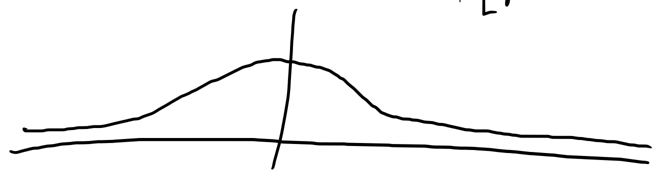
observation: in an ∞-dim. Banach space, closed + bdd $\not\Rightarrow$ cpct.

ex: $f(x) = \frac{1}{1+x^2}$, $f_n(x) := f(x+n)$, so $f_n \in L^p(\mathbb{R})$ $\forall\, p \geq 1$

a 

$\|f_n\|_{L^p}$ is indep. of n, $f_n \to 0$ pointwise as $n \to \infty$

but $\nexists$ L^p-conv. subseq.

(it would have to conv. to 0).

weak convergence

defn: In a normed vector space E, a seq $x_n \in E$ converges weakly to $x \in E$ if $\forall\, \Lambda \in E^*$, $\Lambda(x_n) \longrightarrow \Lambda(x)$. "$x_n \rightharpoonup x$"

rks: (1) strong conv. (i.e. $x_n \to x$) $\Rightarrow$ weak conv. ($x_n \rightharpoonup x$).

(2) If $\dim E < \infty$, can assume $\|\cdot\| = \sqrt{\langle\,,\rangle}$ some inner product, choose O-N basis $e_1, \dots, e_k$ of E, then $x_n \rightharpoonup x \Rightarrow$ $\langle e_j, x_n\rangle \longrightarrow \langle e_j, x\rangle$ as $n \to \infty$ $\forall\, j = 1, \dots, k$ $\Rightarrow$ coords. of x_n $\longrightarrow$ coords of x $\Rightarrow$ $x_n \to x$, i.e. weak $\Rightarrow$ strong.

(3) If $\dim E = \infty$, usually weak $\not\Rightarrow$ strong.

ex (PSET 4): $\mathcal{H}$ an ∞-dim. Hilbert sp. w/ O-N set $\{e_j\}_{j=1}^{\infty}$, then e_j has no conv. subseq as $j \to \infty$, but $e_j \rightharpoonup 0$.

defn: The weak topology on a normed vec. sp. E is the locally convex top. generated by the family of seminorms

$\|x\|_\Lambda := |\Lambda(x)|$ for $\Lambda \in E^*$.

$\Rightarrow$ conv. in weak top. equivalent to $|\Lambda(x - x_n)| \to 0$ as $n \to \infty$ $\forall\, \Lambda \in E^*$

$\Leftrightarrow \Lambda(x_n) \to \Lambda(x)$ $\forall\, \Lambda \in E^*$ $\Leftrightarrow x_n \rightharpoonup x$.

Weak top. = smallest top. containing all sets of form

$\{x \in E \mid |\Lambda(x) - \Lambda(x_0)| < \varepsilon\}$ $\forall\, \Lambda \in E^*$, $x_0 \in E$, $\varepsilon > 0$.

rk: Weak top. is generally not metrizable (PSET 4):

e.g. in Hilbert space $\mathcal{H}$ w/ O-N basis $\{e_j\}_{j \in \mathbb{N}}$,

$0 \in \overline{\{e_1, \sqrt{2}e_2, \sqrt{3}e_3, \dots\}}$ in weak top., but $\nexists$ weakly conv. subseq. to 0.

Defn': For $1 \le p < \infty$ (+ if $p=1$, assume (X,μ) is σ-finite),

a seq. $f_n \in L^p(X)$ **conv. weakly** $f \in L^p(X)$ ($f_n \xrightarrow{L^p} f$)

if $\forall$ $g \in L^q(X)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, $\int_X \langle g, f_n\rangle \, d\mu \longrightarrow \int_X \langle g, f\rangle \, d\mu$.

rk: For $p=\infty$, this is not weak conv. since $L^1(X) \subsetneq (L^\infty(X))^*$.

Defn: For a normed vec. sp. E, the weak*-topology on E^* is the locally convex top. def'd via the seminorms $\{\|\cdot\|_x\}_{x\in E}$, $\|\Lambda\|_x := |\Lambda(x)|$. Then a seq. $\Lambda_n \in E^*$ is weak*-convergent to $\Lambda \in E^*$ iff $\forall$ $x \in E$, $\Lambda_n(x) \longrightarrow \Lambda(x)$.

$\Rightarrow$ **prop**: If E is reflexive Banach space, then weak $\Leftrightarrow$ weak*. $\square$

(e.g. $L^p(X)$ for $1<p<\infty$).

thm (Banach-Alaoglu, separable): Assume E is a separable normed vec. sp. Then bdd seqs. in E^* have weak*-convergent subseqs.

cor: In $L^p(\Omega)$ for $1<p<\infty$, bdd seqs. have weakly conv. subseqs.
$(\Omega \subseteq \mathbb{R}^n)$

pf of thm: Consider $\Lambda_n \in E^*$ s.t. $\|\Lambda_n\| \le C$ $\forall n$.

claim: If $F_1 \subseteq E$ is countable, then after replacing Λ_n w/ a subseq., we can arrange $\Lambda_n(x)$ converges $\forall$ $x \in F_1$.

pf: Let $F_1 = \{x_1, x_2, \dots\} \subseteq E$, $\|\Lambda_n\|$ bdd $\Rightarrow$ $|\Lambda_n(x_1)|$ bdd in $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$)

$\Rightarrow \exists$ subseq. $\Lambda_n^{(1)}$ of Λ_n s.t. $\Lambda_n^{(1)}(x_1)$ conv.

$\Lambda_n^{(1)}$ has a further subseq. $\Lambda_n^{(2)}$ s.t. $\Lambda_n^{(2)}(x_2)$ also conv.

Continue ... seq. of subseqs, the "diagonal" seq. $\Lambda_n^{(n)}$ has $\Lambda_n^{(n)}(x_k)$ conv. as $n \to \infty$ $\forall$ $k \in \mathbb{N}$.

claim: If $F_2 \subseteq E$ is dense & $\Lambda_n(x)$ converges $\forall$ $x \in F_2$, then $\Lambda_n(x)$ converges $\forall$ $x \in E$.

pf: Given $x \in E$, pick $x' \in F_2$, then for large n, m,

$$|\Lambda_m(x) - \Lambda_n(x)| \le \underbrace{|\Lambda_m(x) - \Lambda_m(x')|}_{\le C\|x-x'\|} + \underbrace{|\Lambda_m(x') - \Lambda_n(x')|}_{\text{small since } \Lambda_n(x') \text{ Cauchy}} + \underbrace{|\Lambda_n(x') - \Lambda_n(x)|}_{\le C\|x-x'\|}$$

$\Rightarrow$ $\Lambda_n(x)$ is also Cauchy.

E separable $\Rightarrow$ can assume $F_1 = F_2$, then $\exists$ subseq. s.t. $\lim_{n\to\infty} \Lambda_n(x)$ exists $\forall$ $x \in E$. Defn $\Lambda(x) := \lim_{n\to\infty} \Lambda_n(x)$ $\forall$ $x \in E$, check: $\Lambda \in E^*$. $\square$

rk: Weak top. on E is the smallest s.t. the fnals $\Lambda : E \to \mathbb{K}$ are continuous $\forall\ \Lambda \in E^*$.

rk: $\|\cdot\| : E \to [0,\infty)$ is not generally contin. in weak top.
e.g. in $\mathcal{H}$ w/ O-N set $\{e_n\}_{n\in\mathbb{N}}$, $e_n \to 0$ but $\|e_n\| \not\to 0$.

prop: In any normed vec. sp. E, if $x_n \to x$, then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

pf for L^p, $1 \le p < \infty$: Given $f_n \xrightarrow{L^p} f$, choose $\Lambda = \Lambda_{\frac{g}{\|g\|_{L^q}}} \in (L^p)^*$ for $g := |f|^{p-2} f \in L^q$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, so $\|\Lambda\| = 1$,

$$\Lambda(f) = \int_X \left\langle \frac{g}{\|g\|_{L^q}}, f \right\rangle d\mu = \frac{\int_X \langle g, f\rangle\, d\mu}{\|g\|_{L^q}} = \frac{\|f\|^p_{L^p}}{\|f\|^{p/q}_{L^p}} = \|f\|^{p-\frac{p}{q}}_{L^p} = \|f\|_{L^p}.$$

$|g|^q = |f|^{(p-1)q} = |f|^p$

i.e. $\exists\ \Lambda \in E^*$ s.t. $\|\Lambda\| = 1$ & $\Lambda(f) = \|f\|$.
(in a more general Banach space, this would require Hahn-Banach)

Now $\|f\| = \Lambda(f) = \lim_{n\to\infty} \Lambda(f_n) = \liminf_{n\to\infty} |\Lambda(f_n)| \overset{\|\Lambda\|=1}{\le} \liminf_{n\to\infty} \|f_n\|_{L^p}$. ☒

thm: If E is a unif. convex Banach space & $x_n \in E$ is a seq w/ $x_n \to x$ & $\|x_n\| \to \|x\|$, then $x_n \to x$.

cor: For $1 < p < \infty$, if $f_n \xrightarrow{L^p} f$ but $f_n \not\xrightarrow{L^p} f$, then $\|f_n\|_{L^p} \not\to \|f\|_{L^p}$.
e.g. if $\|f_n\|_{L^p} = c > 0$ const, then $\|f\|_{L^p} < c$.

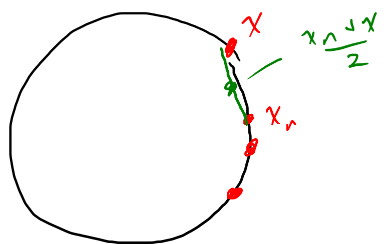
fact (see e.g. Salamon-Bühler):
If $\dim E = \infty$, $\overline{\{\|x\| = 1\}} = \{\|x\| \le 1\}$ in the weak top.

pf of thm: Assume $x \ne 0$ (otherwise easy). Norms conv. $\Rightarrow$
$x_n \to x$ iff $\frac{x_n}{\|x_n\|} \to \frac{x}{\|x\|}$; let's assume WLOG $\|x_n\| = \|x\| = 1$ $\forall n$.

Now $x_n \to x$, $\Rightarrow$ $x_n + x \to 2x$, then

$$2 = \|2x\| \le \liminf_{n\to\infty} \|x_n + x\| \le \limsup_{n\to\infty} \|x_n + x\| \overset{(\Delta\text{-ineq.})}{\le} \limsup_{n\to\infty} (\|x_n\| + \|x\|) = 2$$

$\Rightarrow \lim_{n\to\infty} \|x_n + x\| = 2$, i.e. $\lim_{n\to\infty} \left\| \frac{x_n + x}{2} \right\| = 1$



Unif. convexity $\Rightarrow \lim_{n\to\infty} \|x_n - x\| = 0$. ☒