Recall IMT: $X, Y$ Banach, $A \in \mathcal{L}(X,Y)$ bijective $\Rightarrow A^{-1} \in \mathcal{L}(Y,X)$.

defn: The graph of a linear map $A : X \to Y$ is the subspace

$$\Gamma_A := \{ (x, Ax) \in X \oplus Y \mid x \in X \} \subseteq X \oplus Y$$

i.e. $\operatorname{im} \Phi$ for $\Phi : X \to X \oplus Y : x \mapsto (x, Ax)$.

closed graph thm: a linear map $A : X \to Y$ btwn Banach spaces $X, Y$ is bdd $\Leftrightarrow$ $\Gamma_A$ is a closed subspace of $X \oplus Y$.

pf: $\Rightarrow$: $A$ contin $\Rightarrow$ if $\Gamma_A \ni (x_n, Ax_n) \to (x,y) \in X \oplus Y$

$\Rightarrow x_n \to x$, & since $A$ contin., $Ax_n \to Ax = y \Rightarrow (x,y) \in \Gamma_A$.

$\Leftarrow$: $\Gamma_A$ closed $\Rightarrow$ $\Gamma_A$ is a Banach space w/ norm $\|(x, Ax)\| = \|x\| + \|Ax\|$,

proj. map $\pi_1 : \Gamma_A \to X : (x, Ax) \mapsto x$ is a linear bijection &

$\|\pi_1(x, Ax)\| = \|x\| \leq \|x\| + \|Ax\| = \|(x, Ax)\| \Rightarrow \pi_1$ is bdd

$\overset{IMT}{\Rightarrow}$ $\underset{\Phi}{\underbrace{\pi_1^{-1}}} : X \to \Gamma_A$ is bdd. The other proj. $\pi_2 : \Gamma_A \to Y : (x, Ax) \mapsto Ax$

is also bdd, so $A = \pi_2 \circ \pi_1^{-1}$ is bdd. $\square$

cor ("Hellinger-Toeplitz thm"): $\mathcal{H}$ a Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ linear s.t. $\langle x, Ay \rangle = \langle Ax, y \rangle$ $\forall x, y \in \mathcal{H}$ $\Rightarrow A$ is bdd.

pf: Spse $\Gamma_A \ni (x_n, Ax_n) \to (x,y) \in \mathcal{H} \oplus \mathcal{H}$. Then $z \in \mathcal{H} \Rightarrow$

$$\begin{array}{ccccc} \langle x_n, Az \rangle & \longrightarrow & \langle x, Az \rangle & = & \langle Ax, z \rangle \\ \| & & & // & \\ \langle Ax_n, z \rangle & \longrightarrow & \langle y, z \rangle & & \end{array}$$

$\forall z \in \mathcal{H} \Rightarrow Ax = y$

$\Rightarrow (x,y) \in \Gamma_A \Rightarrow \Gamma_A$ is closed. $\square$

uniform bddness

Q1: If $x_n \longrightarrow x$ weakly, can $\|x_n\|$ be unbdd?

A: No; intuition: if $\|x_n\|$ unbdd, then $\Lambda(x_n)$ is unbdd for "most" bdd lin. fnals $\Lambda$.

Q2: $f \in C^0(S^1) \subseteq L^2(S^1) \Rightarrow \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \hat{f}_k$ conv. in $L^2$ to $f$.

Does it also conv. pointwise?

A: Given $x \in S^1$, for "most" $f \in C^0(S^1)$, Fourier series diverges at $x$.

What does "most" mean?

Recall: $M$ a metric space, $S \subseteq M$ is

- nowhere dense if $\nexists$ ball $\subseteq \bar{S}$.
- meager / "of the first category" if $S$ = countable union of nowhere dense sets.
- comeager / "residual" if $M \setminus S$ is meager.

Residual $\Leftrightarrow$ $M \setminus S = \bigcup_{n \in \mathbb{N}} X_n \subseteq \bigcup_{n \in \mathbb{N}} \bar{X}_n$ for some $X_n \subseteq M$ s.t. $\nexists$ ball $\subseteq \bar{X}_n$

$\Leftrightarrow$ $S \supseteq \bigcap_{n \in \mathbb{N}} (M \setminus \bar{X}_n)$ where $M \setminus \bar{X}_n$ are open & dense

$\Leftrightarrow$ $S$ contains a ctbl. int. of open + dense sets.

Baire category thm: If $M \neq \emptyset$ & is complete, $M$ is not meager.

cor: $M$ complete $\Rightarrow$ residual subsets are dense.

pf: ~~Spse~~ $S \supseteq \bigcap_{n \in \mathbb{N}} U_n$ for $U_n \subseteq M$ open + dense.

If $S$ not dense $\Rightarrow \exists$ a closed ball $B \subseteq M \setminus S$;

$B$ is also a complete metric space $\Rightarrow B \subseteq M \setminus S \subseteq \bigcup_{n \in \mathbb{N}} (M \setminus U_n)$

$\Rightarrow B = \bigcup_{n \in \mathbb{N}} \underbrace{(B \cap (M \setminus U_n))}_{\text{closed, nowhere dense.}}$ contra! $\boxtimes$

rk: If $M_1, M_2, \ldots \subseteq M$ are residual, then so is $\bigcap_{n \in \mathbb{N}} M_n$.

terminology: A property holds for generic $x \in M$ if it holds $\forall x$ in some residual subset.

<u>thm</u> ("principle of condensation of singularities")

Spse $X$ a Banach space, $I$ a set, $\{T_\alpha : X \to Y_\alpha\}_{\alpha \in I}$ is a collection of bdd lin. ops. to normed vec. spaces $Y_\alpha$.

If $\{\|T_\alpha\| \mid \alpha \in I\}$ is unbdd, then $\exists$ a residual set $R \subseteq X$ s.t. $\{\|T_\alpha x\| \mid \alpha \in I\}$ is unbdd $\forall\ x \in R$. (i.e. $\|T_\alpha x\|$ unbdd for "generic" $x$).

Pf: For $n \in \mathbb{N}$, $R_n := \{x \in X \mid \sup_{\alpha \in I} \|T_\alpha x\| > n\}$.

Suff. to show: $R_n$ is open + dense. (Then take $R := \bigcap_{n \in \mathbb{N}} R_n$.)

<u>claim</u> <u>$R_n$ is open</u>: $x_0 \in R_n \Rightarrow \exists\ \alpha \in I$ s.t. $\|T_\alpha x_0\| > n$

$\Rightarrow \|T_\alpha x\| > n$ still $\forall\ x$ suff. close to $x_0$.

<u>claim</u> <u>$R_n$ is dense</u>: If not, $\exists\ x_0 \in X$ & $\varepsilon > 0$ s.t. $B_\varepsilon(x_0) \cap R_n = \emptyset$

$\Rightarrow \forall\ x \in B_\varepsilon(x_0)$, $\sup_{\alpha \in I} \|T_\alpha x\| \leq n$. Then for $x \in X$ with $\|x\| = \frac{\varepsilon}{2}$

$\Rightarrow \|T_\alpha x\| = \|T_\alpha x_0 - T_\alpha(x_0 - x)\| \leq \|T_\alpha x_0\| + \|T_\alpha(x_0 - x)\| \leq n + n = 2n$

$\Rightarrow \|T_\alpha\| \leq \dfrac{2n}{\varepsilon/2}$ $\forall\ \alpha$, contradiction!

☒

**cor** ("princ. of unif. bddness" – PUB): If $\{\|T_\alpha x\| \mid \alpha\in I\}$ bdd $\forall x$,
then $\{\|T_\alpha\| \mid \alpha\in I\}$ is bdd. ☐

**cor**: Weakly conv. seqs. in Banach spaces are bdd.

**pf**: Given $x_n \rightharpoonup x$, $\Lambda(x_n)\to\Lambda(x) \Rightarrow \{|\Lambda(x_n)| \mid n\in\mathbb{N}\}$ bdd $\forall\Lambda\in X^*$

$\Rightarrow$ (using $X \xhookrightarrow{J} X^{**}$) $\{|Jx_n(\Lambda)| \mid n\in\mathbb{N}\}$ bdd $\forall\Lambda\in X^*$

$\overset{PUB}{\Rightarrow}$ the family of lin. fnals $Jx_n : X^*\to\mathbb{K}$ is bdd, i.e.
$\|Jx_n\| = \|x_n\|$ satisfies a bound indep. of $n$.

**thm**: Given any countable set $I\subseteq S^1$, $\exists$ a residual set $R_I\subseteq C^0(S^1)$
s.t. $\forall x\in I$ & $f\in R_I$, $\sum_{k\in\mathbb{N}} e^{2\pi i k x}\hat{f}_k$ diverges.

**pf**: Suff. to prove this for the case $I=\{x\}$, then ctbl $\Rightarrow R_I := \bigcap_{x\in I} R_{\{x\}}$
is also residual. Assume fns. take vals. in a
fin-dim. vec. sp. $V$.

Consider $\{T_N\in\mathcal{L}(C^0(S^1),V) \mid N\in\mathbb{N}\}$, $T_N f := \sum_{k=-N}^{N} e^{2\pi i k x}\hat{f}_k$

$= \sum_{k\in\mathbb{Z}} e^{2\pi i k x}\chi_{[-N,N]}(k)\hat{f}_k = (\rho_N * f)(x)$ where $\rho_N\in C^\infty(S^1)$ is
def'd s.t. $(\hat{\rho}_N)_k = \chi_{[-N,N]}(k)$, i.e. $\rho_N(x) = \sum_{k=-N}^{N} e^{2\pi i k x}$

$= 1 + \sum_{k=1}^{N}(e^{2\pi i x})^k + \sum_{k=1}^{N}(e^{-2\pi i x})^k = 1 + 2\,\mathrm{Re}\sum_{k=1}^{N}(e^{2\pi i x})^k$

$= 1 + 2\,\mathrm{Re}\,\dfrac{e^{2\pi i x}(1-e^{2\pi i N x})}{1-e^{2\pi i x}} = \dots$ algebra ...

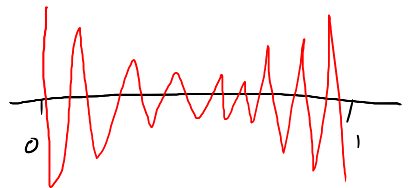
$= \dfrac{\sqrt{2}\,\sin[2\pi(N+\frac12)x]}{\sqrt{1-\cos(2\pi x)}}$ "Nth Dirichlet kernel"

$\|T_N\| = ?$ $\quad |T_N f| = |(\rho_N * f)(x)| = \left|\int_{S^1}\rho_N(x-y)f(y)\,dy\right| \le \|\rho_N\|_{L^1}\cdot\|f\|_{C^0}$

Can make this close to $\|\rho_N\|_{L^1}$ for any $f\in C^0$ that is $L^1$-close to $\dfrac{\rho_N(x-y)}{|\rho_N(x-y)|} = \pm 1$

$\Rightarrow \|T_N\| = \|\rho_N\|_{L^1}$.

If that is unbdd as $N\to\infty$, PCS $\Rightarrow |T_N(f)|$ unbdd as $N\to\infty$ for
generic $f\in C^0(S^1) \Rightarrow$ Fourier series diverges at $x$. $\int_{S^1}\rho_N(x)\,dx = (\hat{\rho}_N)_0 = 1$ $\forall N$.



what about $\int_{S^1}|\rho_N(x)|\,dx$ ?

$$e_N(x) = \frac{\sqrt{2}\,\sin\left[2\pi\left(N+\frac{1}{2}\right)x\right]}{\sqrt{1-\cos 2\pi x}}.$$

Fix $r > 0$ & $a > 0$ s.t.

$$1-\cos(2\pi x) \leq a^2 x^2 \text{ for } x \in [0, r]$$

(Taylor)

Numerator changes sign every $\frac{1}{2N+1}$.

Spse $r_0 = \frac{M}{2N+1} \leq r$ for some $M \in \mathbb{N}$ as large as possible.

Write on $x \in [0, r_0]$, $|e_N(x)| \geq c\frac{\left|\sin\left[2\pi\left(N+\frac{1}{2}\right)x\right]\right|}{x}$ for some const. $c > 0$.

$$\Rightarrow \|e_N\|_{L^1} \geq c\sum_{j=1}^{M}\left|\int_{\frac{j-1}{2N+1}}^{\frac{j}{2N+1}} \frac{\sin\left[2\pi\left(N+\frac{1}{2}\right)x\right]}{x}\,dx\right| = c\sum_{j=1}^{M}\int_{\pi(j-1)}^{\pi j}\frac{\sin u}{u}\,du$$

$$\geq c\sum_{j=1}^{M}\frac{1}{\pi j}\left|\int_{\pi(j-1)}^{\pi(j)}\sin u\,du\right| = \frac{2c}{\pi}\sum_{j=1}^{M}\frac{1}{j} \longrightarrow \infty \text{ as } M \longrightarrow \infty.$$

$$\Rightarrow \lim_{N\to\infty}\|e_N\|_{L^1} = \infty.$$

$\square$