Problem sessions ONLY at 17:15 (O. Müller)
(Wednesdays)
— see link in moodle

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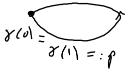
intro. to curvature

$\pi : E \to M$ a V.B., ∇ a conn.

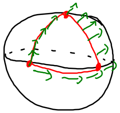
Q1: $\exists$? parallel sections on open nbhds?

Q2:



$\gamma \in M$ $P^1_\gamma : E_p \to E_p$

Is P^1_γ the identity?



Q3: In local coords., $\nabla_i \nabla_j \overset{?}{=} \nabla_j \nabla_i$?

defn: A conn. ∇ on $E \to M$ is *flat* if $\forall$ $p \in M$, $\forall$ $v \in E_p$, $\exists$ a nbhd $\mathcal{U} \subseteq M$ of p & a section $s : \mathcal{U} \to E$ s.t. $s(p) = v$ & $\nabla s \equiv 0$.

(We call s parallel, flat, horizontal, covariantly constant)

prop: ∇ is flat iff every $p \in M$ has a nbhd w/ a triv. of E s.t. ∇ looks like the trivial connection.

pf: Flat $\Rightarrow$ $\forall$ p, $\exists$ nbhd $\mathcal{U} \subseteq M$ of p & a frame $e_1, \ldots, e_m$ for E over $\mathcal{U}$ s.t. $\nabla e_i \equiv 0$ $\forall$ i.

Then for any $s \in \Gamma(E)$, on $\mathcal{U}$, $s = s^i e_i$ satisfies

$$\nabla_X s = ds^i(X)\, e_i + s^i \nabla_X e_i = ds^i(X)\, e_i$$

$\Rightarrow$ ∇ is the trivial conn. w.r.t. this frame. ☒

rk: ∇ flat $\Rightarrow$ answer to (2) is yes $\forall$ loops in some nbhd of any pt.

integrable frames

Q: When does a local frame for TM come from a chart?

thm: For a frame $(X_1, \dots, X_n)$ for TM on some set $U \overset{\text{open}}{\subseteq} M$, following are equivalent:

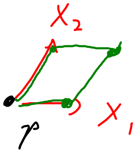
(1) $\forall\, p \in U$, $\exists$ a chart (U', x) with $p \in U' \subseteq U$ s.t. $X_i = \frac{\partial}{\partial x^i}$ on U' for $i = 1, \dots, n$.

(2) $[X_i, X_j] \equiv 0 \quad \forall\, i, j$.

pf: $\Rightarrow$ clear since $[\partial_i, \partial_j] \equiv 0$ always.

$\Leftarrow$: Given $p \in M$, write down inverse of the desired chart as $\psi : (-\varepsilon, \varepsilon)^n \longrightarrow M : (t^1, \dots, t^n) \longmapsto$

$$\varphi_{X_1}^{t_1} \circ \varphi_{X_2}^{t_2} \circ \dots \circ \varphi_{X_n}^{t_n}(p).$$



$[X_i, X_j] \equiv 0 \Rightarrow$ the flows all commute

$\Rightarrow$ can rewrite s.t. for any $j \in \{1, \dots, n\}$, $\varphi_{X_j}^{t_j}$ appears first, then $\frac{\partial \psi}{\partial t^j}(t^1, \dots, t^n) = X_j\big(\psi(t^1, \dots, t^n)\big)$

characterizing flat connections on $E \to M$ / integrable distributions

observe: a section $s: \mathcal{U} \longrightarrow E$ has image
$\mathcal{U} \overset{\text{open}}{\subseteq} M$

$\Sigma := s(\mathcal{U}) \subseteq E$ a submfd s.t. $\forall p \in \mathcal{U}$,

$\Sigma \cap E_p$ = one pt.

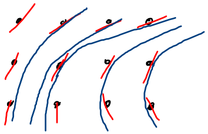
Then $\nabla s \equiv 0 \iff \forall v \in \Sigma, \quad T_v \Sigma = H_v E.$

defn: a k-dimensional distribution on the mfd. M is a rank k subbundle $\xi \subseteq TM$.

An integral submfd for this distribution is a k-dim. submfd $\Sigma \subseteq M$ s.t. $T_p\Sigma = \xi_p \ \forall p \in \Sigma$.

We call ξ integrable if every pt. is contained in an integral submfd.

ex: If $k = 1$, ξ is always integrable.



rk: For $k \geq 2$, much less obvious!

prop: a conn. ∇ on $E \to M$ is flat $\iff$ the horiz. subbundl $HE \subseteq TE$ is an integrable distr. on E.

$\boxtimes$

ex: $VE \subseteq TE$ is an integrable dist. on E.

Integr. submfds = fibers of $E \to M$.

Frobenius integrability thm:

A distr. $\xi \subseteq TM$ on M is integrable $\Longleftrightarrow$ $\forall$ $X, Y \in \Gamma(\xi) \subseteq \mathfrak{X}(M)$, $[X,Y]$ is also in $\Gamma(\xi)$.

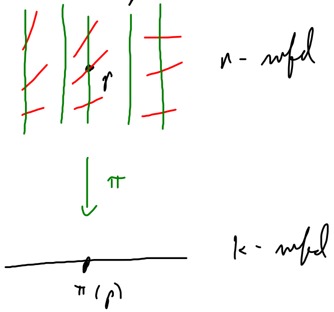
pf of $\Rightarrow$: Given $p \in M$ & an integ. submfd $\Sigma \subseteq M$ containing p, $X, Y \in \Gamma(\xi)$ then restrict to vec. flds. on Σ,

$$[X|_\Sigma, Y|_\Sigma] = [X,Y]|_\Sigma$$ is also a vec. fld on Σ

$$\Longrightarrow [X,Y](p) \in T_p\Sigma = \xi_p .$$ ▨

For converse, consider special case where $E \to M$ a vec. bndl, $\xi := HE \subseteq TE$ a horizontal subbndl, i.e. $TE = VE \oplus HE$.

rk: Every distr. on any mfd. is locally diffeo to this case:



The horiz. subbdl $HE \to TE$ determines projections

$$TE \xrightarrow{V} VE, \qquad TE \xrightarrow{H} HE$$

& horizontal lift isos.

$$\mathrm{Hor}_v : T_{\pi(v)} M \xrightarrow{\cong} H_v E \subseteq T_v E$$

For any vec. fld $X \in \mathfrak{X}(M)$, defn. a vec. fld $X^h \in \mathfrak{X}(E)$ by $X^h(v) := \mathrm{Hor}_v\big(X(\pi(v))\big)$.

EX 1: $\forall\ X \in \mathfrak{X}(M)$ & $f \in C^\infty(M)$, $\mathcal{L}_{X^h}(f \circ \pi) = \mathcal{L}_X f \circ \pi$.

EX 2: For any $\eta, \xi \in \Gamma(HE) \subseteq \mathfrak{X}(E)$,

$$\mathcal{L}_\eta(f \circ \pi) = \mathcal{L}_\xi(f \circ \pi)\ \ \forall\ f \in C^\infty(M) \iff \eta \equiv \xi.$$

Lemma: $\forall\ X, Y \in \mathfrak{X}(M)$, $[X,Y]^h = H([X^h, Y^h])$.

pf: For any $f \in C^\infty(M)$,

$$\mathcal{L}_{H([X^h,Y^h])}(f \circ \pi) = \mathcal{L}_{[X^h,Y^h]}(f \circ \pi)$$

(since difference is $\mathcal{L}_Z(f\circ\pi)$ for some vertical vec. fld Z)

$$= \mathcal{L}_{X^h}\mathcal{L}_{Y^h}(f\circ\pi) - \mathcal{L}_{Y^h}\mathcal{L}_{X^h}(f\circ\pi)$$

$$\overset{(EX1)}{=} \mathcal{L}_{X^h}(\mathcal{L}_Y f \circ \pi) - \mathcal{L}_{Y^h}(\mathcal{L}_X f \circ \pi)$$

$$= (\mathcal{L}_X f_Y f)\circ\pi - (\mathcal{L}_Y \mathcal{L}_X f)\circ\pi = (\mathcal{L}_{[X,Y]} f)\circ\pi$$

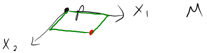
$$= \mathcal{L}_{[X,Y]^h}(f\circ\pi),$$ EX 2 ⟹ done. ▨

pf of Frobenius ⟸: We assume $HE \subseteq TE$ a horiz. subbndl s.t. $\forall\ \eta, \xi \in \Gamma(HE)$, $[\eta, \xi] \in \Gamma(HE)$.

Given p, choose a frame $X_1, \dots, X_n$ on a nbhd $\mathcal{U} \subseteq M$ of p s.t. $[X_i, X_j] \equiv 0$. Then $[X_i^h, X_j^h] = H([X_i^h, X_j^h])$

$$\overset{(\text{lemma})}{=} [X_i, X_j]^h = 0.$$

Now for any $v \in E_p$, an integral submfd through v can be parametrized by $\psi(t^1, \dots, t^n) = \varphi^{t^1}_{X_1^h} \circ \dots \circ \varphi^{t^n}_{X_n^h}(v)$.



▨

defn: $\widehat{\Omega}_K : \mathfrak{X}(E) \times \mathfrak{X}(E) \longrightarrow \Gamma(VE)$

$$\widehat{\Omega}_K(\eta, \xi) := -V([H(\eta), H(\xi)])$$

Frobenius ⟹ ∇ is flat iff $\widehat{\Omega}_K = 0$.

EX: $\widehat{\Omega}_K$ is C^∞-linear, i.e. it is a "bndl-valued 2-form" $\widehat{\Omega}_K \in \Omega^2(E, VE)$.