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Differentialgeometrie I
WiSe 2021–22

Problem Set 12

To be discussed: 9.02.2022

Problem 1
Prove that in local coordinates on some open subset $\mathcal{U}$ of a pseudo-Riemannian 2-manifold (Σ, g), the Riemann tensor $R \in \Gamma(T^1_3\Sigma)$ is determined on $\mathcal{U}$ by the component R_{1122}.

Problem 2
The *Ricci tensor* $\mathrm{Ric} \in \Gamma(T^0_2 M)$ can be defined on a Riemannian n-manifold (M, g) by

$$\mathrm{Ric}(Y, Z) := \sum_{j=1}^{n} \langle e_j, R(e_j, Y)Z\rangle = \sum_{j=1}^{n} \mathrm{Riem}(e_j, e_j, Y, Z) \in \mathbb{R}, \qquad \text{for } Y, Z \in T_pM, \tag{1}$$

where $e_1, \ldots, e_n$ is any choice of orthonormal basis of T_pM at a point $p \in M$. The following sequence of exercises is aimed showing that this definition does not depend on the choice of basis $e_1, \ldots, e_n$, and also generalizing it to the pseudo-Riemannian case:

(a) Use the Einstein summation convention to give a one-line proof that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ for all pairs of square matrices $\mathbf{A}$ and $\mathbf{B}$.

(b) Define $\mathrm{tr}(A)$ for any linear map $A : V \to V$ on a finite-dimensional vector space V. (There is only one reasonable definition. Show that it is independent of choices.)

(c) Show that $\mathrm{Ric}(Y, Z)$ according to (1) is the trace of the linear map $T_pM \to T_pM : X \mapsto R(X, Y)Z$.

(d) If (M, g) is a pseudo-Riemannian manifold, then the trace in part (c) can be taken as a definition of Ric, but the formula (1) is not quite right if g is indefinite. Fix it.

(e) Show that in local coordinates, the components $R_{k\ell}$ of Ric are given by $R_{k\ell} = R^i{}_{ik\ell}$.

The trick used above to turn a type $(1, 3)$ tensor into a type $(0, 2)$ tensor is called *contraction.* One can contract further to define the *scalar curvature*, a function $\mathrm{Scal} : M \to \mathbb{R}$ that, on a Riemannian manifold (M, g), can be written as

$$\mathrm{Scal}(p) := \sum_{j=1}^{n} \mathrm{Ric}(e_j, e_j) = \sum_{j,k=1}^{n} \mathrm{Riem}(e_j, e_j, e_k, e_k) \in \mathbb{R}, \tag{2}$$

where $e_1, \ldots, e_n \in T_pM$ again denotes an orthonormal basis.

(f) Show that (2) is independent of the choice of orthonormal basis $e_1, \ldots, e_n \in T_pM$ by reinterpreting it as a contraction (i.e. trace) of the tensor $\mathrm{Ric}^\sharp \in \Gamma(T^1_1 M)$ defined via the relation $\langle Y, \mathrm{Ric}^\sharp(Z)\rangle = \mathrm{Ric}(Y, Z)$.

(g) Taking the trace in part (f) as a general definition of $\mathrm{Scal} : M \to \mathbb{R}$ for pseudo-Riemannian manifolds (M, g), rewrite (2) so that it is also valid when g is indefinite.

(h) Show that in local coordinates, $\mathrm{Scal} = g^{k\ell} R^i{}_{ik\ell}$.

(i) Prove that if $\dim M = 2$, then $R \in \Gamma(T^1_3 M)$ is fully determined by $\mathrm{Scal} : M \to \mathbb{R}$. *Hint: Use Problem 1 in well-chosen coordinates near a given point $p \in M$.*

(j) Show that on a Riemannian 2-manifold, Scal is twice the Gaussian curvature K_G.

Problem 3
Prove: A closed surface Σ in Euclidean $\mathbb{R}^3$ cannot have $K_G \leqslant 0$ everywhere.
Hint: For some $R > 0$, Σ must lie inside the closed ball of radius R and touch its boundary tangentially at some point.

Problem 4
Prove that for the hyperboloid $H := \{x^2 + y^2 - z^2 = 1\}$ in Euclidean $\mathbb{R}^3$, $K_G(p) = -\dfrac{1}{|p|^4}$.
Hint: This can be a horrible computation, but it doesn't have to be. For instance, there are some obvious isometries that make it sufficient to consider a point of the form $(r, 0, z) \in H$ with $r^2 - z^2 = 1$, which is the intersection of the smooth curves $\alpha(t) = (\cosh t, 0, \sinh t)$ and $\beta(t) = (r \cos t, r \sin t, z)$ in H. Since H is a level set of $f(x, y, z) = x^2 + y^2 - z^2$, there is a unit normal vector field of the form $\nu = g \cdot \nabla f$ for some function $g : H \to (0, \infty)$. Try to convince yourself without any calculations that the curves α and β are tangent to the principal directions. Then consider the following: if you know $\gamma(t) \in H$ satisfies $\frac{d}{dt}\nu(\gamma(t)) = \lambda\dot{\gamma}(t)$ for some $\lambda \in \mathbb{R}$, what happens if you take the inner product of both sides with $\dot{\gamma}(t)$? Write $\nu = g \cdot \nabla f$ and use this observation to compute the two principal curvatures at $(r, 0, z)$. You will need to write down the function g for this, but you should not need to differentiate it.
Final remark: It's also possible there's an easier way to do this that I haven't thought of.

Problem 5
In Problem 5 on the take-home midterm, we established that the geodesic curves on the Poincaré half-plane $(\mathbb{H}, h)$, defined as $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with $h := \frac{1}{y^2}(dx^2 + dy^2)$, are the vertical lines and the semicircles that meet the x-axis orthogonally.

(a) Write down the Riemannian volume form on $(\mathbb{H}, h)$, and show that any region of the form $[a, b] \times [c, \infty) \subset \mathbb{H}$ for $-\infty < a < b < \infty$ and $c > 0$ has finite area, while regions of the form $[a, b] \times (0, c] \subset \mathbb{H}$ have infinite area.

(b) By drawing pictures, show that the sum of the angles in a geodesic triangle in $(\mathbb{H}, h)$ can be arbitrarily small. (By "geodesic triangle" we mean a compact region in $\mathbb{H}$ bounded by three geodesic segments.)

(c) Pretend for the moment that you don't know $(\mathbb{H}, h)$ is isometric to the hyperbolic plane, and compute its Gaussian curvature.
Note: Since $(\mathbb{H}, h)$ is not given as a submanifold of $\mathbb{R}^3$, one should define $K_G : \mathbb{H} \to \mathbb{R}$ in this case as the unique function satisfying $R(X, Y)Z = -K_G \,\mathrm{dvol}(X, Y)JZ$.

Problem 6
Suppose $\pi : E \to M$ is a complex line bundle with a bundle metric $\langle\ ,\ \rangle$, so it has structure group $\mathrm{U}(1)$. Since $\mathrm{U}(1)$ is abelian, we showed in lecture that any metric connection ∇ on $E \to M$ gives rise to a globally-defined imaginary-valued curvature 2-form $F \in \Omega^2(M, \mathfrak{u}(1)) = \Omega^2(M, i\mathbb{R})$, which matches dA_α on $\mathcal{U}_\alpha \subset M$ for any $\mathrm{U}(1)$-compatible local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$ with connection 1-form $A_\alpha \in \Omega^1(M, \mathfrak{u}(1))$. Show that if $\widehat{\nabla}$ is a second metric connection on $E \to M$ with curvature 2-form $\widehat{F} \in \Omega^2(M, \mathfrak{u}(1))$, then $\widehat{F} - F$ is exact. The cohomology class $c_1(E) := \left[-\frac{1}{2\pi i}F\right] \in H^2_{\mathrm{dR}}(M)$ is thus independent of the choice of connection; it is known as the *first Chern class* of E.
Hint: The two connections differ by a bililinear bundle map $B : TM \oplus E \to E$ satisfying $B(X, v) = \widehat{\nabla}_X v - \nabla_X v$. Reinterpret this as an $\mathrm{End}(E)$-valued 1-form, and then as a complex-valued 1-form, using the fact that fibers of E are 1-dimensional.