PROBLEM SET 13
To be discussed: 14.02.2024

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Use the nonsingularity of the intersection form to establish the following isomorphisms of $\mathbb{Z}$-graded rings.
   (a) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 2$
   (b) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$

   Hint for both: You need to show in both cases that if $\alpha \in H^k(M;R) \cong R$ and $\beta \in H^\ell(M;R) \cong R$ are generators with $k + \ell \leqslant \dim M$, then $\alpha \cup \beta$ is a generator of $H^{k+\ell}(M;R) \cong R$. Start with the case $k + \ell = \dim M$, and then deduce the general case from this using the fact that for each $m = 0,\dots,n$, there are natural inclusions $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$ and $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$ which are also cellular maps. (The induced homomorphisms on cohomology should be easy to compute.)

   Now use the obvious (cellular) inclusions $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ and $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ to compute:
   (c) $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$ with $|\alpha| = 2$
   (d) $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ with $|\alpha| = 1$

2. A closed and connected 3-manifold M is called a **rational homology sphere** if $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. Prove that this condition holds if and only if M is orientable and $H_1(M;\mathbb{Z})$ is torsion.

3. In lecture we defined the **compactly supported cohomology** $H^*_c(X)$ of a space X via the direct limit
$$H^k_c(X) := \varinjlim \left\{ H^k(X \,|\, K) \right\}_K ,$$
   where $H^k(X \,|\, K)$ is an abbreviation for $H^k(X, X\backslash K)$, and K ranges over the set of all compact subsets of X. These subsets are ordered by inclusion $K \subset K' \subset X$ and form a direct system via the maps $H^k(X \,|\, K) \to H^k(X \,|\, K')$ induced by inclusions $(X, X\backslash K') \hookrightarrow (X, X\backslash K)$.
   (a) Letting G denote the (arbitrarily chosen) coefficient group, construct a canonical isomorphism between $H^*_c(X)$ and the homology of the subcomplex $C^*_c(X) \subset C^*(X)$ consisting of every cochain $\varphi : C_k(X) \to G$ for which there exists a compact subset $K \subset X$ with $\varphi|_{C_k(X\backslash K)} = 0$. (Note that K may depend on φ).
   Hint: The properties underlying the fact that direct limits preserve the homologies of chain complexes are also relevant here.
   (b) Prove that $H^n_c(\mathbb{R}^n)$ is isomorphic to the coefficient group, and $H^k_c(\mathbb{R}^n) = 0$ for all $k \neq n$.
   Hint: You can restrict your attention to compact sets $K \subset \mathbb{R}^n$ that are disks of arbitrarily large radius. (Why?)
   (c) Recall that a continuous map $f : X \to Y$ is called **proper**[1] if for every compact set $K \subset Y$, $f^{-1}(K) \subset X$ is also compact. Show that proper maps $f : X \to Y$ induce homomorphisms $f^* : H^*_c(Y) \to H^*_c(X)$, making H^*_c into a contravariant functor on the category of topological spaces with morphisms defined as proper maps.

[1] This definition of properness is standard in differential geometry, though for certain purposes, it is sometimes considered an inadequate definition if considering spaces that are not assumed second countable and Hausdorff (the general definition of properness is then a slightly stronger condition). As far as I can tell, it's still an adequate definition for the present exercise.

(d) Deduce from part (c) that H_c^* is a topological invariant, i.e. $H_c^*(X)$ and $H_c^*(Y)$ are isomorphic whenever X and Y are homeomorphic. Give an example showing that this need not be true if X and Y are only homotopy equivalent.

(e) In contrast to part (c), show that H_c^* does *not* define a functor on the usual category of topological spaces with morphisms defined to be continuous (but not necessarily proper) maps.
Hint: Think about maps between $\mathbb{R}^n$ and the one-point space.

(f) We say that two proper maps $f, g : X \to Y$ are **properly homotopic** if there exists a homotopy $h : I \times X \to Y$ between them that is also a proper map. Show that under this assumption, the induced maps $f^*, g^* : H_c^*(Y) \to H_c^*(X)$ in part (c) are identical.[2] In other words, H_c^* defines a contravariant functor on the category whose objects are topological spaces and whose morphisms are proper homotopy classes of proper maps.
Hint: If you express H_c^* as the homology of the cochain complex in part (a), then your main task is to show that the usual dualized chain homotopy $h^* : C^*(Y) \to C^{*-1}(X)$ induced by h sends $C_c^*(Y)$ to $C_c^{*-1}(X)$. Alternatively, it should also be possible to work with the definition of H_c^* via direct limits and use the universal property to characterize the maps $f^*, g^* : H_c^*(Y) \to H_c^*(X)$. In both approaches, you may find it helpful to know that every compact subset of $I \times X$ is contained in $I \times K'$ for some compact $K' \subset X$.[3]

4. In the following, all (co)homology groups use coefficients in a fixed ring R, and M is a compact n-manifold with boundary endowed with an R-orientation, which determines a relative fundamental class $[M] \in H_n(M, \partial M)$. The relative cap product with $[M]$ then gives rise to two natural maps

$$\mathrm{PD} : H^k(M, \partial M) \to H_{n-k}(M), \tag{1}$$

$$\mathrm{PD} : H^k(M) \to H_{n-k}(M, \partial M), \tag{2}$$

both defined by $\mathrm{PD}(\varphi) = \varphi \cap [M]$. The theorem that both are isomorphisms is sometimes called *Lefschetz duality.*

(a) Find a cofinal family of compact subsets $A \subset \mathring{M} := M \backslash \partial M$ such that the natural maps in the diagram

$$H^*(\mathring{M} \,|\, A) \longleftarrow H^*(M \,|\, A) \longrightarrow H^*(M, \partial M)$$

are isomorphisms. Use this to find a natural isomorphism

$$H_c^*(M) \cong H^*(M, \partial M),$$

and deduce via the duality map $H_c^k(\mathring{M}) \to H_{n-k}(\mathring{M})$ that (1) is an isomorphism.

(b) Show that the long exact sequenes of the pair $(M, \partial M)$ in homology and cohomology fit together into a commutative diagram of the form

$$\begin{tikzcd}
\ldots \arrow[r] & H^k(M,\partial M) \arrow[r, "j^*"] \arrow[d, "\cdot\cap[M]"] & H^k(M) \arrow[r, "i^*"] \arrow[d, "\cdot\cap[M]"] & H^k(\partial M) \arrow[r, "\delta^*"] \arrow[d, "\cdot\cap[\partial M]"] & H^{k+1}(M,\partial M) \arrow[r] \arrow[d, "\cdot\cap[M]"] & \ldots \\
\ldots \arrow[r] & H_{n-k}(M) \arrow[r, "j_*"] & H_{n-k}(M,\partial M) \arrow[r, "\partial_*"] & H_{n-k-1}(\partial M) \arrow[r, "i_*"] & H_{n-k-1}(M) \arrow[r] & \ldots,
\end{tikzcd}$$

where $i : \partial M \hookrightarrow M$ and $j : (M, \varnothing) \hookrightarrow (M, \partial M)$ denote the usual inclusions.
Hint: Work directly with chains and cochains. Problem Set 12 #4 implies the (intuitively unsurprising) fact that if $c \in C_n(M)$ is a relative n-cycle representing $[M] \in H_n(M, \partial M)$, then the $(n-1)$-cycle $\partial c \in C_{n-1}(\partial M)$ represents $[\partial M] \in H_{n-1}(\partial M)$.

(c) Deduce from the diagram in part (b) that the map in (2) is also an isomorphism.

(d) If M has a triangulation, interpret the isomorphisms (1) and (2) in terms of the dual cell decomposition.

[2]The last time I assigned this problem in a previous year, I added the assumption that X is Hausdorff and locally compact, but I can no longer convince myself that those assumptions are necessary. Let me know if you think they are.

[3]This seems to be the detail on which I once thought it was important to assume X is Hausdorff and locally compact, but actually it appears to be easier than that.