(X, μ) measure space, $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$L^q(X) \longrightarrow (L^p(X))^* : g \longmapsto \Lambda_g, \qquad \Lambda_g := \int_X \langle g, \cdot \rangle \, d\mu$$

Hölder $\Rightarrow$ $\frac{|\Lambda_g(f)|}{\|f\|_{L^p}} \leq \|g\|_{L^q}$, equality attained for $f := |g|^{q-2} g$ except in case $\begin{cases} q = \infty \\ p = 1 \end{cases}$.

Lemma: $\sup_{f \in L^1(X) \setminus \{0\}} \frac{|\Lambda_g(f)|}{\|f\|_{L^p}} = \|g\|_{L^\infty} \quad \forall \; g \in L^\infty(X)$. (if (X, μ) is σ-finite)

Pf: If not, then $\exists \; c > \sup_{f \in L^1 \setminus \{0\}} \frac{|\Lambda_g(f)|}{\|f\|_{L^1}}$ s.t. $c < \|g\|_{L^\infty}$, so

$A' := \{ x \in X \mid |g(x)| \geq c \}$ has $\mu(A') > 0$.

X σ-finite $\Rightarrow$ can replace A' with a possibly smaller set $A \subseteq A'$ s.t. $0 < \mu(A) < \infty$.

Let $f := \frac{g}{|g|}$ on A & $f := 0$ on $X \setminus A$, then

$\|f\|_{L^1} = \mu(A) < \infty$. Since $|g| \geq c$ on A,

$$\left| \int_X \langle g, f \rangle \, d\mu \right| = \int_A |g| \, d\mu \geq c \mu(A) = c \|f\|_{L^1} > \left| \int_X \langle g, f \rangle \, d\mu \right|$$

contr. ☒

Cor: The map $L^q(X) \longrightarrow (L^p(X))^*$, $g \longmapsto \Lambda_g$ is also an isometry for $p, q \in \{1, \infty\}$, assuming (in case $p = 1$) X is σ-finite. ☒

Thm: If X is σ-finite, then $L^\infty(X) \longrightarrow (L^1(X))^*$ is also an isomorphism.

EX (PSET 4): For $f \in L^\infty(X)$ s.t. $|f| < \|f\|_{L^\infty}$ a.e.,

$$\left| \int_X \langle g, f \rangle \, d\mu \right| < \|g\|_{L^1} \cdot \|f\|_{L^\infty} \quad \forall \; g \in L^1(X) \setminus \{0\}.$$

We will later prove (via Hahn-Banach):

For every normed vec. sp. E & $x \in E$, $\exists \; \Lambda \in E^*$ s.t. $\|\Lambda\| = 1$ & $|\Lambda(x)| = \|x\|$. $\Longrightarrow$ $L^1(X) \longrightarrow (L^\infty(X))^*$ is not surjective.

Lemma: $\forall\ 1 < p \leq \infty$, $L^p(X) \cap L^1(X)$ is dense in $L^1(X)$.

pf: Given $f \in L^1(X)$, $n \in \mathbb{N}$, denote $A_n := \{ x \in X \mid |f(x)| \leq n \}$,

then defn $f_n := \chi_{A_n} f$. $\left(\chi_A(x) := \begin{cases} 1 & \text{at } x \in A \\ 0 & \text{at } x \in X \setminus A \end{cases} \right)$

Since $f \in L^1(X)$ & $|f| > 1$ on $X \setminus A_1$,

$\infty > \int_X |f|\, d\mu \geq \int_{X \setminus A_1} |f|\, d\mu \geq \mu(X \setminus A_1)$. Note $|f_n| \leq n$ everywhere,

& on A_1, $|f|^p \leq |f|$,

$$\|f_n\|_{L^p}^p = \int_{X \setminus A_1} |f_n|^p\, d\mu + \int_{A_1} |f_n|^p\, d\mu \leq n^p \underbrace{\mu(X \setminus A_1)}_{< \infty} + \underbrace{\int_{A_1} |f|\, d\mu}_{\leq \|f\|_{L^1}} < \infty.$$

$\|f - f_n\|_{L^1} = \int_{X \setminus A_n} |f|\, d\mu \longrightarrow 0$ because $\underbrace{X \setminus A_1}_{\mu < \infty} \supseteq X \setminus A_2 \supseteq X \setminus A_3 \supseteq$

$\bigcap_{n \in \mathbb{N}} X \setminus A_n = \emptyset$ ∎

pf of thm for $p = 1$, $\mu(X) < \infty$

$\forall\ p' > p \geq 1$, set $r \geq p$ s.t. $\frac{1}{p'} + \frac{1}{r} = \frac{1}{p}$, then minor generalization of Hölder, $\forall\ f \in L^{p'}(X)$,

$$\|f\|_{L^p} = \|1 \cdot f\|_{L^p} \leq \|1\|_{L^r} \cdot \|f\|_{L^{p'}} = \mu(X)^{1/r} \cdot \|f\|_{L^{p'}}$$

$\Rightarrow L^{p'}(X) \subseteq L^p(X)$.

Spse $\Lambda \in (L^1(X))^*$, then for $f \in L^p(X)$ with $1 < p < \infty$, also $f \in L^1$

& $|\Lambda(f)| \leq \|\Lambda\|_{(L^1)^*} \cdot \|f\|_{L^1} \leq \|\Lambda\|_{(L^1)^*} \cdot \mu(X)^{1/q} \cdot \|f\|_{L^p}$

$\Rightarrow \Lambda \in (L^p(X))^*$ & $\|\Lambda\|_{(L^p)^*} \leq \|\Lambda\|_{(L^1)^*}\, \mu(X)^{1/q}$. $(*)$

Riesz thm for $p > 1$ $\Rightarrow \exists\ g_p \in L^q(X)$ $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ s.t.

$\Lambda(f) = \int_X \langle g_p, f \rangle\, d\mu \quad \forall\ f \in L^p(X)$.

claim: g_p is the same fn. $\forall\ p \in (1, \infty)$ & $g := g_p \in L^\infty(X)$.

Spse $p < p' < \infty$, set $q' < q$ s.t. $\frac{1}{p'} + \frac{1}{q'} = 1$.

Then $\int_X \langle g_p, f \rangle\, d\mu = \Lambda(f) = \int_X \langle g_{p'}, f \rangle\, d\mu \quad \forall\ f \in L^{p'}(X)$

$\Rightarrow g_p - g_{p'} \in L^{q'}(X)$ satisfies $\int_X \langle g_p - g_{p'}, f \rangle\, d\mu = 0 \quad \forall\ f \in L^{p'}(X)$

i.e. $g_p - g_{p'}$ represents the trivial element of $(L^{p'}(X))^* \Rightarrow g_p - g_{p'} = 0 \in L^q$,

i.e. $g_p - g_{p'} = 0$ a.e.

$\Rightarrow$ all the g_p are WLOG a single fn. $g \in \bigcap_{1 < q < \infty} L^q(X)$.

$\|g\|_{L^q} = \|\Lambda\|_{(L^p)^*} \overset{(*)}{\leq} \|\Lambda\|_{(L^1)^*} \cdot \mu(X)^{1/q} \quad \forall\ q \in (1, \infty)$.

claim: $\|g\|_{L^\infty} \leq \|\Lambda\|_{(L^1)^*}$. **pf**: For $c > 0$, let $A_c := \{ x \in X \mid |g(x)| \geq c \}$

$$\underbrace{c\,\mu(A_c)^{1/q}}_{\leq \left(\int_{A_c} |g|^q\, d\mu \right)^{1/q}} \leq \|g\|_{L^q} \leq \|\Lambda\|_{(L^1)^*} \cdot \mu(X)^{1/q}$$

If $\mu(A_c) > 0$, this ineq. converges as $q \to \infty$ to $c \leq \|\Lambda\|_{(L^1)^*}$.

$\Rightarrow |g| \leq \|\Lambda\|_{(L^1)^*}$ a.e.

We've proved: given $\Lambda \in (L^1(X))^*$, $\exists\ g \in L^\infty(X)$ s.t. $\Lambda(f) = \int_X \langle g, f \rangle\, d\mu$

$\forall\ f \in L^p(X)$ for $p > 1$.

Given $f \in L^1(X)$, choose $f_n \in L^1(X) \cap L^p(X)$ s.t. $f_n \xrightarrow{L^1} f$,

then Hölder $\Rightarrow \int_X \langle g, f_n \rangle\, d\mu \longrightarrow \int_X \langle g, f \rangle\, d\mu$ since $g \in L^\infty(X)$,

Λ contin. on L^1 $\Rightarrow \underset{\parallel}{\Lambda(f_n)} \longrightarrow \Lambda(f) \Rightarrow \Lambda(f) = \int_X \langle g, f \rangle\, d\mu$. ∎

<u>pf for $p=1$ & (X,μ) σ-finite:</u>

$X = \bigcup_{n\in\mathbb{N}} X_n$ s.t. $\mu(X_n)<\infty$, WLOG $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots \subseteq \bigcup_n X_n = X$.

Given $\Lambda \in (L^1(X))^*$, def. $\Lambda_n \in (L^1(X_n))^*$ by $\Lambda_n(f) := \Lambda(f_n)$

where $f_n := \begin{cases} f & \text{on } X_n \\ 0 & \text{on } X \setminus X_n \end{cases}$.

Case of finite measure $\Rightarrow$ $\exists$ $g_n \in L^\infty(X_n)$ s.t. $\Lambda_n(f) = \int_{X_n} \langle g_n, f\rangle \, d\mu$

$\forall$ $f \in L^1(X_n)$.

<u>check</u>: $\forall$ n, m, g_n & g_m match a single fn. $g \in L^\infty(X)$

where they are def'd, & $\Lambda(f) = \int_X \langle g, f\rangle \, d\mu$ $\forall$ $f \in L^1(X)$.

(<u>Use</u>: every $f \in L^1(X)$ can be L^1-approximated by fns. $\chi_{X_n} f$ as $n \to \infty$.)

$\blacksquare$

<u>separability</u>: Assume $(X, \mu) = \Omega \subseteq \mathbb{R}^n$ Lebesgue measurable, $\mu = m =$ Leb. meas.

<u>thm</u>: $\forall\ 1 \le p < \infty$, $L^p(\Omega)$ contains a countable dense subset (i.e. is separable).

<u>note</u>: Suff. to prove for $\Omega = \mathbb{R}^n$. We consider fns. $f : \mathbb{R}^n \to V$ s.t. $\dim V < \infty \Rightarrow \exists$ countable dense set $V_0 \subseteq V$ (e.g. $V = \mathbb{R}^n$, take $V_0 = \mathbb{Q}^n$).

<u>defn</u>: A <u>dyadic cube</u> in $\mathbb{R}^n$ is any set of the form

$$Q := \left[\frac{m_1}{2^N}, \frac{m_1+1}{2^N}\right] \times \dots \times \left[\frac{m_n}{2^N}, \frac{m_n+1}{2^N}\right] \quad \forall\ N \in \mathbb{N},\ m_1, \dots, m_n \in \mathbb{Z}.$$

$\exists$ countably many.

Let $Q(\mathbb{R}^n) := \{$ finite linear combins. of characteristic fns. of dyadic cubes w/ coeffs. in V_0, i.e. $\sum_{j=1}^{k} \chi_{Q_j} v_j$, $v_j \in V_0$, Q_j dyadic cubes, $k \in \mathbb{N}$ $\}$

<u>prop</u>: $Q(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $\forall\ p \in [1, \infty)$.
(is also countable)

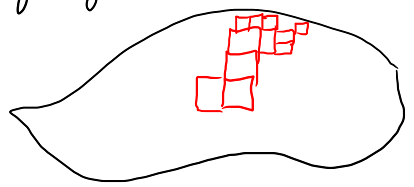
<u>pf</u>: Let $S(\mathbb{R}^n) := \{$ fin. lin. combins. of char. fns. of sets of finite measure w/ coffs. in V $\}$
$= \{$ integrable simple fns. $\} \subseteq L^p(\mathbb{R}^n)$.

<u>Lemma</u>: $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

<u>pf for $V = \mathbb{R}$</u>: $L^p(\mathbb{R}^n) \ni f = f^+ - f^-$ for $f^\pm : \mathbb{R}^n \to [0, \infty)$ s.t. $|f| = f^+ + f^-$;
$f^\pm$ = limit of monotone seqs. of simple fns $f_n^\pm$.
<u>check</u>: $f_n^+ - f_n^- \xrightarrow{L^p} f$. $\square$

Now suff. to prove: Lemma: any $A \subseteq \mathbb{R}^n$ with finite measure & any $v \in V$, the fn $\chi_A v$ can be approximated in L^p arbitrarily well by fns. in $Q(\mathbb{R}^n)$.

<u>Lemma</u>: If $A \subseteq \mathbb{R}^n$ is open, then A = union of dyadic cubes w/ empty intersection of their interiors. $\square$



pf of Lemma for $A \overset{\text{open}}{\subseteq} \mathbb{R}^n$:
Say $A = \bigcup_{n \in \mathbb{N}} Q_n$, Q_n dyadic cubes w/ interiors not intersecting,
so $m(A) = \sum_n m(Q_n)$. choose $v_0 \in V_0$ close to v, then estimate

$$\left\| \chi_A v - \sum_{j=1}^{k} \chi_{Q_j} v_0 \right\|_{L^p} \quad \text{for } k \gg 0 \ \&\ |v_0 - v| \text{ small}.$$

If $A \subseteq \mathbb{R}^n$ not open but measurable, use <u>outer regularity</u>:
$\exists$ a seq. of open sets $A_k' \subseteq \mathbb{R}^n$ containing A s.t.
$m(A_k' \setminus A) \to 0$ as $k \to \infty$. $\square$