refs for Hilbert spaces:

- unif. convexity + $\perp$-complements: lecture notes, Sec. 1
- all else: Reed–Simon, Secs. II.1–3

next several weeks: lecture notes, Secs. 2 – ...

Riesz repr. thm. for Hilbert spaces: $\mathcal{H} \to \mathcal{H}^* : x \mapsto \Lambda_x := \langle x, \cdot \rangle$ is a $\mathbb{R}$-linear isometric isomorphism.

today: $(X, \mu)$ measure space, $1 < p, q < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

$(V, \langle\,,\rangle)$ a fin.-dim. inner prod. sp. over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in which fns on $X$ take values.

Riesz repr. thm for $L^p$: The natural map $L^q(X) \to (L^p(X))^* : g \mapsto \Lambda_g$ def'd by

$$\Lambda_g(f) := \int_X \langle g(x), f(x) \rangle \, d\mu(x)$$ is an isometric $\mathbb{R}$-linear isomorphism.

Why $g \mapsto \Lambda_g$ is well-def'd?

$$|\Lambda_g(f)| = \left| \int_X \langle g, f \rangle \, d\mu \right| \le \int_X |\langle g, f \rangle| \, d\mu \le \int_X |g| \cdot |f| \, d\mu \overset{\text{(Hölder)}}{\le} \|g\|_{L^q} \cdot \|f\|_{L^p}$$

$\Rightarrow \Lambda_g \in (L^p(X))^*$ & $\|\Lambda_g\| \le \|g\|_{L^q}$.

Why isometric?

prop: $\forall\, g \in L^q(X)$, $\exists\, f \in L^p(X)$ s.t. $f \neq 0$ s.t. $\left| \int_X \langle g, f \rangle \, d\mu \right| = \|g\|_{L^q} \cdot \|f\|_{L^p}$.

cor: $\|\Lambda_g\| = \|g\|_{L^q}$.

pf: $f := \begin{cases} |g|^{q-2} g & \text{whenever } g \neq 0 \\ 0 & \text{whenever } g = 0 \end{cases}$

computation: $|g|^q = |f|^p$

$\int_X \langle g, f \rangle \, d\mu = \int |g|^q \, d\mu = \ldots = \|g\|_{L^q} \cdot \|f\|_{L^p}$. ∎

Still to prove: $\forall\, \Lambda \in (L^p(X))^*$, $\exists\, g \in L^q(X)$ s.t. $\Lambda = \Lambda_g$.

Recall: $\exists$ natural map $\Phi : L^p(X) \to (\underbrace{L^p(X)^*}_{\cong L^q(X)})^* \cong L^p(X)$

$$\Phi(f)\Lambda = \Lambda(f)$$

means as a map $\Phi : L^p(X) \to L^p(X)$, identifying $\Lambda = \Lambda_g$ for some $g \in L^q(X)$,

$$\int_X \langle \Phi(f), g \rangle \, d\mu = \int_X \langle g, f \rangle \, d\mu \quad \forall\, g \in L^q(X) \quad \Rightarrow$$

$$\mathrm{Re} \int_X \langle \Phi(f) - f, g \rangle \, d\mu = 0 \quad \forall\, g \in L^q(X).$$

In case $\mathbb{K} = \mathbb{C}$, can also plug in $ig \in L^q(X)$ $\Rightarrow$ imaginary part also $= 0$

$$\Rightarrow \int_X \langle \Phi(f) - f, g \rangle \, d\mu = 0 \quad \forall\, g \in L^q(X) \Rightarrow \Lambda_{\Phi(f) - f} = 0 \in (L^q(X))^*$$

$\Rightarrow \Phi(f) - f = 0 \in L^p(X)$.

$\Rightarrow \Phi$ is an iso.

cor: $\forall\, 1 < p < \infty$, $L^p(X)$ is reflexive. ∎

differentiability of $\|\cdot\|_{L^p}$: $p>1$. Wanted: $\frac{d}{dt}\|f+tg\|_{L^p}^p\Big|_{t=0}$ for $f,g\in L^p(X)$.

For $v,w\in V$,

$$\frac{d}{dt}|v+tw|^p = \frac{d}{dt}\langle v+tw,v+tw\rangle^{p/2} = \frac{p}{2}\langle v+tw,v+tw\rangle^{\frac{p}{2}-1}\cdot\frac{d}{dt}\left(\|v\|^2+2\operatorname{Re}\langle v,w\rangle t + t^2\|w\|^2\right)$$

$$= \dots = p|v+tw|^{p-2}\cdot\operatorname{Re}\langle v+tw,w\rangle.$$

note: Cauchy-Schwarz $\Rightarrow |RHS|\le p\,|v+tw|^{p-1}\cdot|w|$ where $p-1>0$

$\Rightarrow$ sensible to defn. $RHS := 0$ whenever $v+tw=0$

check: $\frac{d}{dt}|v+tw|^p=0$ also in that case.

Now, by differentiation under the integral, for $f,g\in L^p(X)$,

$$\frac{d}{dt}\|f+tg\|_{L^p}^p = \frac{d}{dt}\int_X |f(x)+tg(x)|^p\,d\mu(x) = \int_X \frac{\partial}{\partial t}|f(x)+tg(x)|^p\,d\mu(x)$$

$$=\int_X p|f+tg|^{p-2}\cdot\operatorname{Re}\langle f+tg,g\rangle\,d\mu$$

$\Rightarrow$ prop: $\boxed{\frac{d}{dt}\|f+tg\|_{L^p}^p\Big|_{t=0} = p\int_X |f|^{p-2}\operatorname{Re}\langle f,g\rangle\,d\mu.}$

Justification of diff. under integral sign:

(1) $|f(x)+tg(x)|^p \le (|f(x)|+|g(x)|)^p \quad \forall\, t\in[-1,1]$

$$\int(|f|+|g|)^p\,d\mu = \||f|+|g|\|_{L^p}^p \overset{\text{(Minkowski)}}{\le} (\|f\|_{L^p}+\|g\|_{L^p})^p<\infty$$

$\Rightarrow (|f|+|g|)^p$ is a Lebesgue-integrable fn.

(2) $\frac{\partial}{\partial t}|f(x)+tg(x)|^p = p\int|f+tg|^{p-2}\cdot\operatorname{Re}\langle f+tg,g\rangle\,d\mu$

$$\left||f+tg|^{p-2}\cdot\operatorname{Re}\langle f+tg,g\rangle\right| \le |f+tg|^{p-1}\cdot|g| \le (|f|+|g|)^{p-1}\cdot|g|$$

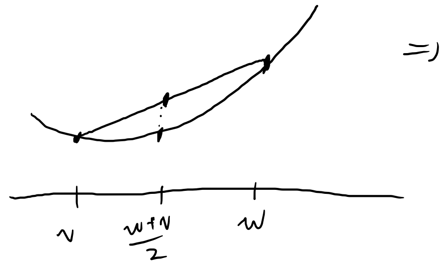
$\forall\, t\in[-1,1]$. This is L.-integrable due to Hölder:

$\frac{p-1}{p}+\frac{1}{p}=1$, so $\int(|f|+|g|)^{p-1}\cdot|g|\,d\mu \le \underbrace{\|(|f|+|g|)^{p-1}\|_{L^{\frac{p}{p-1}}}}_{<\infty \text{ since } (|f|+|g|)^p \text{ integrable.}}\cdot\|g\|_{L^p}$

☒

uniform convexity of $L^p(X)$: $p > 1$

$V \longrightarrow \mathbb{R} : v \longmapsto |v|^p$ is strictly convex ($\Leftarrow$ Hessian is pos.-def.)




$\Rightarrow$ the fn. $\psi : V \times V \longrightarrow \mathbb{R} : (v, w) \longmapsto$

$$\frac{|v|^p + |w|^p}{2} - \left|\frac{v+w}{2}\right|^p \geq 0, \quad > 0 \text{ unless } v = w.$$

Fix $\varepsilon > 0$, let

$$K := \{ (v, w) \in V \times V \mid |v - w|^p \geq \varepsilon \text{ \& } |v|^p + |w|^p \leq 1 \}$$

since $V$ is fin-dim, $K$ is cpct $\Rightarrow \exists\, \delta > 0$ s.t. $\psi|_K \geq \delta$.

Sppe $v, w \in V$ s.t. $v \neq w$, set $\tau := (|v|^p + |w|^p)^{1/p} > 0$,

$v' := \frac{v}{\tau}$, $w' := \frac{w}{\tau}$, $|v'|^p + |w'|^p = 1$, $|v' - w'|^p \geq \varepsilon \iff |v - w|^p \geq \varepsilon \tau^p$

$\Rightarrow \psi(v', w') \geq \delta \Rightarrow \psi(v, w) \geq \delta \tau^p$

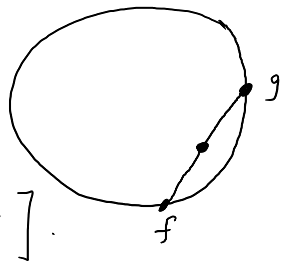
$\Rightarrow$ Lemma: Given $1 < p < \infty$ & $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\psi$ satisfies

$$|v - w|^p \geq \varepsilon(|v|^p + |w|^p) \Rightarrow \psi(v, w) \geq \delta(|v|^p + |w|^p) \quad \forall\, v, w \in V.$$

thm: Given $1 < p < \infty$ & $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, f, g \in L^p(X)$,

$$\frac{\|f\|_{L^p}^p + \|g\|_{L^p}^p}{2} - \left\|\frac{f+g}{2}\right\|_{L^p}^p \geq \delta\left[\|f - g\|_{L^p}^p - \varepsilon(\|f\|_{L^p}^p + \|g\|_{L^p}^p)\right].$$

cor: $L^p(X)$ is unif. convex




Pf: If $\|f\|_{L^p}, \|g\|_{L^p} \leq 1$,

thm $\Rightarrow 1 - \left\|\frac{f+g}{2}\right\|_{L^p}^p \geq \delta\left[\|f - g\|_{L^p}^p - 2\varepsilon\right]$.

Pf of thm: Let $A := \{ x \in X \mid |f(x) - g(x)|^p \geq \varepsilon(|f(x)|^p + |g(x)|^p) \}$

$A^c := X \setminus A$.

Then for $x \in A$, $\psi(f(x), g(x)) \geq \delta(|f(x)|^p + |g(x)|^p)$.

for $x \in A^c$, $|f(x) - g(x)|^p < \varepsilon(|f(x)|^p + |g(x)|^p)$.

$\left|\frac{f-g}{2}\right|^p = \left|\frac{f + (-g)}{2}\right|^p \leq \frac{|f|^p + |g|^p}{2}$ by convexity of $v \longmapsto |v|^p$.

Now,
$$\frac{\|f\|_{L^p}^p + \|g\|_{L^p}^p}{2} - \left\|\frac{f+g}{2}\right\|_{L^p}^p = \int_X \left( \frac{|f(x)|^p + |g(x)|^p}{2} - \left|\frac{f(x)+g(x)}{2}\right|^p \right) d\mu$$

$$= \int_X \psi(f(x), g(x))\, d\mu \geq \int_A \psi(f, g)\, d\mu \quad (\text{since } \psi \geq 0)$$

$$\geq \delta \int_A (|f|^p + |g|^p)\, d\mu \geq \frac{\delta}{2^{p-1}} \int_A |f - g|^p\, d\mu$$

$$= \frac{\delta}{2^{p-1}}\left( \int_X |f - g|^p\, d\mu - \int_{A^c} |f - g|^p\, d\mu \right) \geq \frac{\delta}{2^{p-1}}\left( \int_X |f - g|^p\, d\mu - \varepsilon \int_{A^c} (|f|^p + |g|^p)\, d\mu \right)$$

$$\geq \frac{\delta}{2^{p-1}}\left( \|f - g\|_{L^p}^p - \varepsilon(\|f\|_{L^p}^p + \|g\|_{L^p}^p) \right). \quad \blacksquare$$

pf of Riesz: Given $\Lambda \in (L^p(X))^*$, assume $\Lambda \neq 0$ since otherwise could set $g = 0$. Then $\ker \Lambda \subseteq L^p(X)$ is a closed subspace of codim. 1, & $\exists\ h \in L^p(X)$ s.t. $\Lambda(h) = 1$.

$\ker \Lambda =: K$ is a closed convex subset in a unif. convex Banach space

(last week) $\Longrightarrow$ $\exists\ k_0 \in \ker \Lambda$ minimizing $\|h - k_0\|_{L^p}^p$.

Then for any $k \in \ker \Lambda$,

$$0 = \frac{d}{dt} \|h - (k_0 + tk)\|_{L^p}^p \Big|_{t=0} = p \int_X |h - k_0|^{p-2} \cdot \operatorname{Re}\langle h - k_0, k\rangle \, d\mu$$

$$\Rightarrow \forall\ k \in \ker \Lambda, \ \operatorname{Re} \int_X \langle \underbrace{|h - k_0|^{p-2}(h - k_0)}_{\in L^q}, \underbrace{k}_{\in L^p} \rangle \, d\mu = 0$$

In case $\mathbb{K} = \mathbb{C}$, can plug in $ik$

$$\Longrightarrow \int_X \langle |h - k_0|^{p-2}(h - k_0), k \rangle \, d\mu = 0 \quad \forall\ k \in \ker \Lambda.$$

check: $\exists!\ c \in \mathbb{K}$ s.t. the fn. $g := c\,|h - k_0|^{p-2}(h - k_0)$ is in $L^q(X)$ & satisfies $\Lambda_g = \Lambda$. $\square$