Recall: $\mathcal{H} \overset{\text{dense}}{\supseteq} \mathcal{D} = \mathcal{D}(A) \xrightarrow{A} \mathcal{H}$ is symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle \ \forall\ x, y \in \mathcal{D}$.

A is self-adjoint if symmetric & $\mathcal{D}(A^*) = \mathcal{D}(A)$, i.e.

$\forall\ x, y \in \mathcal{H}, \quad \langle y, z\rangle = \langle x, Az\rangle \ \forall\ z \in \mathcal{D} \implies x \in \mathcal{D} \quad (\implies Ax = y)$.

We saw ex 2: $L^2(\mathbb{R}^n) \supseteq H^2(\mathbb{R}^n) \xrightarrow{-\Delta} L^2(\mathbb{R}^n)$ is self-adj. ("regularity of weak sols.")

ex 3 ("unbdd diagonal op."): (X, μ) measure space, $F: X \to \mathbb{R}$ measurable & finite a.e., $\mathcal{D} := \{ u \in L^2(X,\mu) \mid Fu \in L^2(X,\mu)\}$,

(EX: $\mathcal{D}$ is dense in $\underbrace{L^2(X,\mu)}_{\text{cpx-val'd fns.}}$). Then $L^2(X,\mu) \supseteq \mathcal{D} \xrightarrow{T_F} L^2(X,\mu) : u \longmapsto Fu$ is a self-adj. op.

pf: $F(X) \subseteq \mathbb{R} \implies \langle u, Fv\rangle_{L^2} = \langle Fu, v\rangle_{L^2} \implies$ symmetric.

To show: $\forall\ u, v \in L^2$, $\langle v, w\rangle_{L^2} = \langle u, Fw\rangle_{L^2} \ \forall\ w \in \mathcal{D} \implies u \in \mathcal{D}$.

For $N \in \mathbb{N}$, let $\chi_N := \chi_{F^{-1}([-N,N])} : X \to [0,\infty)$, then $\chi_N F$ is bdd

$\implies \forall\ w \in \mathcal{D}$, $\langle \chi_N F u, w\rangle_{L^2} = \langle u, \chi_N F w\rangle_{L^2} = \langle u, F \cdot \chi_N w\rangle_{L^2} = \langle v, \chi_N w\rangle_{L^2}$

$= \langle \chi_N v, w\rangle_{L^2}$; since $\mathcal{D}$ is dense, $\implies \chi_N F u = \chi_N v \ \forall\ N$,

monotone conv. thm. $\implies$ as $N \to \infty$, $\|\chi_N v\|_{L^2} \to \|v\|_{L^2}$,

$\|\chi_N F u\|_{L^2} \to \|Fu\|_{L^2} \implies \|Fu\|_{L^2} = \|v\|_{L^2} < \infty \implies u \in \mathcal{D}$. $\boxtimes$

spectral thm for unbdd self-adjoint operators

$\mathcal{H} \overset{\text{dense}}{\supseteq} \mathcal{D} \xrightarrow{A} \mathcal{H}$ self-adjoint $\iff$ $\exists$ a measure space (X, μ) (can assume $\mu(X) < \infty$ if $\mathcal{H}$ is separable), measurable fn. $F : X \to \mathbb{R}$, finite a.e., a unitary iso. $U : \mathcal{H} \to L^2(X, \mu)$ s.t.

$U(\mathcal{D}) = \{ u \in L^2(X) \mid Fu \in L^2(X) \}$ &

$U A U^{-1} = T_F : U(\mathcal{D}) \to L^2(X) : u \mapsto Fu$.

rk: Ex. 3 shows this cannot be true for symmetric but non-self-adj. ops.

Lemma 1: $\mathcal{H} \overset{\text{dense}}{\supseteq} \mathcal{D} \xrightarrow{A} \mathcal{H}$ $\Rightarrow$ A^* is closed.

cor: densely def'd + symmetric $\Rightarrow$ closable (A^* is a closed ext. of A)

self-adjoint $\Rightarrow$ closed.

Pf: Spse $x_n \in \mathcal{D}(A^*)$, $x_n \to x \in \mathcal{H}$ & $A^* x_n \to y \in \mathcal{H}$. Then

$\langle A^* x_n, z \rangle = \langle x_n, Az \rangle \quad \forall z \in \mathcal{D}(A) \Rightarrow \langle y, z \rangle = \langle x, Az \rangle \Rightarrow x \in \mathcal{D}(A^*),\ y = A^* x.$ $\square$

Lemma 2: $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ symmetric $\Rightarrow$ all approximate e-vals of A are real.

Pf: Let $\lambda = \alpha + i\beta \in \mathbb{C}$ for $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. Then for $x \in \mathcal{D}$,

$$\|(A - \lambda)x\|^2 = \langle (A-\alpha)x - i\beta x, (A-\alpha)x - i\beta x \rangle$$

$$= \|(A-\alpha)x\|^2 + \beta^2 \|x\|^2 \underbrace{- i\beta \langle (A-\alpha)x, x \rangle + i\beta \langle x, (A-\alpha)x \rangle}_{=0}$$

$\geq \beta^2 \|x\|^2 \Rightarrow \nexists$ "approx e-vectors" for λ. $\square$

Recall ex 1 : $T_2 = i\dfrac{d}{dt}$ on $\mathcal{D}_2 := \left\{ f : [0,1] \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ abs. contin., } f' \in L^2 \\ f(0) = f(1) = 0 \end{array} \right\}$

is closed, symmetric, not self-adjoint, $\sigma(T_2) = \mathbb{C}$ (all residual).

Lemma 3 : $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ self-adj. $\Rightarrow$ $\nexists$ residual spectrum ($\Rightarrow \sigma(A) \subseteq \mathbb{R}$).

Pf: Spse $\lambda \in \sigma(A)$ not an e-val. but $\operatorname{im}(A-\lambda) \subseteq \mathcal{H}$ not dense,

then $(\operatorname{im}(A-\lambda))^\perp \neq \{0\} \Rightarrow \exists\ v \neq 0 \in \mathcal{H}$ s.t. $\langle (A-\lambda)x, v\rangle = 0$

$\forall\ x \in \mathcal{D}$. Then $\langle v, Ax\rangle = \langle v, \lambda x\rangle = \langle \bar\lambda v, x\rangle$ $\forall\ x \in \mathcal{D}$

$\Rightarrow v \in \mathcal{D}(A^*)$ & $A^* v = \bar\lambda v \Rightarrow v \in \mathcal{D}$, $Av = \bar\lambda v$.

If $\lambda \notin \mathbb{R}$, contra to Lemma 2 since $\bar\lambda \notin \mathbb{R}$.

If $\lambda \in \mathbb{R}$, contra. since λ is an e-val. ☐

Lemma 4 ($\Rightarrow$ spectral thm): Spse $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ closed, symmetric &

$\exists\ \lambda \in \mathbb{C}$ s.t. $\lambda, \bar\lambda \notin \sigma(A)$. Then A satisfies the conclusion of the spectral thm.

Pf: Let $T_+ := -R_\lambda(A) = (A-\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$, $T_- := -R_{\bar\lambda}(A) = (A-\bar\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$

claim: $T_+^* = T_-$ & $T_+T_- = T_-T_+$, thus $T_\pm$ are normal.

Pf: $x, y \in \mathcal{H}$, $\langle x, T_+ y\rangle = \langle (A-\bar\lambda)\underbrace{T_- x}_{\in \mathcal{D}}, \underbrace{T_+ y}_{\in \mathcal{D}}\rangle = \langle T_- x, (A-\lambda)T_+ y\rangle$

$= \langle T_- x, y\rangle$.

Commutativity: fix $x \in \mathcal{H}$, let $y = T_+T_- x$, $z = T_-T_+ x$.

Observe: $\operatorname{im} T_\pm = \mathcal{D} \Rightarrow (A-\lambda)y = T_- x \in \mathcal{D} \Rightarrow Ay \in \mathcal{D} \Rightarrow A^2 y \in \mathcal{H}$

is well def'd; similarly $A^2 z$ is def'd, now

$x = (A-\bar\lambda)(A-\lambda)y = (A^2 + |\lambda|^2 - 2(\operatorname{Re}\lambda)A)y$

$= \underbrace{(A-\lambda)}_{\text{inj}}\underbrace{(A-\bar\lambda)}_{\text{inj}} z = \underbrace{(A^2 + |\lambda|^2 - 2(\operatorname{Re}\lambda)A)}_{\text{inj}} z$ $\Rightarrow y = z$.

Spectral thm for normal ops $\Rightarrow$ $\exists$ measure space (X, μ),

unitary $U : \mathcal{H} \to L^2(X,\mu)$, bdd measurable $G : X \to \mathbb{C}$ s.t.

$U T_+ U^{-1} = T_G : u \mapsto Gu$. T_+ inj $\Rightarrow$ 0 not an e-val. of T_+

$\Rightarrow \mu(G^{-1}(0)) = 0$. $\mathcal{D} = \operatorname{im} T_+ \Rightarrow U(\mathcal{D}) = \{ Gu \mid u \in L^2(X,\mu) \}$.

For $u \in U(\mathcal{D})$, $U(A-\lambda)U^{-1} u = \frac{1}{G} u = UAU^{-1}u - \lambda u$

$\Rightarrow UAU^{-1}u = \underbrace{\left(\frac{1}{G} + \lambda\right)}_{=: F} u$. Now $u \in U(\mathcal{D}) \Leftrightarrow u = Gv$ for some $v \in L^2 \Leftrightarrow Fu \in L^2$.

$G \neq 0$ a.e. $\Rightarrow F$ finite a.e.

EX: Every $\mu \in$ essential range of $F : X \to \mathbb{C}$ is an approx. e-val. of T_F

$\Rightarrow$ outside a set of measure 0, $F(X) \subseteq \mathbb{R}$. $\square$

ex: $L^2(\mathbb{R}^n) \supseteq H^2(\mathbb{R}^n) \xrightarrow{-\Delta} L^2(\mathbb{R}^n)$ has spectral repr. $L^2(\mathbb{R}^n) \xrightarrow{\mathcal{F}} L^2(\mathbb{R}^n)$,

$\mathcal{F}(-\Delta)\mathcal{F}^{-1} : T_F$ for $F(p) = 4\pi^2 |p|^2$, $\mathcal{F}(H^2(\mathbb{R}^n)) = \{ u \in L^2 \mid |p|^2 u \in L^2 \}$.

EX: $\sigma(A) = \sigma(T_F) =$ essential range of F: in case $-\Delta$, $\sigma(-\Delta) = [0,\infty)$,

all approx e-val.

cor ("basic criterion for self-adjointness"): A closed symmetric op. $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ is self-adj. $\iff$ $\underbrace{\operatorname{im}(A \pm i)}_{\text{both}} = \mathcal{H}$ $\boxtimes$

defn: $\mathcal{H} \overset{\text{dense}}{\supseteq} \mathcal{D} \xrightarrow{A} \mathcal{H}$ symmetric is essentially self-adjoint if its closure $\bar{A}$ is self-adj.

EX: A symmetric & closable $\Rightarrow$ $\bar{A}$ also symmetric.

useful lemma: A densely def'd & symmetric, then ess. self-adj. $\iff$ $\operatorname{im}(A \pm i) \subseteq \mathcal{H}$ are both dense.

Pf: $\bar{A}$ symmetric & closed, $\|(A \pm i)x\|^2 = \|Ax\|^2 + \|x\|^2 \geq c\,\|x\|_A^2$ (graph norm)

$\bar{A}$ closed $\Rightarrow$ $\|\cdot\|_A$ complete $\xrightarrow{\text{take-home \#1 (a)}}$ $\operatorname{im}(\bar{A} \pm i)$ is closed.

If also dense, then $\bar{A} \pm i$ surj. $\Rightarrow$ $\bar{A}$ self-adj. Converse: EX. $\boxtimes$

ex: $L^2(\mathbb{R}^n) \supseteq \mathcal{S}(\mathbb{R}^n) \xrightarrow{i\partial_j} L^2(\mathbb{R}^n)$ is symmetric: $\langle i\partial_j f, g\rangle_{L^2} = \langle f, i\partial_j g\rangle_{L^2}$ $\forall f, g \in \mathcal{S}(\mathbb{R}^n)$. claim: $\mathcal{S}(\mathbb{R}^n) \xrightarrow{i\partial_j \pm i} L^2(\mathbb{R}^n)$ has dense image.

If not, $\exists$ $g \in L^2$ s.t. $\langle i\partial_j \varphi \pm i\varphi, g\rangle_{L^2} = 0$ $\forall \varphi \in \mathcal{S}$ $\iff$

$$0 = \langle -2\pi p_j \hat{\varphi} \pm i\hat{\varphi}, \hat{g}\rangle_{L^2} = \langle \underbrace{(-2\pi p_j \pm i)\hat{\varphi}}_{\text{arbitrary } \varphi \in \mathcal{S}}, \hat{g}\rangle_{L^2} \Rightarrow \hat{g} = 0$$

$\Rightarrow$ $g = 0$.

$\Rightarrow$ Closure of $\mathcal{S} \xrightarrow{i\partial_j} L^2$ is self-adj. Brainteaser: what is the domain of its closure?