Following formulation is equivalent:
In a complete metric space M, every $\underbrace{\text{countable intersection of open \& dense sets}}_{\text{"comeager"}}$ is also dense.)

pf of Baire: idea: if $M = \bigcup_{n \in \mathbb{N}} M_n$ & M_n nowhere dense $\forall n$, construct a Cauchy seq. that stays away from M_n s.t. $x_k \longrightarrow x \notin M_n \ \forall n$.

Pick $x_1 \in M \setminus \bar{M}_1$ & $\varepsilon_1 > 0$ s.t. $B_{\varepsilon_1}(x_1) \cap \bar{M}_1 = \emptyset$, & $\varepsilon_1 < 1$.

Pick $x_2 \in B_{\varepsilon_1}(x_1) \setminus \bar{M}_2$ (possible since M_2 nowhere dense), $\varepsilon_2 \in (0, \frac{1}{2})$ s.t. $\overline{B_{\varepsilon_2}(x_2)} \subseteq B_{\varepsilon_1}(x_1)$ but $B_{\varepsilon_2}(x_2) \cap \bar{M}_2 = \emptyset$.



Induction ... $\exists$ x_n & $\varepsilon_n \in (0, \frac{1}{2^n})$ s.t. $\overline{B_{\varepsilon_n}(x_n)} \subseteq B_{\varepsilon_{n-1}}(x_{n-1})$ but $B_{\varepsilon_n}(x_n) \cap \bar{M}_n$. Now x_n is Cauchy, $x_n \longrightarrow$ some $x \in M$ with $x \in \bigcap_{n \in \mathbb{N}} \overline{B_{\varepsilon_n}(x_n)}$

$\Rightarrow x \notin M_n \ \forall n$, contradiction. $\boxtimes$

open mapping thm (OMT): If X, Y are Banach spaces & $T \in \mathcal{L}(X,Y)$ is surjective, then T is an open map, i.e. $\forall\ \mathcal{U} \overset{\text{open}}{\subseteq} X$, $T(\mathcal{U})$ is open in Y.

cor (IMT): If T also inj., then T^{-1} is continuous.

pf: T^{-1} is contin iff $\forall\ \mathcal{U} \overset{\text{open}}{\subseteq} X$, the preimage $\underbrace{(T^{-1})^{-1}(\mathcal{U})}_{=T(\mathcal{U})}$ is open in Y. ☐

cor (see PSET 1 #1(c)): For Banach spaces X, Y, the set of invertible bdd lin. maps $X \to Y$ is open in $\mathcal{L}(X,Y)$. ☐

pf of OMT:

Remark 1: Suff to prove $T(B_r(0)) \supseteq B_R(0)$ for some $r, R > 0$.

Indeed, then $T(B_\varepsilon(x_0)) = Tx_0 + \frac{\varepsilon}{r} T(B_r(0)) \supseteq Tx_0 + B_{\frac{\varepsilon R}{r}}(0)$

$= B_{\frac{\varepsilon R}{r}}(Tx_0) \Rightarrow T(\text{open}) = \text{open}$.

claim 1: $\overline{T(B_r(0))}$ has nonempty interior for some $r > 0$.

pf: T surj. $\Rightarrow Y = \bigcup_{n=1}^\infty T(B_n(0)) \overset{\text{Baire}}{\Longrightarrow}$ not all $T(B_n(0))$ are nowhere dense, i.e. $\exists\ n \in \mathbb{N}$ s.t. $\overline{T(B_n(0))}$ contains a ball $B_\varepsilon(y) \subseteq Y$.

claim 2: $\overline{T(B_1(0))}$ contains a ball about 0.

pf: $B_\varepsilon(y) \subseteq \overline{T(B_n(0))} \Rightarrow \forall\ y' \in B_\varepsilon(0)$, $\exists$ seqs. $x_k, x_k' \in B_n(0)$ s.t. $Tx_n \to y$ & $Tx_n' \to y + y' \in B_\varepsilon(y)$. then $T(x_k' - x_k) \to y'$, where $\|x_k' - x_k\| < 2n \Rightarrow B_\varepsilon(0) \subseteq \overline{T(B_{2n}(0))}$.

$\Rightarrow B_{\varepsilon/2n}(0) \subseteq \overline{T(B_1(0))}$. Relabel $\frac{\varepsilon}{2n}$ as ε; WLOG $B_\varepsilon(0) \subseteq \overline{T(B_1(0))}$.

claim 3: $\overline{T(B_1(0))} \subseteq T(B_2(0))$. ($\Rightarrow$ done!)

Given $y \in \overline{T(B_1(0))}$, $\exists\ x_1 \in B_1(0)$ s.t. $y - Tx_1 \in B_{\varepsilon/2}(0) \subseteq \overline{T(B_{1/2}(0))}$.

Pick $x_2 \in B_{1/2}(0)$ s.t. $y - Tx_1 - Tx_2 \in B_{\varepsilon/4}(0) \subseteq \overline{T(B_{1/4}(0))}$.

Induction $\Rightarrow \exists$ seq. $x_n \in B_{\frac{1}{2^{n-1}}}(0)$ s.t. $y - Tx_1 - Tx_2 - \dots - Tx_n \in B_{\varepsilon/2^n}(0)$.

$\|x_n\| < \frac{1}{2^{n-1}} \Rightarrow \sum_{n=1}^\infty \|x_n\| < \infty \Rightarrow \exists\ x := \sum_{n=1}^\infty x_n \in B_2(0)$, $y = Tx$.

☐