Unbounded operators

<u>motivational ex</u>: $-\Delta := -\sum_{j=1}^{n} \partial_j^2$ is a "symmetric" & "positive" op. on "$L^2(\mathbb{R}^n)$"

$\forall\ \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \varphi, -\Delta\psi\rangle_{L^2} = -\sum_j \int_{\mathbb{R}^n} \langle \varphi, \partial_j^2 \psi\rangle\, dm = \langle \nabla\varphi, \nabla\psi\rangle_{L^2}$

$= \langle -\Delta\varphi, \psi\rangle_{L^2}$, in particular, $\langle \varphi, -\Delta\varphi\rangle_{L^2} = \|\nabla\varphi\|_{L^2}^2 \geq 0 \quad \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)$.

<u>Trouble</u>: Δ def'd on a <u>dense</u> <u>subspace</u> of L^2, but $\nexists$ contin. extension to L^2.

<u>defn</u>: X, Y Banach spaces, an "<u>unbounded</u> <u>linear</u> op. <u>from</u> X <u>to</u> Y"
consists of a subspace $\mathcal{D} := \mathcal{D}(T) \subseteq X$ & a linear map $T : \mathcal{D} \to Y$.

<u>notation</u>: $X \supseteq \mathcal{D} \xrightarrow{T} Y$.

It is <u>closed</u> if its <u>graph</u> $\Gamma_T := \{(x, Tx) \mid x \in \mathcal{D}\}$ is a closed subspace of $X \times Y$; <u>closable</u> if $\exists$ a closed extension $X \supseteq \mathcal{D}' \xrightarrow{T'} Y$, meaning $\mathcal{D} \subseteq \mathcal{D}'$ & $T'|_{\mathcal{D}} = T$. <u>Closure</u> $\overline{T}$:= smallest closed extension of T.

<u>remarks</u>: (1) T is closable $\iff$ $\overline{\Gamma}_T \subseteq X \times Y$ is the graph of an operator $\iff$
$\nexists\ (x, y), (x', y') \in \overline{\Gamma}_T$ s.t. $x = x'$ but $y \neq y'$ $\iff$
$\forall\ x \in \overline{\mathcal{D}}$, $\exists$ <u>at</u> <u>most</u> <u>one</u> $y \in Y$ arising as $\lim_{n\to\infty} Tx_n$ for seqs. $\mathcal{D} \ni x_n \to x$.

(2) If $\mathcal{D} = X$ & T closed, closed graph thm $\Rightarrow$ T is bdd.

(3) If $\mathcal{D} \subsetneq X$ dense & T closed, $\Rightarrow$ $T : \mathcal{D} \to Y$ is discontinuous!
(Else T has bdd extension to $X \to Y$, so $\overline{\Gamma}_T$ = graph of extension
$\Rightarrow$ T not closed.)

(4) Can defn on $\mathcal{D}$ the <u>graph</u> <u>norm</u> $\|x\|_T := \|x\|_X + \|Tx\|_Y$, then
$T \in \mathcal{L}((\mathcal{D}, \|\cdot\|_T), Y)$.

<u>Easy EX</u> (PSET 12): T is closed $\iff$ $(\mathcal{D}, \|\cdot\|_T)$ is a Banach space.

(5) If $\mathcal{D} \subsetneq X$ <u>not</u> dense & $X = \overline{\mathcal{D}} \oplus X'$ for $X' \overset{\text{closed}}{\subseteq} X$ (e.g. always possible if X = Hilbert sp.)
can extend T to $X \supseteq \mathcal{D} \oplus X' \xrightarrow{T'} Y$ s.t. $T'|_{X'} = 0$, then $\mathcal{D} \oplus X' \subseteq X$
is dense & <u>EX</u>: T closed $\iff$ T' closed.

We will usually assume $\mathcal{D}(T) \subseteq X$ is dense.

ex: Consider $T_0, T_1 := -\Delta$ on dense domains $D_0 := \mathcal{S}(\mathbb{R}^n)$, $D_1 := H^2(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$.

claim: $\overline{\Gamma_{T_0}} \supseteq \Gamma_{T_1}$ ($\Rightarrow T_0$ not closed). Pf: $(f, -\Delta f) \in \Gamma_{T_1}$ means

$f \in H^2$, $\mathcal{S} \subseteq H^2$ dense $\Rightarrow \exists$ seq. $\mathcal{S} \ni f_j \xrightarrow{H^2} f \Rightarrow$ since

$-\Delta : H^2 \to L^2$ is bdd, $-\Delta f_j \xrightarrow{L^2} -\Delta f$, also $f_j \xrightarrow{L^2} f \Rightarrow$

$\Gamma_{T_0} \ni (f_j, -\Delta f_j) \longrightarrow (f, -\Delta f)$.

claim: T_1 is closed ($\Rightarrow \overline{T_0} = T_1$). Spse $\Gamma_{T_1} \ni (f_j, -\Delta f_j) \longrightarrow (f, g)$

i.e. $f_j \xrightarrow{L^2} f$, $-\Delta f_j \xrightarrow{L^2} g$, $\Rightarrow \hat{f}_j \xrightarrow{L^2} \hat{f}$,

$\widehat{-\Delta f_j} = 4\pi^2 |p|^2 \hat{f}_j \xrightarrow{L^2} \hat{g} \Rightarrow (1+|p|^2)\hat{f}_j = \hat{f}_j - \frac{1}{4\pi^2}\widehat{\Delta f_j}$

$\xrightarrow{L^2} \hat{f} + \frac{1}{4\pi^2}\hat{g} \Rightarrow (1+|p|^2)\hat{f}_j$ is L^2-Cauchy $\Rightarrow f_j$ is H^2-Cauchy

$\Rightarrow f \in H^2$, $f_j \xrightarrow{H^2} f$, since $-\Delta : H^2 \to L^2$ is bdd,

$\Rightarrow -\Delta f_j \xrightarrow{L^2} -\Delta f$, $\Rightarrow g = -\Delta f \Rightarrow (f, g) \in \Gamma_{T_1}$. $\square$

spectrum: X a cpx Banach space. "resolvent set"

defn: For $X \supseteq \mathcal{D} \xrightarrow{T} X$ closed, $\rho(T) := \{\lambda \in \mathbb{C} \mid \mathcal{D} \xrightarrow{\lambda - T} X \text{ is bijective}\}$

$\rightsquigarrow$ resolvent $R_\lambda(T) : X \longrightarrow X : x \longmapsto (\lambda - T)^{-1} x$, spectrum $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

important EX (PSET 12): By closed graph thm, T closed $\Rightarrow$ $R_\lambda(T) : X \to X$ is bdd.

convention: If T is closable, $\sigma(T) := \sigma(\overline{T})$.

$\{\lambda \in \sigma(T) \mid \ker(\lambda - T) \neq \{0\} \subseteq \mathcal{D}\} =:$ point spectrum (eigenvalues)

$\{$ " $\mid \operatorname{im}(\lambda - T) \subseteq X$ not dense but λ not an e-val.$\} =:$ residual spectrum.

EX (PSET 12): Rest of $\sigma(T)$ consists of approximate e-val: $\exists$ $x_n \in \mathcal{D}$ s.t. $\|x_n\|_X = 1$ & $(\lambda - T) x_n \longrightarrow 0$.

thm: $\rho(T) \subseteq \mathbb{C}$ is open & $\rho(T) \longrightarrow \mathcal{L}(X) : \lambda \longmapsto R_\lambda(T)$ is analytic.

Pf: $\lambda_0 \in \rho(T)$ & $\mu \in \mathbb{C}$ w/ $|\mu|$ small. $\lambda := \lambda_0 + \mu$,
$\lambda - T = (\lambda_0 - T) + \mu$ is a bdd small pert. of $\lambda_0 - T$.

main idea: If $A_0 : \mathcal{D} \longrightarrow X$ has bdd inverse & $B \in \mathcal{L}(X)$ is small,

$\|A_0^{-1} B\| < 1$ & $\|B A_0^{-1}\| < 1$ $\Rightarrow$ $\exists$ $(1 + A_0^{-1} B)^{-1} = \sum_{k=0}^{\infty} (-1)^k (A_0^{-1} B)^k$

$\Rightarrow$ $(1 + A_0^{-1} B) A_0^{-1} = A_0^{-1} - A_0^{-1} B A_0^{-1} + A_0^{-1} B A_0^{-1} B A_0^{-1} - \dots = A_0^{-1} (1 + B A_0^{-1})^{-1}$

is the inverse of $A_0 + B : \mathcal{D} \longrightarrow X$. $\Rightarrow$ $R_\lambda(T) = \sum_{k=0}^{\infty} (-1)^k \mu^k R_{\lambda_0}(T)^{k+1}$.

$\Rightarrow$ $\sigma(T) \subseteq \mathbb{C}$ is closed. $\square$

Contrast to bdd case: $\lambda - T = \lambda\left(1 - \frac{1}{\lambda} T\right) \not\Rightarrow$ invtble $\forall$ $|\lambda|$ suff. large,

nor is $\|R_\lambda(T)\|$ small as $|\lambda| \to \infty$.

ex: $T_0, T_1 := i \frac{d}{dt}$ on domains in $L^2([0,1])$:

$\mathcal{D}_0 := \{f : [0,1] \to \mathbb{C} \mid f \text{ abs. contin. \& } f' \in L^2[0,1]\}$

then $\forall$ $\lambda \in \mathbb{C}$, $e^{-i\lambda t} \in \mathcal{D}$ is an e-vec. of T_0 w/ e-val. λ $\Rightarrow$ $\sigma(T_0) = \mathbb{C}$.

$\mathcal{D}_1 := \{f \in \mathcal{D}_0 \mid f(0) = 0\}$. also contains $C_0^\infty((0,1))$ $\Rightarrow$ dense in $L^2([0,1])$.

Then $\forall$ $\lambda \in \mathbb{C}$, $(\lambda - T_1) f = g \iff \lambda f - i f' = g \iff f' = i(-\lambda f + g)$

Can adapt Picard-Lindelöf to prove $\forall$ $g \in L^2$, $\lambda \in \mathbb{C}$, $\exists!$ sol. f abs. contin. & $f(0) = 0$, i.e. $f \in \mathcal{D}_1$ $\Rightarrow$ $\sigma(T_1) = \emptyset$.

<u>adjoints</u>: $\mathcal{H}$ = a cpx Hilbert sp.

<u>defn</u>: If $\mathcal{H} \supseteq \mathcal{D}(T) \xrightarrow{T} \mathcal{H}$ is densely def'd ($\mathcal{D}(T) \subseteq \mathcal{H}$ is dense),

its <u>adjoint</u> $\mathcal{H} \supseteq \mathcal{D}(T^*) \xrightarrow{T^*} \mathcal{H}$ is the unique op. s.t.

(i) $\langle T^* x, y\rangle = \langle x, Ty\rangle \quad \forall \ x \in \mathcal{D}(T^*), \ y \in \mathcal{D}(T)$,

(ii) T^* cannot be extended to any larger domain s.t. (i) holds

$\Rightarrow \mathcal{D}(T^*) := \{x \in \mathcal{H} \mid \exists \ z \in \mathcal{H} \ \text{ s.t. } \langle z, y\rangle = \langle x, Ty\rangle \ \ \forall \ y \in \mathcal{D}(T)\}$

<u>observe</u>: $x \in \mathcal{D}(T^*) \Rightarrow \Lambda(y) := \langle x, Ty\rangle$ satisfies $|\Lambda(y)| \le \|z\| \cdot \|y\|$

$\Rightarrow$ since $\mathcal{D}(T) \subseteq \mathcal{H}$ is dense, $\Lambda : \mathcal{D}(T) \to \mathbb{C}$ has ! extension to $\Lambda \in \mathcal{H}^*$

$\Rightarrow z =: T^* x$ is uniquely det'd by $\Lambda = \langle z, \cdot\rangle$. (i.e. $\mathcal{D}(T)$ not dense, T^* not unique)

<u>defn</u>: T is <u>symmetric</u> if $\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall \ x, y \in \mathcal{D}(T)$.

$\Leftrightarrow$ T^* is an extension of T (but maybe $\mathcal{D}(T) \subsetneq \mathcal{D}(T^*)$)

T is self-adjoint if $T = T^*$ (meaning also $\mathcal{D}(T) = \mathcal{D}(T^*)$.)

<u>ex 1</u>: $T_2 := i\dfrac{d}{dt}$ on $\mathcal{D}_2 := \{f \in \mathcal{D}_0 \mid f(0) = f(1) = 0\}$.

Then $f, g \in \mathcal{D}_2$, $\Rightarrow \langle f, T_2 g\rangle_{L^2} = \langle f, i g'\rangle_{L^2} = -\langle i f, g'\rangle_{L^2}$

$$= \langle i f', g\rangle_{L^2} - f(t) g(t)\Big|_{t=0}^{t=1} = \langle T_2 f, g\rangle_{L^2} \Rightarrow T_2 \text{ is symmetric.}$$

This also works if $g \in \mathcal{D}_2$ but $f \in \mathcal{D}_0 \supsetneq \mathcal{D}_2 \Rightarrow$ domain of T_2^* contains $\mathcal{D}_0 \Rightarrow T_2$ <u>not</u> self-adjoint

<u>EX</u>: all of $\mathbb{C}$ is residual spectrum of T_2^*.

ex 2 : $L^2(\mathbb{R}^n) \supseteq H^2(\mathbb{R}^n) \xrightarrow{-\Delta} L^2(\mathbb{R}^n)$ is self-adjoint.

Pf: $-\Delta$ is symmetric on $\mathcal{S}(\mathbb{R}^n)$, which is dense in H^2 $\Rightarrow$ also symmetric on H^2.

To show: $\mathcal{D}(-\Delta^*) \subseteq \mathcal{D}(\Delta) = H^2$, i.e. $\forall f \in L^2$ s.t. $\exists g \in L^2$ satisfying $\langle g, h\rangle_{L^2} = \langle f, -\Delta h\rangle_{L^2}$ $\forall h \in H^2$, it follows that $f \in H^2$.

$$\int_{\mathbb{R}^n} \langle g, h\rangle \, dm = \int_{\mathbb{R}^n} \langle f, -\Delta h\rangle \, dm \quad \forall h \in C_0^\infty(\mathbb{R}^n) \Rightarrow -\Delta f = g \text{ weakly},$$

$$\Rightarrow -\Delta f = g \in \mathcal{S}'(\mathbb{R}^n) \xrightarrow{(F.T.)} 4\pi^2 |p|^2 \hat{f} = \hat{g} \in \mathcal{S}'(\mathbb{R}^n)$$

$\Rightarrow 4\pi^2 |p|^2 \hat{f} = \hat{g}$ a.e. Then $\|f\|_{H^2}^2 = \int_{\mathbb{R}^n} (1 + |p|^2) |\hat{f}(p)|^2 \, dp$

$$= \|\hat{f}\|_{L^2}^2 + \frac{1}{4\pi^2} \|\hat{g}\|_{L^2}^2 < \infty \Rightarrow f \in H^2.$$

$\square$