**PROBLEM SET 10**
**To be discussed: 10.01.2024**

## Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Using product CW-complexes, describe a cell decomposition of the torus $\mathbb{T}^n$ for every $n \in \mathbb{N}$ such that the cellular boundary map vanishes.[1] Use this to prove that for any axiomatic homology theory $h_*$ with coefficient group $G$,
$$h_k(\mathbb{T}^n) \cong G^{\binom{n}{k}}$$
for all $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$.

2. As in Problem 1, describe a cell decomposition of $\Sigma_g \times S^1$ for which the cellular boundary map vanishes. One can use this to compute $H_*(\Sigma_g \times S^1; \mathbb{Z})$, but I would like a more concrete description of $H_2(\Sigma_g \times S^1; \mathbb{Z})$ in particular, meaning the following: show that $H_2(\Sigma_g \times S^1; \mathbb{Z})$ is a free abelian group generated by $2g + 1$ submanifolds of the form
$$\gamma_1 \times S^1, \ldots, \gamma_{2g} \times S^1 \quad \text{and} \quad \Sigma_g \times \{\text{const}\} \subset \Sigma_g \times S^1,$$
where $\gamma_1, \ldots, \gamma_{2g}$ are closed 1-dimensional submanifolds of $\Sigma_g$. Here, the homology class represented by a closed orientable 2-dimensional submanifold $S \subset \Sigma_g \times S^1$ is understood to mean $i_*[S] \in H_2(\Sigma_g \times S^1; \mathbb{Z})$, with $i : S \hookrightarrow \Sigma_g \times S^1$ denoting the inclusion and $[S] \in H_2(S; \mathbb{Z}) \cong \mathbb{Z}$ a chosen generator.
*Hint: You can choose your cell decomposition so that each of these submanifolds is presentable as a subcomplex, and the inclusion is then a cellular map. There is no need to talk about triangulations.*

3. Recall that the topology of a CW-complex $X$ is defined normally as the strongest topology for which the characteristic maps of all cells $\Phi_\alpha : \mathbb{D}^k \to X$ are continuous. Given another CW-complex $Y$, let $Z$ and $Z'$ denote the set $X \times Y$ with two (potentially) different topologies: we assign to $Z$ the product topology, and to $Z'$ the topology of the product CW-complex induced by the cell decompositions of $X$ and $Y$.
   (a) Prove that every open set in $Z$ is also an open set in $Z'$, i.e. the identity map $Z' \to Z$ is continuous. *Remark: In general, the identity map $Z' \to Z$ might not be a homeomorphism!*[2]
   (b) Prove that the identity map $Z' \to Z$ is a homeomorphism if $X$ and $Y$ are both compact.
   (c) Prove that a subset $K \subset Z$ is compact if and only if it is compact in $Z'$, and the two subspace topologies induced by $Z$ and $Z'$ on $K$ are the same. Deduce from this that $Z$ and $Z'$ have the same singular homology groups.

4. Assume $\Sigma$ is a compact oriented surface with nonempty boundary. Prove $H_2(\Sigma \times S^2; \mathbb{Z}) \cong \mathbb{Z}$. Can you describe (e.g. in terms of a closed oriented 2-dimensional submanifold) a specific homology class that generates $H_2(\Sigma \times S^2; \mathbb{Z})$?
*Hint: The classification of surfaces gives a countable list of possibilities of what $\Sigma$ could be. Show that $\Sigma$ is homotopy equivalent to a 1-dimensional cell complex, draw whatever conclusions you can from that, and then use the Künneth formula.*

5. In Problem Set 6 #2, you computed $H_*(\mathbb{RP}^2; \mathbb{Z})$: the nontrivial groups are $H_0(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}_2$.

[1] In both this and Problem 2, it is possible to apply the Künneth formula, but not necessary, due to the fact that the cellular boundary map vanishes.

[2] This is easily said, but writing down actual counterexamples is surprisingly difficult, e.g. it turns out that they must involve uncountable many cells. For more on such bizarre issues, see `https://arxiv.org/abs/1710.05296`.

(a) Use this and the Künneth formula to prove

$$H_n(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n = 1, \\ \mathbb{Z}_2 & \text{for } n = 2, \\ \mathbb{Z}_2 & \text{for } n = 3, \\ 0 & \text{for all other } n \in \mathbb{Z}. \end{cases}$$

(b) For $n = 0, 1, 2$, describe explicit oriented submanifolds that represent homology classes generating $H_n(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$.
*Hint: As in Problem 2, it is easiest to describe specific homology classes via submanifolds that are also subcomplexes in a cell decomposition, so that the inclusion is a cellular map.*

(c) Describe an explicit cell decomposition of $\mathbb{RP}^2 \times \mathbb{RP}^2$ and a specific element of the corresponding cellular chain complex that represents the nontrivial element of $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$.
*Remark: It is not so easy to describe this one as a submanifold, so do not get frustrated if you can't.*

6. The goal of this exercise is to prove the associativity of the cross product on singular homology, i.e.

$$(A \times B) \times C = A \times (B \times C) \in H_*(X \times Y \times Z)$$

for all $A \in H_*(X)$, $B \in H_*(Y)$ and $C \in H_*(Z)$, with coefficients in a commutative ring $R$.

(a) Use acyclic models to prove that for triples of spaces $X, Y, Z$, all natural chain maps

$$\Psi : C_*(X; \mathbb{Z}) \otimes C_*(Y; \mathbb{Z}) \otimes C_*(Z; \mathbb{Z}) \to C_*(X \times Y \times Z; \mathbb{Z})$$

that act on 0-chains by $\Psi(x \otimes y \otimes z) = (x, y, z)$ are chain homotopic.
*Remark: The statement implicitly assumes that there is a well-defined notion of the tensor product of three chain complexes, which of course is true since there is a canonical chain isomorphism between $(C_*(X) \otimes C_*(Y)) \otimes C_*(Z)$ and $C_*(X) \otimes (C_*(Y) \otimes C_*(Z))$. Right?*

(b) Given $A \in H_*(X)$, $B \in H_*(Y)$ and $C \in H_*(Z)$ with coefficients in a commutative ring, show that the products $(A \times B) \times C$ and $A \times (B \times C) \in H_*(X \times Y \times Z)$ can each be expressed via natural chain maps as in part (a), and conclude that they are identical.