X, Y normed vec. spaces, $A : X \to Y$ linear; continuous $\iff$ bounded

<u>Q</u>: If $\dim X = \infty$, $\exists$ <u>unbdd</u> lin. ops $A : X \to Y$?

<u>A</u>: Yes, but $\nexists$ concrete examples (!!!)

Choose a <u>Hamel basis</u> on X, i.e. a maximal lin. indep. subset $\{e_\alpha\}_{\alpha \in I}$

$\Rightarrow$ every $x \in X$ can be written as $x = \sum_{\alpha \in I} c_\alpha e_\alpha$ for unique $c_\alpha \in \mathbb{K}$, only fin.-many nonzero ($\Rightarrow$ sum is finite)

Choose a seq $\{\alpha_n \in I\}_{n=1}^\infty$ & defn $A : X \to Y$ as unique lin map satisfying $Ae_{\alpha_n} := n\|e_{\alpha_n}\| y$ ($y \neq 0 \in Y$ some fixed vector) $\forall\, n \in \mathbb{N}$,

$Ae_\alpha := 0$ $\forall\, \alpha$ that are not α_n for any $n \in \mathbb{N}$.

$$\frac{\|Ae_{\alpha_n}\|}{\|e_{\alpha_n}\|} = n\|y\| \longrightarrow \infty \text{ as } n \to \infty \Rightarrow A \text{ is unbdd.}$$

<u>Lemma</u>: Every vec. space admits a Hamel basis. ($X :=$ vector space)

<u>Pf</u>: Let $S := \{$linearly independent subsets of $X\}$, defn. a partial order $\prec$ on S by $\mathcal{A} \prec \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$.

<u>EX (easy)</u>: Spse $S_0 \subseteq S$ is totally ordered subset, i.e. $\forall\, \mathcal{A}, \mathcal{B} \in S_0$, $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{B} \subseteq \mathcal{A}$. Then $\bigcup_{\mathcal{B} \in S_0} \mathcal{B} =: \mathcal{B}_\infty$ is also in S & is an upper bound for S_0 (i.e. $\forall\, \mathcal{B} \in S_0$, $\mathcal{B} \prec \mathcal{B}_\infty$).

Then Zorn's lemma $\Rightarrow$ S has a maximal element: $\exists\, \mathcal{A}_\infty \in S$ s.t. if $\mathcal{B} \in S$ satisfies $\mathcal{A}_\infty \prec \mathcal{B}$, then $\mathcal{B} = \mathcal{A}_\infty$. $\blacksquare$

<u>Zorn's lemma</u>: For every nonempty partially ordered set $(S, \prec)$ in which every totally ordered subset has an upper bound, that upper bound can be chosen to be a maximal element of S.

$\Updownarrow$ (see e.g. Salamon-Bühler)

<u>Axiom of choice</u>: For any set X, $\exists$ a "choice" fn. $f : 2^X \setminus \{\emptyset\} \to X$ s.t. $\forall$ nonempty $A \in X$, $f(A) \in A$.

<u>rk</u>: Existence of Hamel bases & unbdd lin ops <u>cannot</u> be proved w/o AOC.

Hilbert spaces

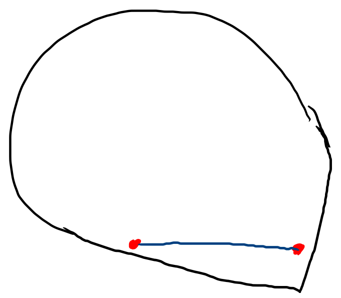
<u>defn</u>: a Hilbert sp. $(\mathcal{H}, \langle\,,\rangle)$ is a complete inner product, i.e.
a Banach space $(\mathcal{H}, \|\cdot\|)$ with an inner prod. $\langle\,,\rangle$ s.t. $\|x\| = \sqrt{\langle x,x\rangle}$.

<u>main thm</u>: For any closed subspace V in a Hilbert sp. $\mathcal{H}$,
$\mathcal{H} = V \oplus V^\perp$, i.e. $\forall\ x \in \mathcal{H}$, one can write $x = v + w$ for unique elements $v \in V$, $w \in V^\perp := \{ y \in \mathcal{H} \mid \langle y, v\rangle = 0\ \ \forall\ v \in V\}$.

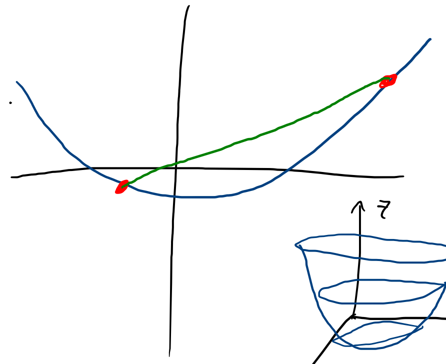
<u>rk</u>: If V is not closed in $\mathcal{H}$, then $\bar{V}$ is also a subspace, &
$V^\perp = \bar{V}^\perp$; in particular, if $V \subseteq \mathcal{H}$ is dense, $V^\perp = \{0\}$.

<u>convexity</u>

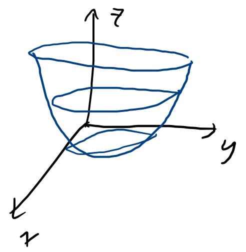
X a vec. sp., $K \subseteq X$ is a <u>convex</u> subset if
$\forall\ x, y \in K$, $tx + (1-t)y \in K \quad \forall\ t \in [0,1]$.



a fn. $f : K \longrightarrow \mathbb{R}$ is <u>convex</u> if $\forall\ x, y \in K$,
$\forall\ t \in [0,1]$, $f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$.



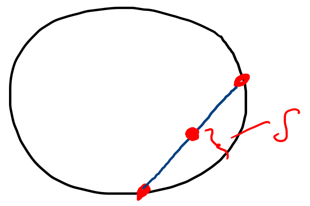
f is <u>strictly convex</u> this inequality is strict
$\forall\ t \in (0,1)$ if $x \neq y$.



<u>ex</u>: Any normed vec. space $(X, \|\cdot\|)$,
$\bar{B} := \{ x \in X \mid \|x\| \leq 1\}$ is a convex subset.
$\partial\bar{B} := \{ x \in X \mid \|x\| = 1\}$.
For subsets $\mathcal{U}, V \subseteq X$, $\operatorname{dist}(\mathcal{U}, V) := \inf\{ \|x - y\| \mid x \in \mathcal{U},\ y \in V\}$

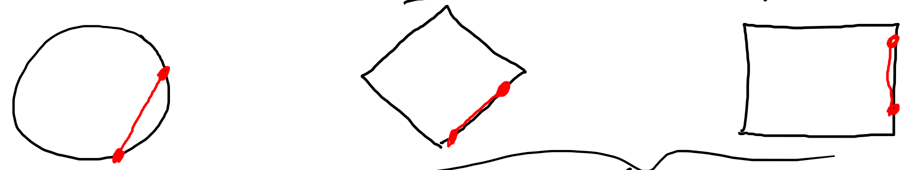
<u>defn</u>: a normed vec. sp. $(X, \|\cdot\|)$ is <u>strictly convex</u> if
$x, y \in \bar{B}$ with $x \neq y \implies tx + (1-t)y \in \bar{B} \setminus \partial\bar{B} \quad \forall\ t \in (0,1)$.

<u>defn</u>: $(X, \|\cdot\|)$ is <u>uniformly convex</u> if $\forall\ \varepsilon > 0$,
$\exists\ \delta > 0$ s.t.
$x, y \in \bar{B}$ with $\|x - y\| \geq \varepsilon \implies \operatorname{dist}\left(\frac{x+y}{2}, \partial\bar{B}\right) \geq \delta$.



<u>ex</u>: $\mathbb{R}^2 \ni v = (x, y)$, $|v|_p := \left(|x|^p + |y|^p\right)^{1/p}$ for $1 \leq p < \infty$
$|v|_\infty := \max\{|x|, |y|\}$.

<u>unit ball</u>: $\underline{p=2}$ $\quad\underline{p=1}$ $\quad p = \infty$



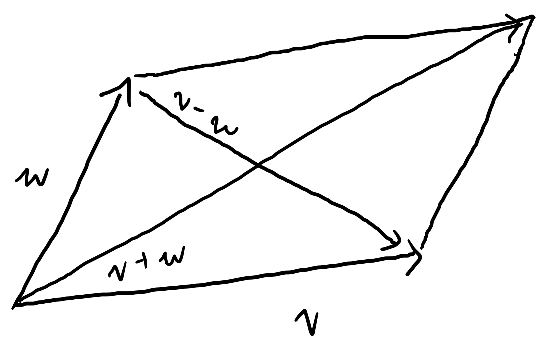
unif. convex $\quad$ not strictly ($\Rightarrow$ not unif.) convex

prop: Every inner prod. space $(X, \langle\,,\,\rangle)$ is unif. convex.

pf: $v, w \in X$,

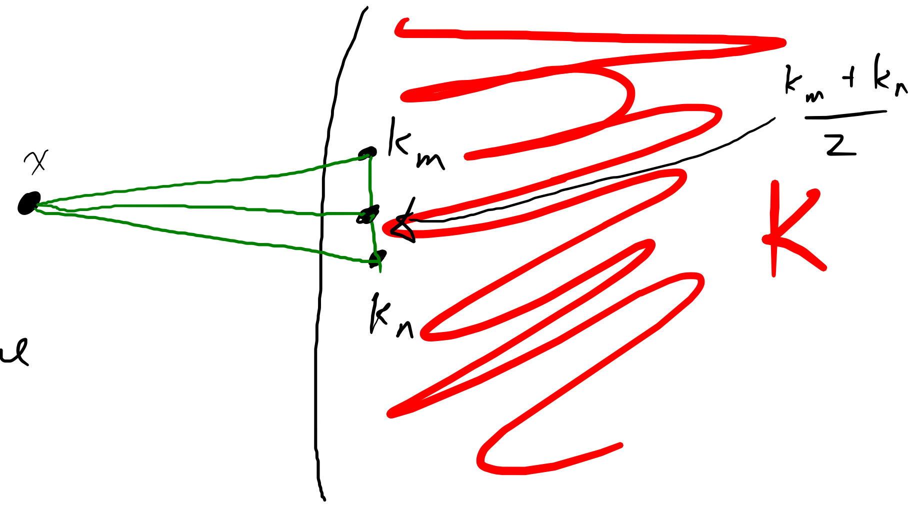
$$\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle = 2\|v\|^2 + 2\|w\|^2$$

"parallelogram identity"



$\Rightarrow$ if $v, w \in \overline{B}$, then $\frac{\|v-w\|^2}{4} \le 1 - \left\|\frac{v+w}{2}\right\|^2$. $\square$

thm: Assume $(X, \|\cdot\|)$ is a unif. convex Banach space, $K \subseteq X$ is a closed convex subset, $x \in X \setminus K$. Then the fn $K \to (0,\infty) : k \mapsto \|k - x\|$ attains a unique global minimum.



pf sketch: Choose a seqn. $k_n \in K$ s.t. $\|k_n - x\| \to I := \inf_{k\in K} \|k - x\|$.

Rescale s.t. WLOG $\forall$ n large, $\|k_n - x\| \le 1$ but close to 1.

$\frac{k_m - k_n}{2} \in K$ since K is convex, $\frac{k_m + k_n}{2} - x = \frac{(k_m - x) + (k_n - x)}{2} \in \overline{B}$.

n large $\Rightarrow \|k_n - x\| - I$ small $\Rightarrow \left\|\frac{k_m+k_n}{2} - x\right\|$ cannot be much smaller than 1,

$\Rightarrow \frac{k_m+k_n}{2} - x$ is close to $\partial\overline{B} \overset{\text{unif. conv.}}{\Longrightarrow} k_n - x$ & $k_m - x$ are close

$\Leftrightarrow \|k_n - k_m\|$ becomes arbitrarily small for m, n large

$\Rightarrow k_n$ is Cauchy $\Rightarrow \exists\ k_\infty := \lim_{n\to\infty} k_n \in K$ (since K is closed),

so $\|k_\infty - x\| = I$. $\square$

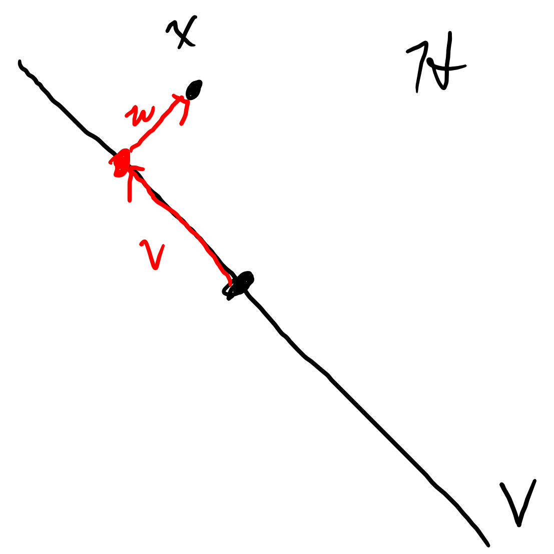
ref: Lecture notes, Sec. 1 (on website) pf of main thm on Hilbert spaces:

To show: given $x \in \mathcal{H}$ & a closed subspace $V \subseteq \mathcal{H}$, then $x = v + w$ for unique $v \in V$ & $w \perp V$.

uniqueness: $x = v + w = v' + w'$ with $v, v' \in V$, $w, w' \in V^\perp$,

then $v - v' = w' - w \in V \cap V^\perp = \{0\}$.

existence: Assume $x \in \mathcal{H} \setminus V$ (else problem is trivial).



$\mathcal{H}$ is a unif. convex Banach space, $V \subseteq \mathcal{H}$ is closed convex subset

$\Longrightarrow \exists!\ v \in V$ ~~closest~~ closest to x. Let $w := x - v$.

claim: $w \in V^\perp$.

Given $h \in V$, v is a minimum of the fn. $V \longrightarrow \mathbb{R} : k \longmapsto \|x - k\|^2$

$$\Rightarrow 0 = \frac{d}{dt} \|x - (v + th)\|^2 \Big|_{t=0} = \frac{d}{dt} \|w - th\|^2 \Big|_{t=0} = \frac{d}{dt} \langle w - th, w - th \rangle \Big|_{t=0}$$

$$= \frac{d}{dt} \left(\|w\|^2 - 2t \operatorname{Re}\langle w, h \rangle + t^2 \|h\|^2 \right) \Big|_{t=0} = -2 \operatorname{Re} \langle w, h \rangle \qquad \left(\begin{array}{c} \text{in case} \\ \mathbb{K} = \mathbb{C} \end{array} \right)$$

In case $\mathbb{K} = \mathbb{R}$, result is $0 = -2 \langle w, h \rangle \Rightarrow w \perp V$.

In case $\mathbb{K} = \mathbb{C}$, can also replace h with $ih \in V$,

$\Rightarrow 0 = -2 \operatorname{Re} \langle w, ih \rangle = 2 \operatorname{Im} \langle w, h \rangle \Rightarrow \langle w, h \rangle = 0 \Rightarrow w \perp V$. $\boxtimes$