PROBLEM SET 8
To be discussed: 13.12.2023

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next week's lectures, as the problems will often serve as mental preparation for the lecture material. Solutions will be discussed in the Übung.

1. Let Σ_g denote the closed oriented surface of genus g and suppose $\Gamma \subset \Sigma_g$ is a finite subset with $m \geqslant 0$ points. Prove $\chi(\Sigma_g \backslash \Gamma) = 2 - 2g - m$.
   Hint: Do not compute any homology, but use the fact that singular homology (and therefore also the Euler characteristic) only depends on the homotopy type of the space.

2. (a) Suppose (X, A) and (Y, B) are compact CW-pairs and there exists a homeomorphism $\Phi : A \to B$ that is also a cellular map. Show that
   $$\chi(X \cup_\Phi Y) = \chi(X) + \chi(Y) - \chi(A),$$
   where $X \cup_\Phi Y$ is defined by gluing X and Y together such that A and B are identified via Φ.
   (b) What does part (a) imply if X and Y are manifolds of even dimension $2n$ and A and B are disjoint unions of spheres S^{2n-1}, e.g. when X and Y are compact surfaces with A and B as their boundaries? Use this to give an alternative proof of the formula in Problem 1.
   (c) Given closed n-dimensional triangulable manifolds M and N, how are $\chi(M)$ and $\chi(N)$ related to $\chi(M \# N)$? (The answer should look slightly different depending on whether n is even or odd. Check in the case $n = 2$ that it is consistent with $\chi(\Sigma_g) = 2 - 2g$.)
   Hint: The simplest way I can think of to remove a ball from a triangulated n-manifold is to remove the interior of an n-simplex.

   Remark (and also hint): By now, you should be starting to notice that you do not know any way of constructing a closed odd-dimensional manifold M with $\chi(M) \neq 0$. When we study Poincaré duality, we will see that this is not a coincidence.

3. The standard $(n+1)$-simplex $\Delta^{n+1} \subset \mathbb{R}^{n+2}$ has boundary homeomorphic to S^n, and this boundary comes with an obvious triangulation. Use this observation to give a topological proof of the combinatorial formula
   $$\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+1} = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

4. (a) Prove the following statement that was mentioned in lecture: if X is a compact CW-complex and $\pi : Y \to X$ is a covering map of finite degree $d \in \mathbb{N}$, then Y admits a cell decomposition such that $Y^n = \pi^{-1}(X^n)$ for every $n \geqslant 0$ and each n-cell $e^n_\alpha \subset X^n$ with characteristic map $\Phi_\alpha : \mathbb{D}^n \to X$ corresponds to exactly d cells in Y^n whose characteristic maps are lifts of Φ_α to the cover. Deduce from this the formula
   $$\chi(Y) = d \cdot \chi(X).$$
   Hint: Maps $\mathbb{D}^n \to X$ can always be lifted to the cover since $\mathbb{D}^n$ is simply connected.
   (b) If $\pi : Y \to \Sigma_g$ is a covering map of degree $d \in \mathbb{N}$, and $g \geqslant 1$, identify the topological type of Y.
   Hint: Y must be a compact oriented 2-dimensional manifold. (Why?) The orientability condition here can be expressed either in terms of oriented triangulations, or using the notion of orientations of surfaces that we defined in Lecture 20 last semester.

5. A space X is called **acyclic** if $\widetilde{H}_*(X; \mathbb{K}) = 0$ for some field $\mathbb{K}$. For example, contractible spaces are clearly acyclic, but the freedom to choose the coefficient field allows for some more interesting examples.

(a) Prove that if X is a compact polyhedron that is acyclic, then every map $f : X \to X$ has a fixed point.

(b) Prove that $\mathbb{RP}^n$ is acyclic for every even n.

6. Suppose $A \subset X$ is a subspace with inclusion $i : A \hookrightarrow X$ and a retraction $r : X \to A$, and X has finite-dimensional homology with coefficients in some field $\mathbb{K}$.

(a) Prove that $H_*(A; \mathbb{K})$ is also finite dimensional.

(b) Prove that for any map $f : A \to A$, the induced maps $f_* : H_n(A; \mathbb{K}) \to H_n(A; \mathbb{K})$ and $(i \circ f \circ r)_* : H_n(X; \mathbb{K}) \to H_n(X; \mathbb{K})$ for every $n \in \mathbb{Z}$ satisfy

$$\operatorname{tr}(f_*) = \operatorname{tr}((i \circ f \circ r)_*).$$

Hint: Write $(i \circ f \circ r)_* = i_* f_* r_*$ *as the composition of two homomorphisms* $f_* r_* : H_n(X; \mathbb{K}) \to H_n(A; \mathbb{K})$ *and* $i_* : H_n(A; \mathbb{K}) \to H_n(X; \mathbb{K})$, *and recall the formula* $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.

(c) Deduce that the Lefschetz numbers of $f : A \to A$ and $i \circ f \circ r : X \to X$ are the same.

(d) Conclude that if X is a compact polyhedron, then A also satisfies the Lefschetz fixed point theorem.

Remark: This is the main part of the argument needed for extending the Lefschetz fixed point theorem from compact polyhedra to arbitrary compact Euclidean neighborhood retracts, a class of spaces that includes all compact topological manifolds.