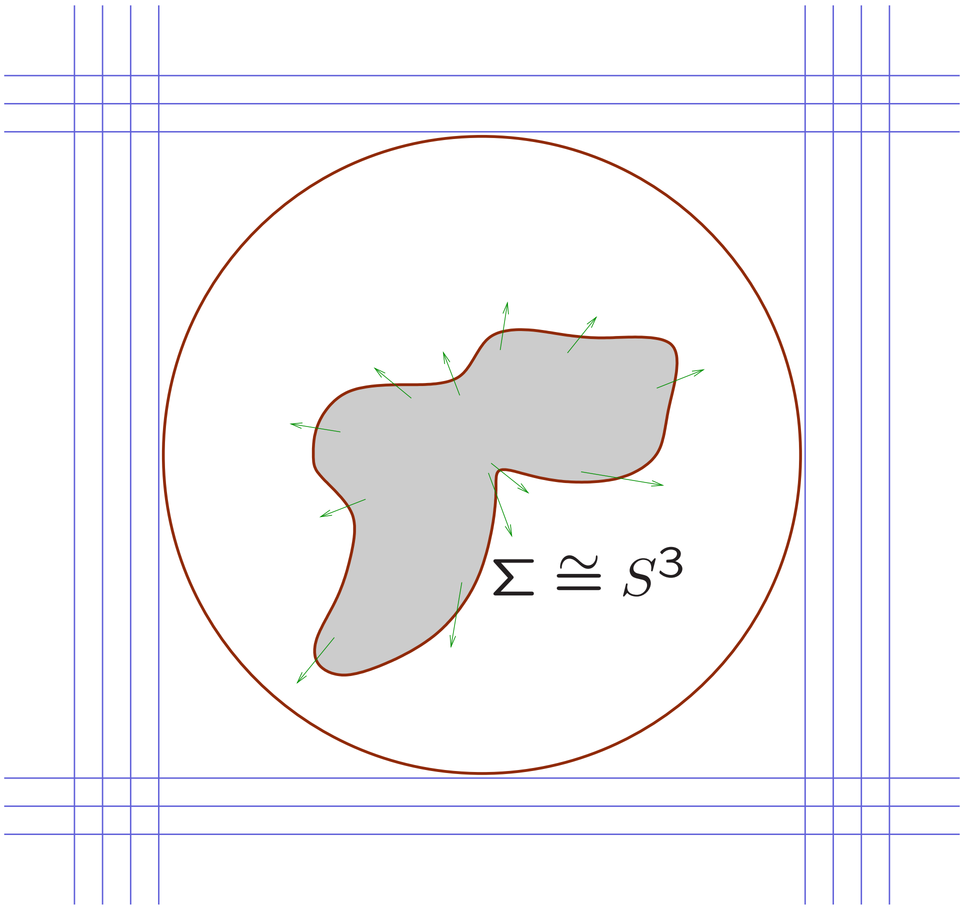
What can have a 3-sphere as its boundary, and why should you ask Isaac Newton?



Chris Wendl

University College London

Talk for the UCL AdM Maths Society, 3rd March, 2014

Slides available at:

http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks

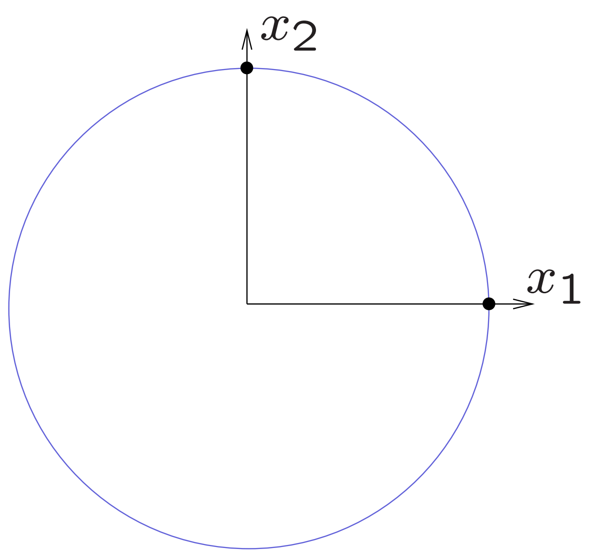
PART 1: Differential topology

The n-dimensional sphere

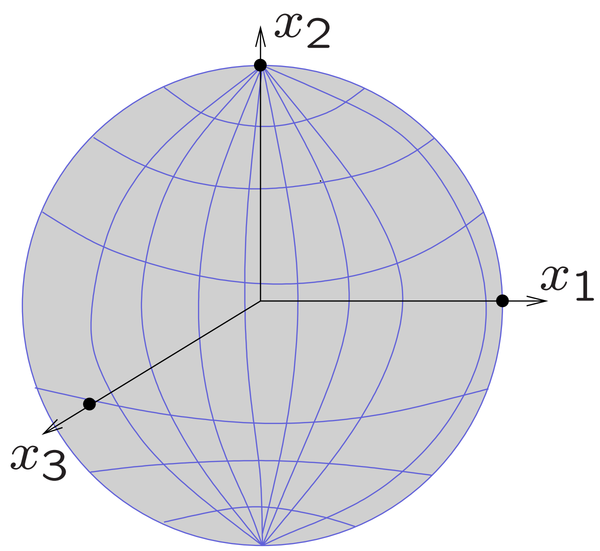
$$S^n := \left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots x_{n+1}^2 = 1\right\}$$

$=$ boundary of the $(n+1)$-dimensional ball

$$B^{n+1} := \left\{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_1^2 + \dots x_{n+1}^2 \leq 1\right\}.$$



$$S^1 = \partial B^2$$



$$S^2 = \partial B^3$$

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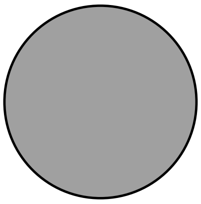

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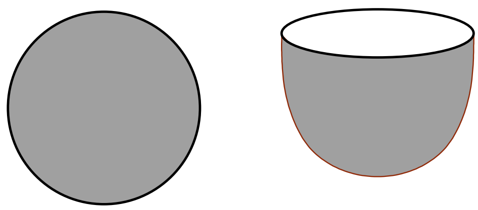

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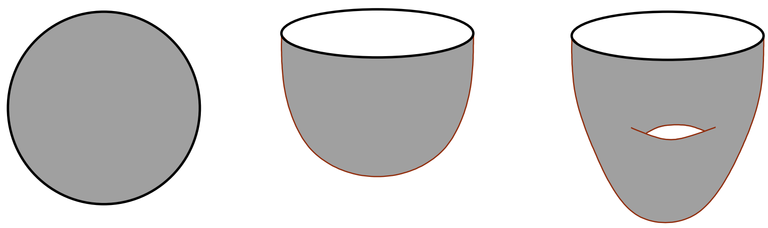

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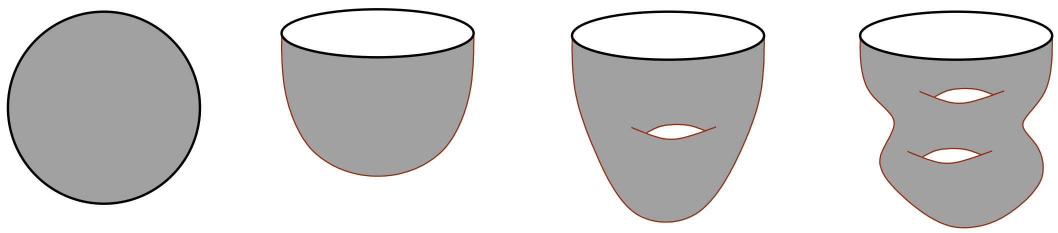

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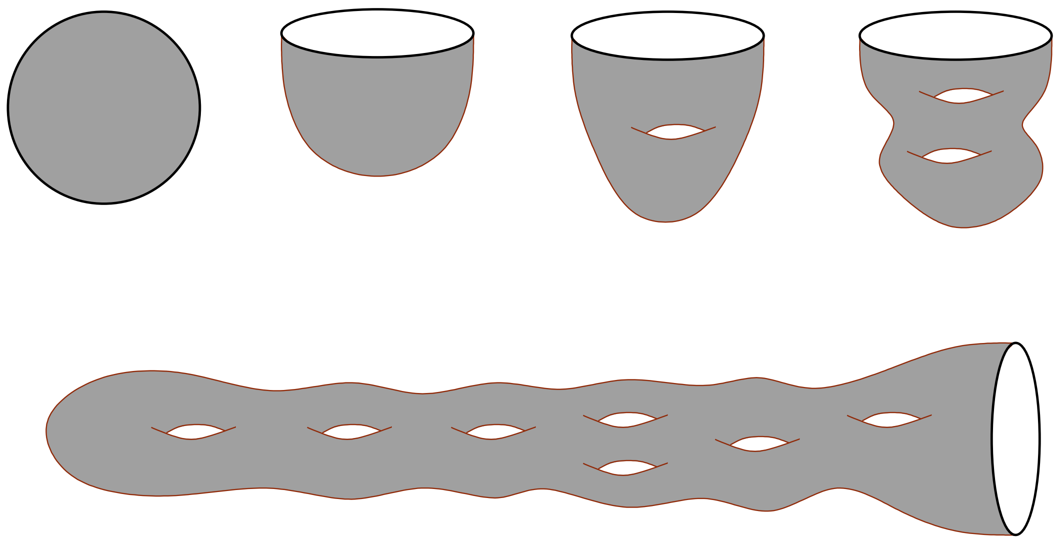

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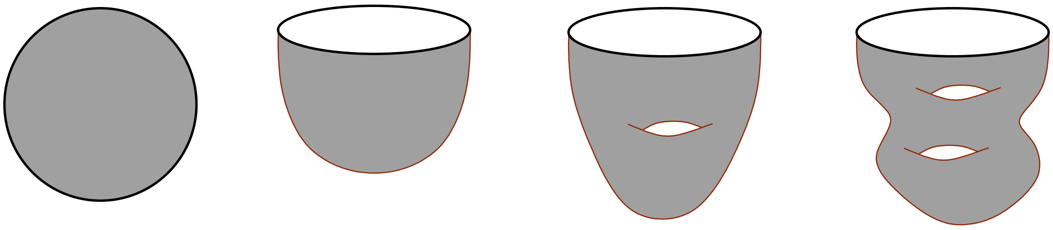

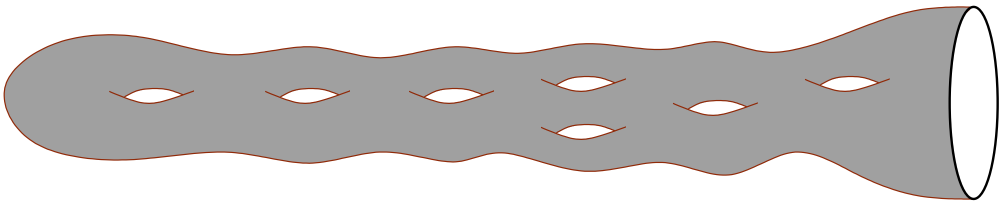

Definition

Suppose $M \subset \mathbb{R}^N$ is a subset, $\mathcal{U} \subset M$ is open.

An n-dimensional coordinate chart on $\mathcal{U}$ is a set of functions $x_1, \ldots, x_n : \mathcal{U} \to \mathbb{R}$ such that the mapping

$$(x_1, \ldots, x_n) : \mathcal{U} \to \mathbb{R}^n$$

is bijective onto some open subset of $\mathbb{R}^n$.

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Proposition
If $M \cong M'$, then they have the same dimension, and M compact $\Leftrightarrow$ M' compact.

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- The universe?
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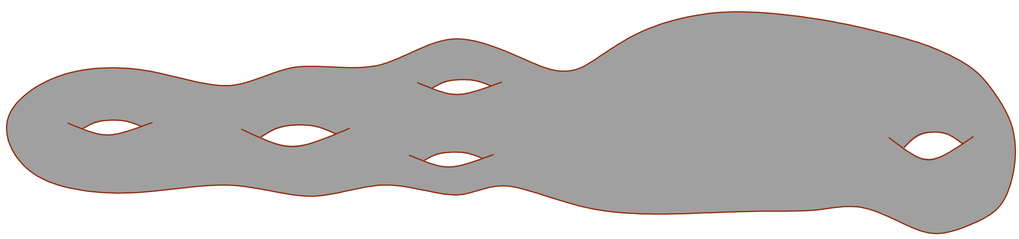

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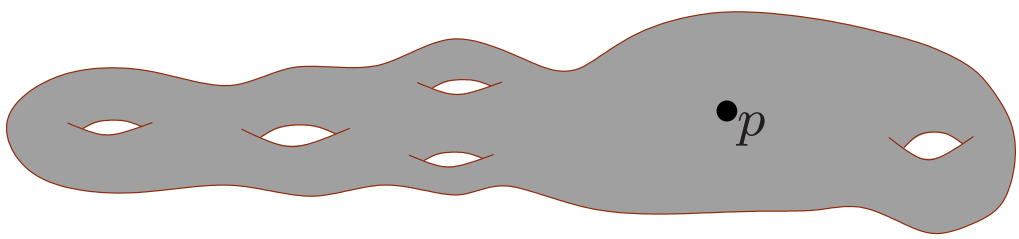

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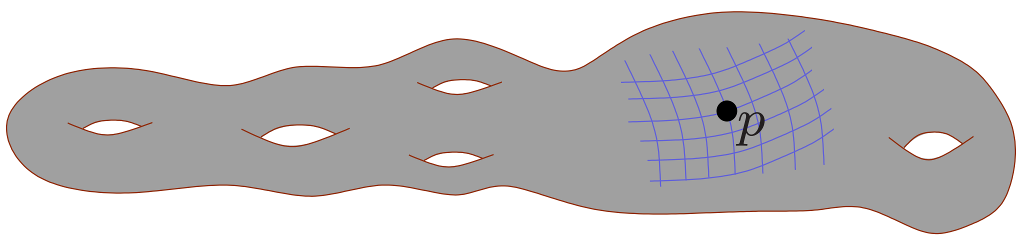

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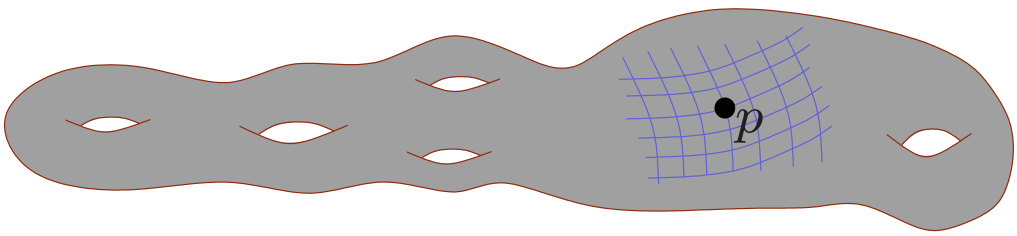

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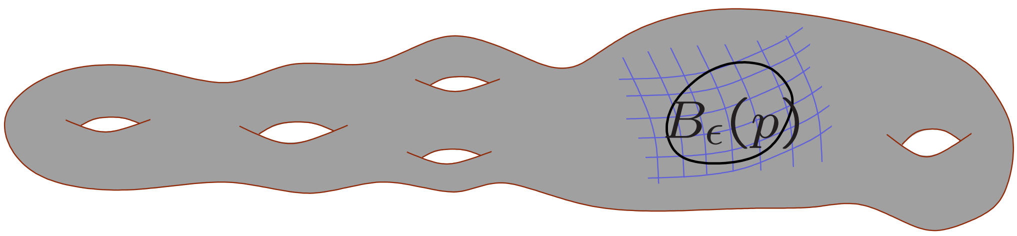

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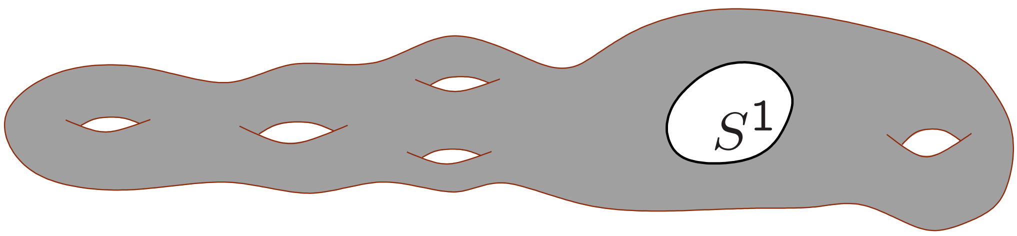

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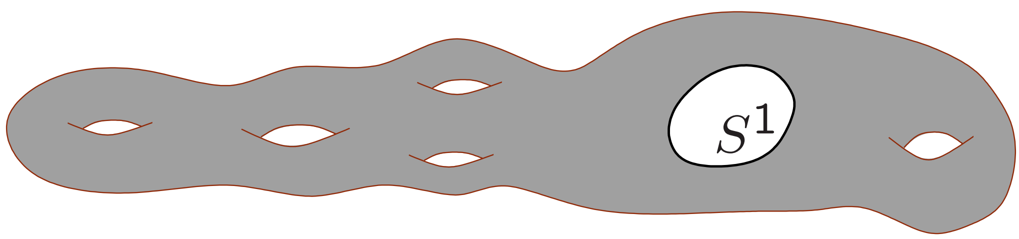

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Conclusion: We asked the wrong question.
The answer was too easy!

PART 2: Dynamics

Newton (18th century):

A system of particles moving with n degrees of freedom is described by a path in $\mathbb{R}^n$,

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a system of n second-order ordinary differential equations (ODE). Its total energy

$$E = \sum_{j=1}^{n} \frac{1}{2} m_j \dot{q}_j^2 + V(\mathbf{q})$$

is conserved, i.e. $\frac{dE}{dt} = 0$.

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and Newton's second-order system becomes **Hamilton's** (first-order!) **equations**:

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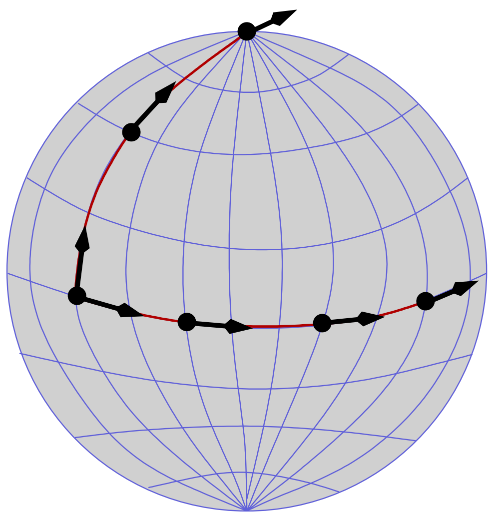
Idea: To study motion of systems satisfying constraints, we can treat $(\mathbf{q}, \mathbf{p})$ as local coordinates of a point moving in a manifold.



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- Not symplectic: S^{2n} for $n > 1$
  (can prove using *de Rham cohomology*)

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Theorem (Rabinowitz-Weinstein '78)
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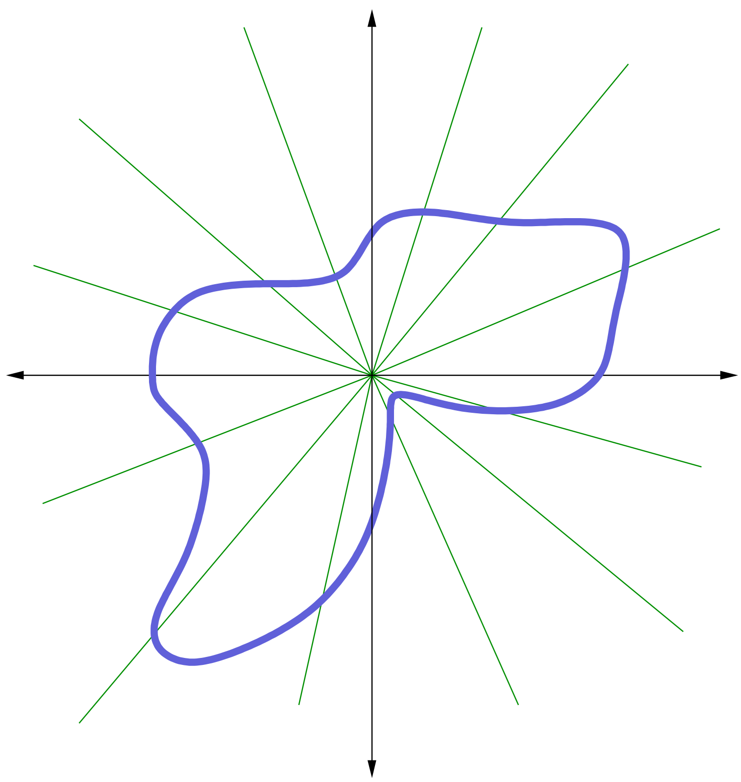

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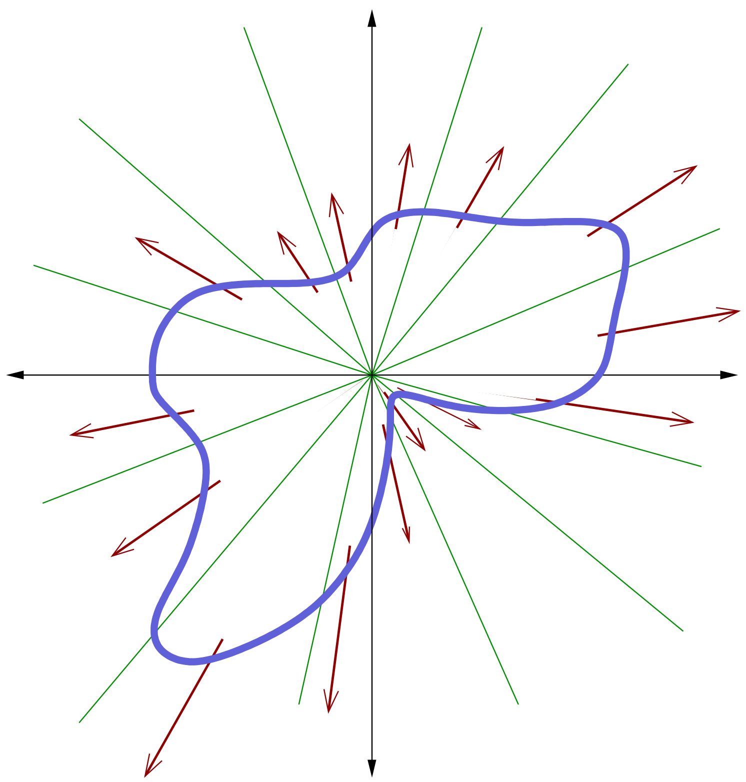

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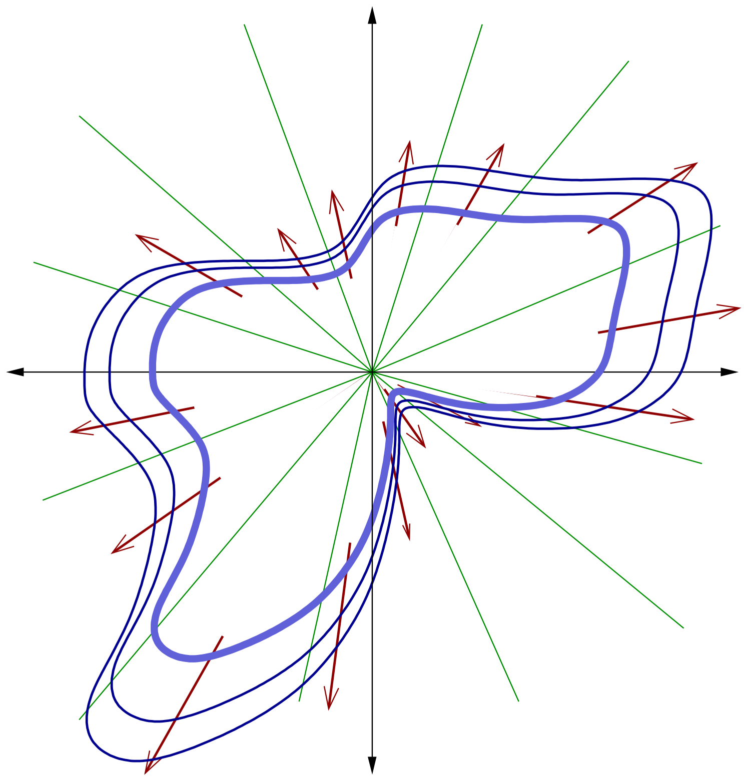

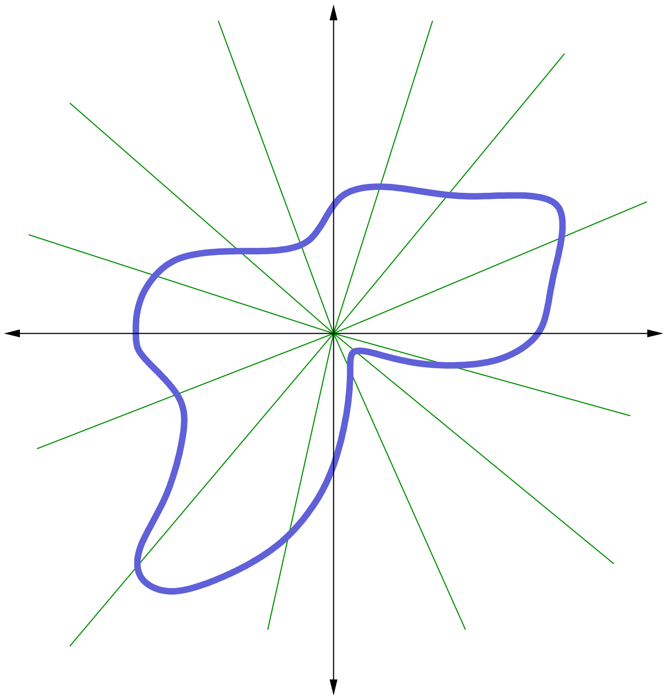

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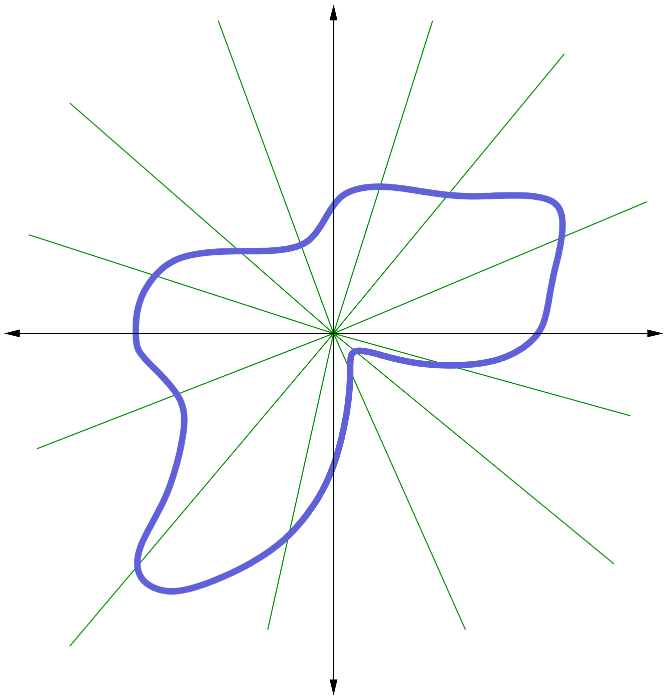

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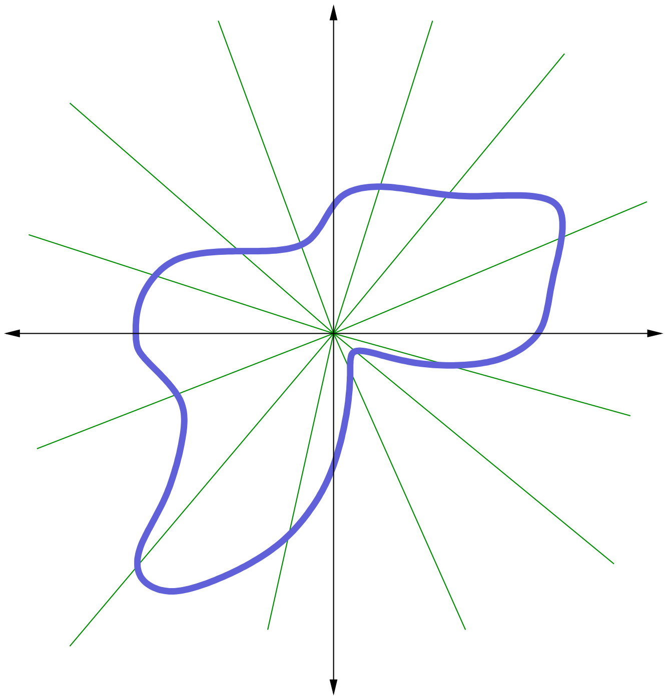

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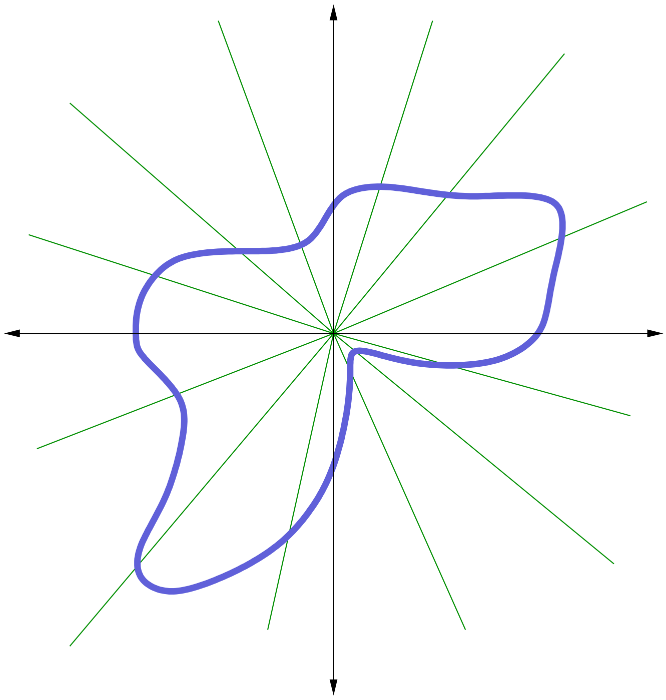

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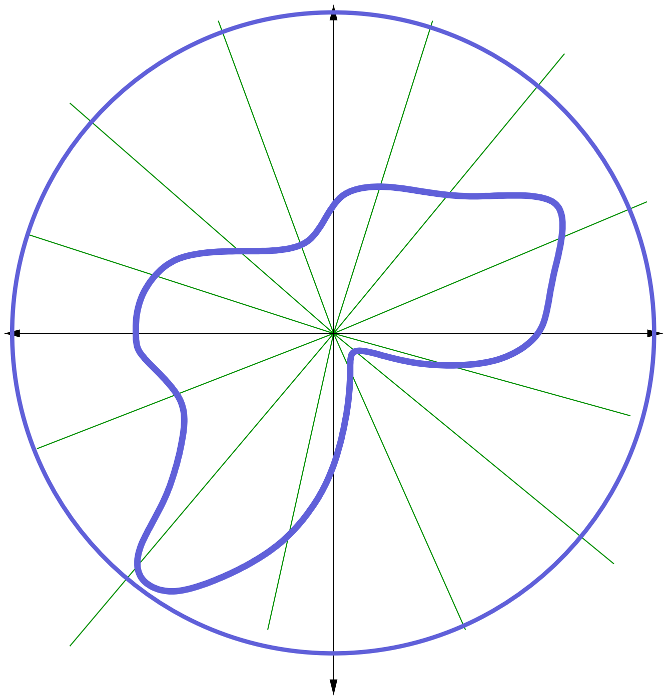

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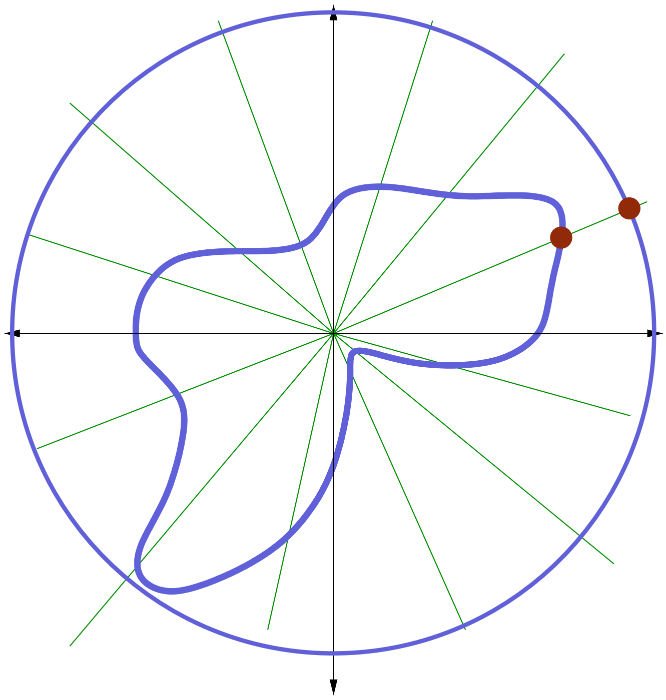

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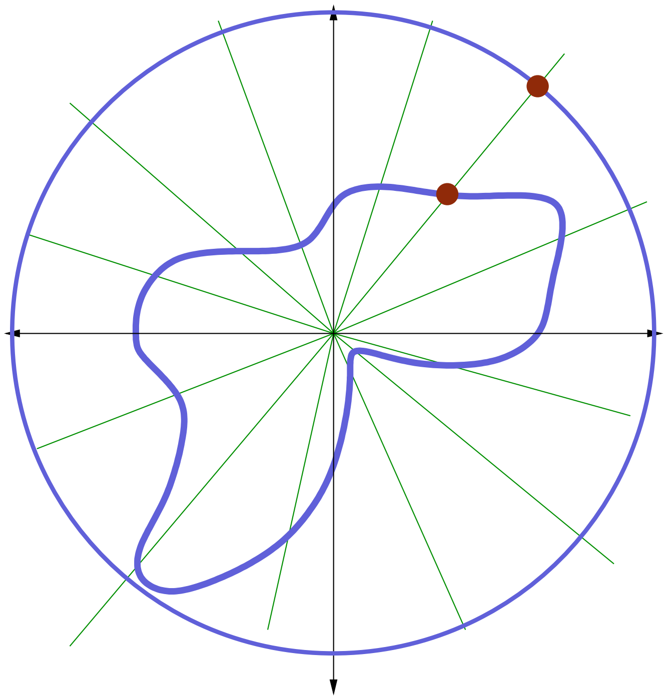

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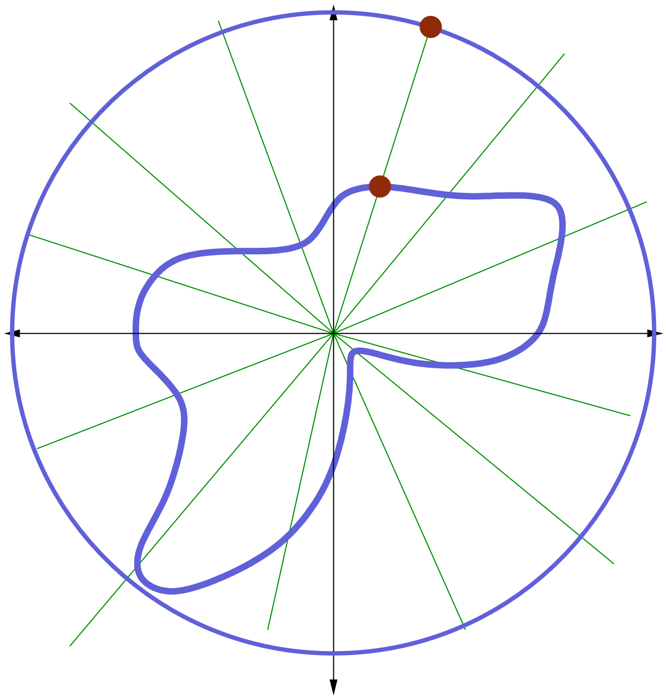

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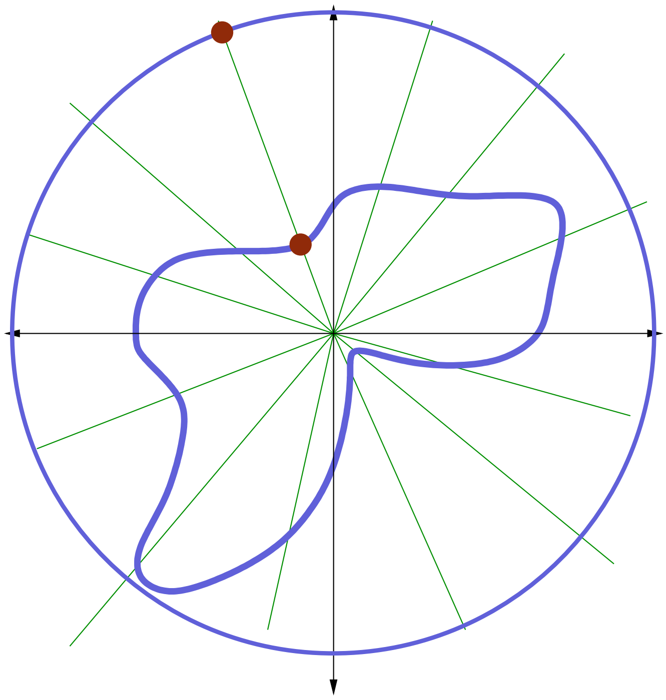

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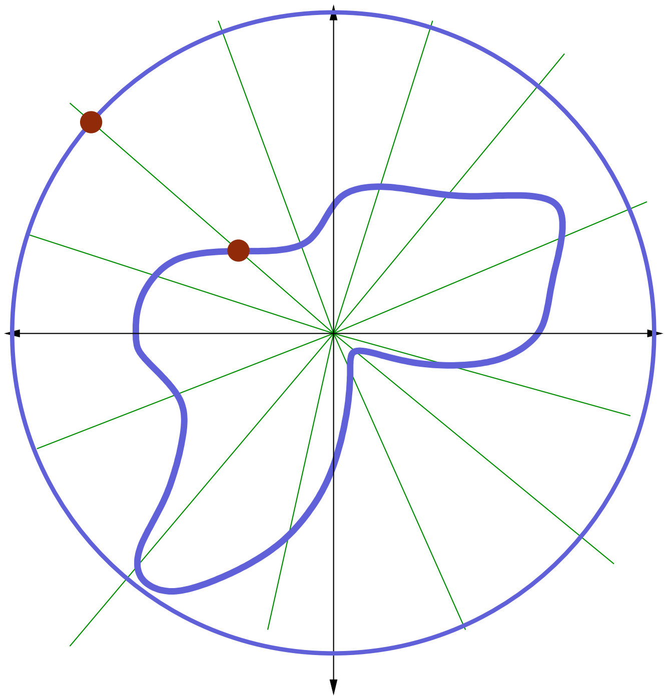

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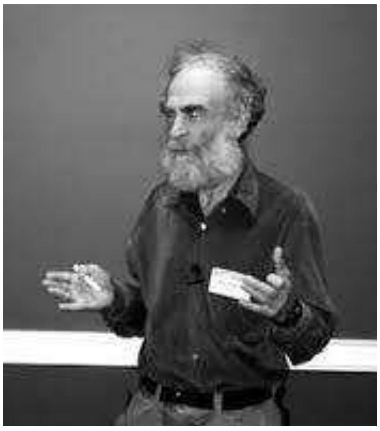

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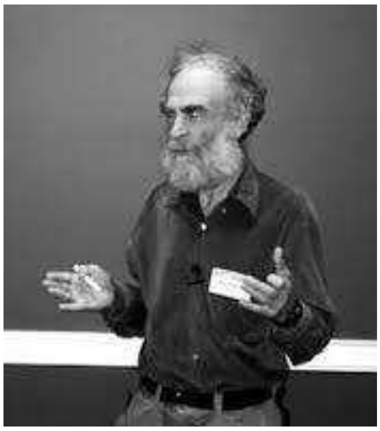

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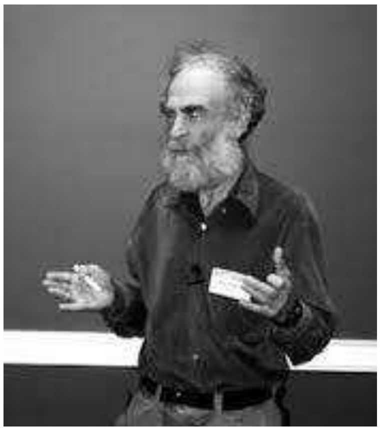

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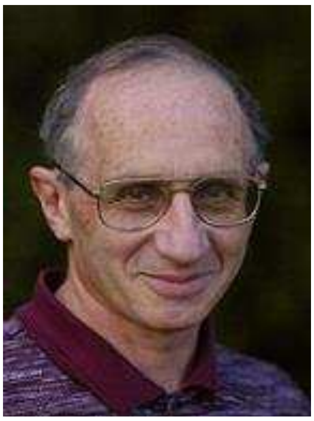

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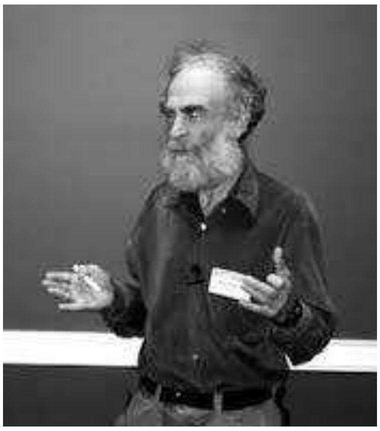

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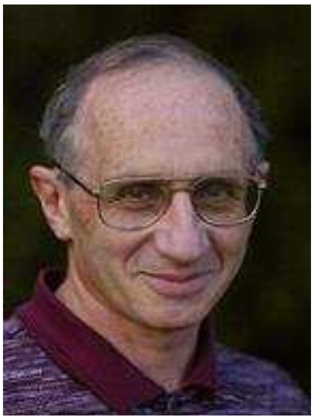

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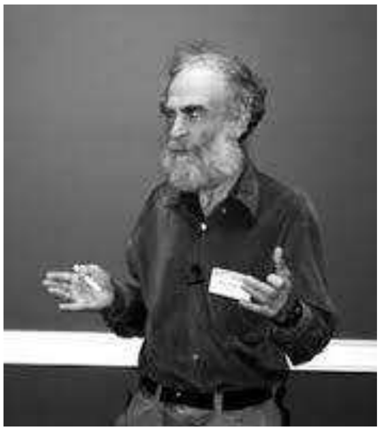

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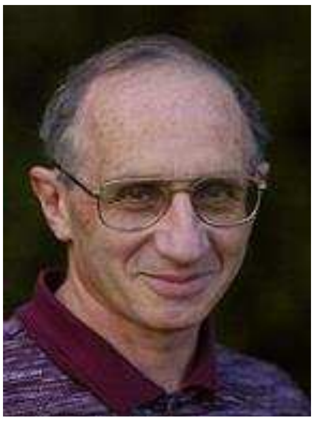

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Some preparation from complex analysis

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Fundamental lemma:
Every symplectic manifold admits a special class of *compatible* almost complex structures.

A decomposition of the standard $B^4 \subset \mathbb{R}^4$

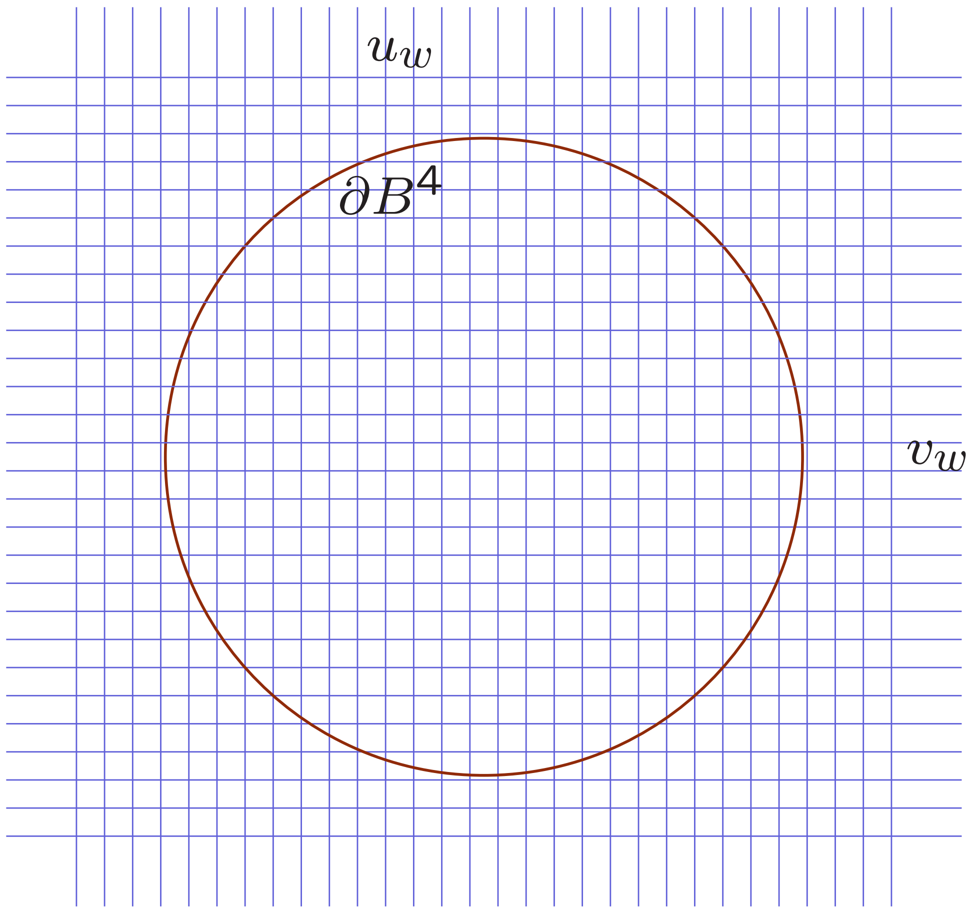
Identify $\mathbb{R}^4 = \mathbb{C}^2$ and define

$$J_0(p) := i \qquad \text{for all } p \in \mathbb{R}^4.$$

We now see two obvious 2-dimensional families of pseudoholomorphic curves:

$$u_w : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (z, w) \qquad \text{for } w \in \mathbb{C},$$
$$v_w : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (w, z) \qquad \text{for } w \in \mathbb{C}.$$

They form two transverse *foliations* of $\mathbb{C}^2$:



Proof of the main theorem

Given $\partial M = \Sigma \subset \mathbb{R}^4$ star-shaped, construct a symplectic manifold W by *surgery*:

(1) Remove from $\mathbb{R}^4 = \mathbb{C}^2$ the interior of Σ;
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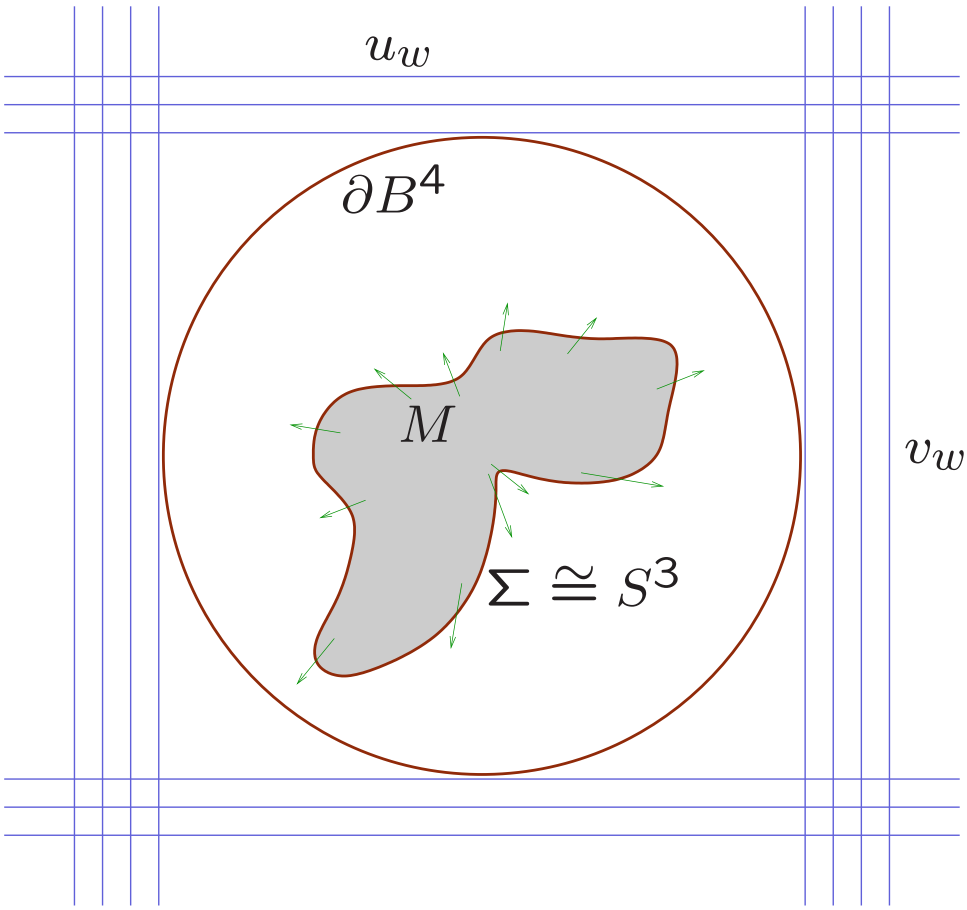

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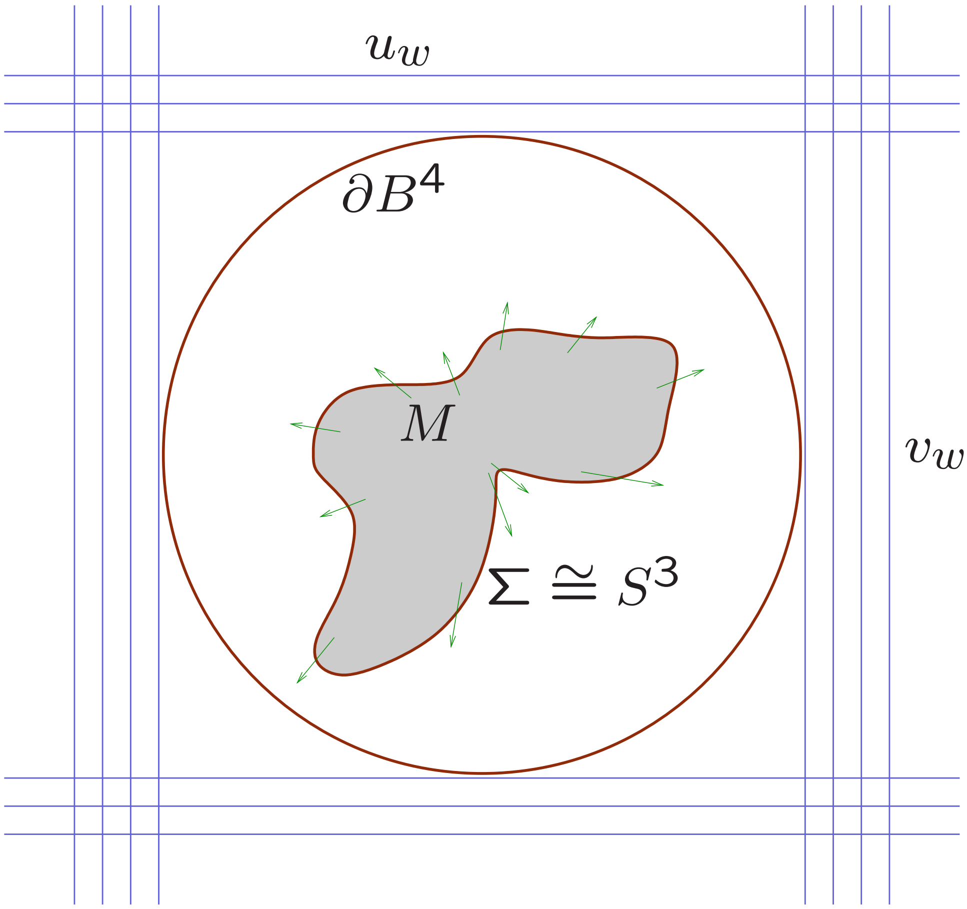

Choose J matching J_0 outside a large ball. Then for large $|w|$, the pseudoholomorphic curves u_w and v_w also exist in W.

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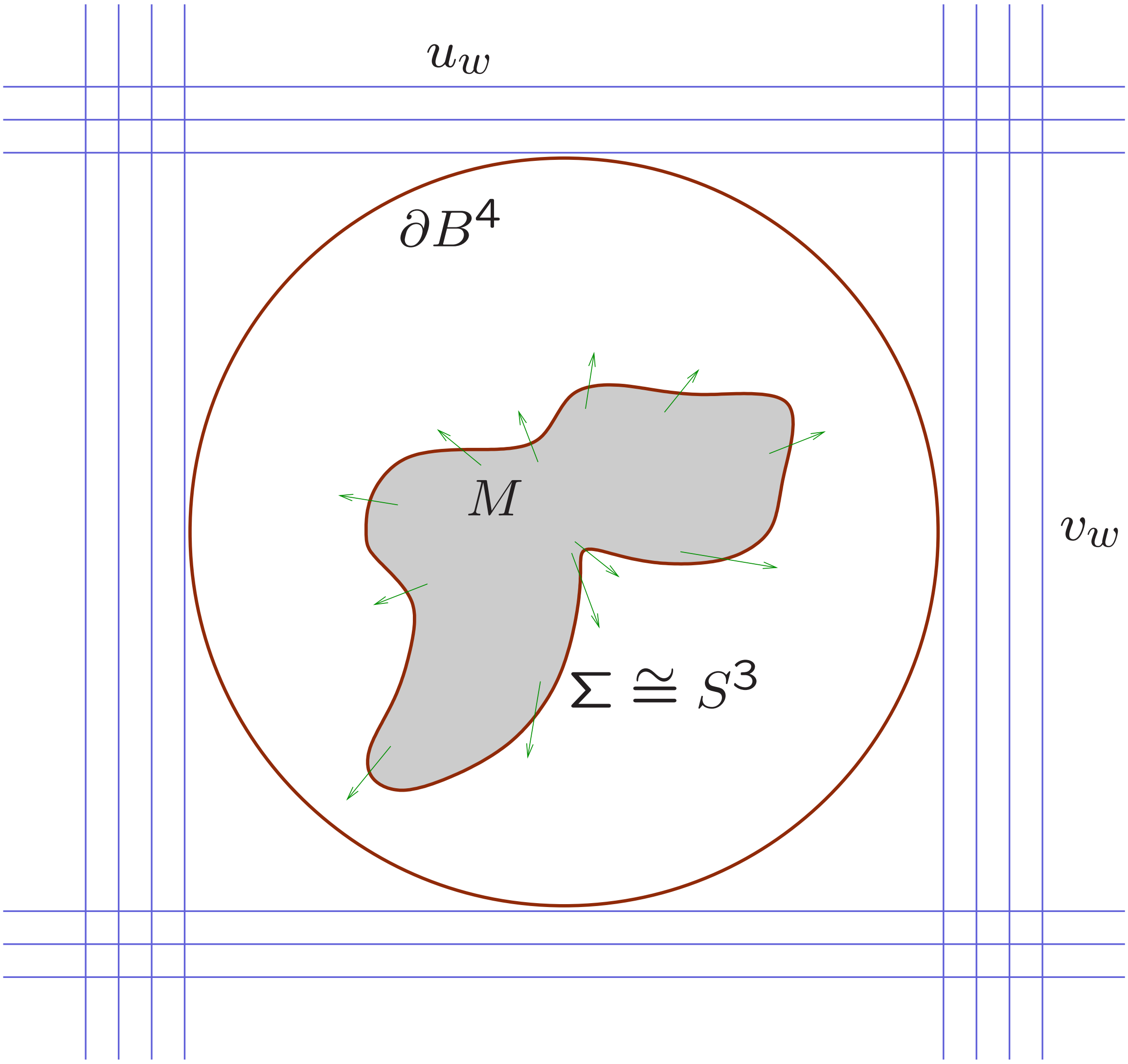
These lemmas concern general properties of solution spaces.

One can prove them without knowing how to *solve* the PDE, and without knowing what M actually is!

Final step: "turn on the machine. . . "



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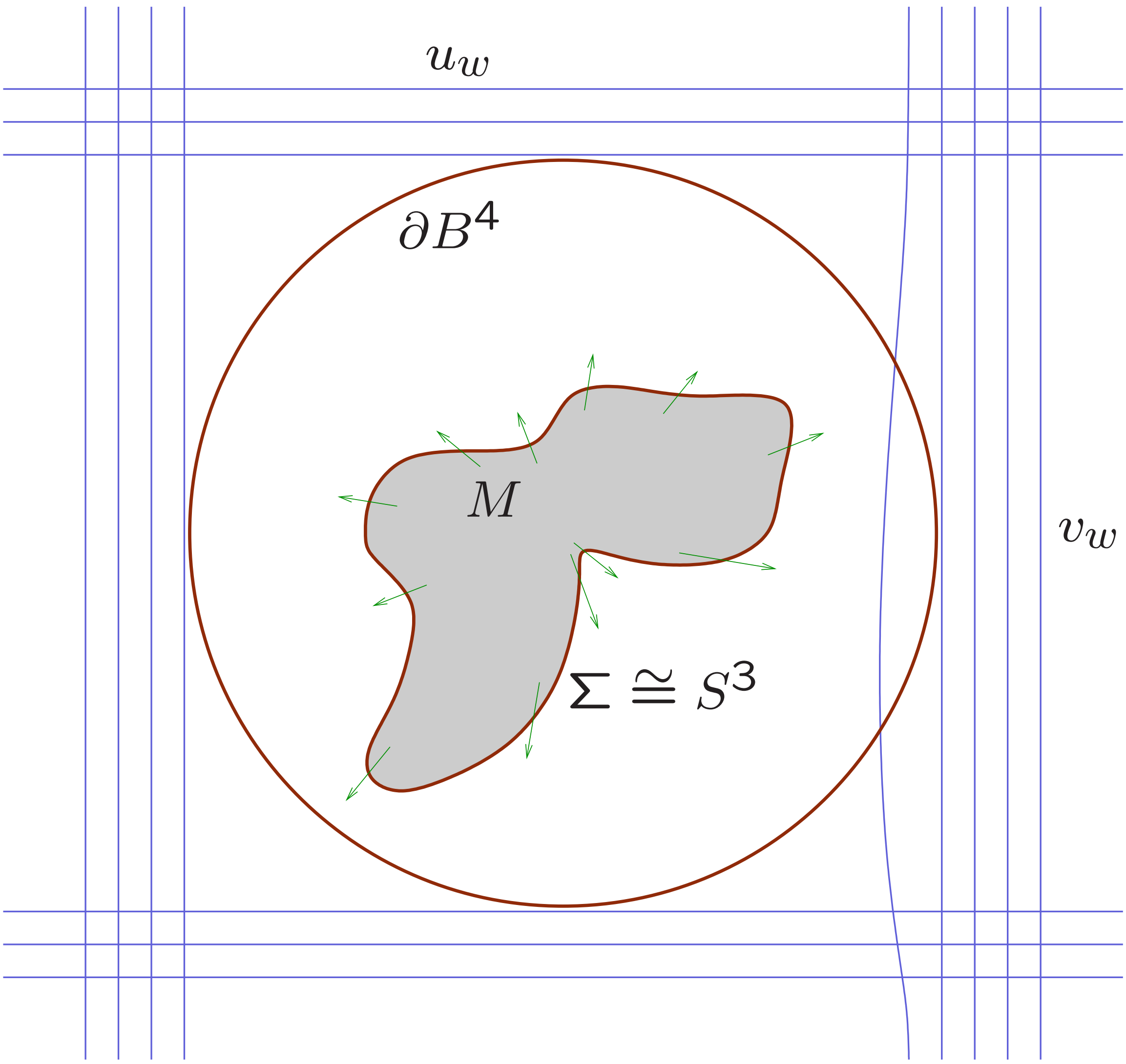

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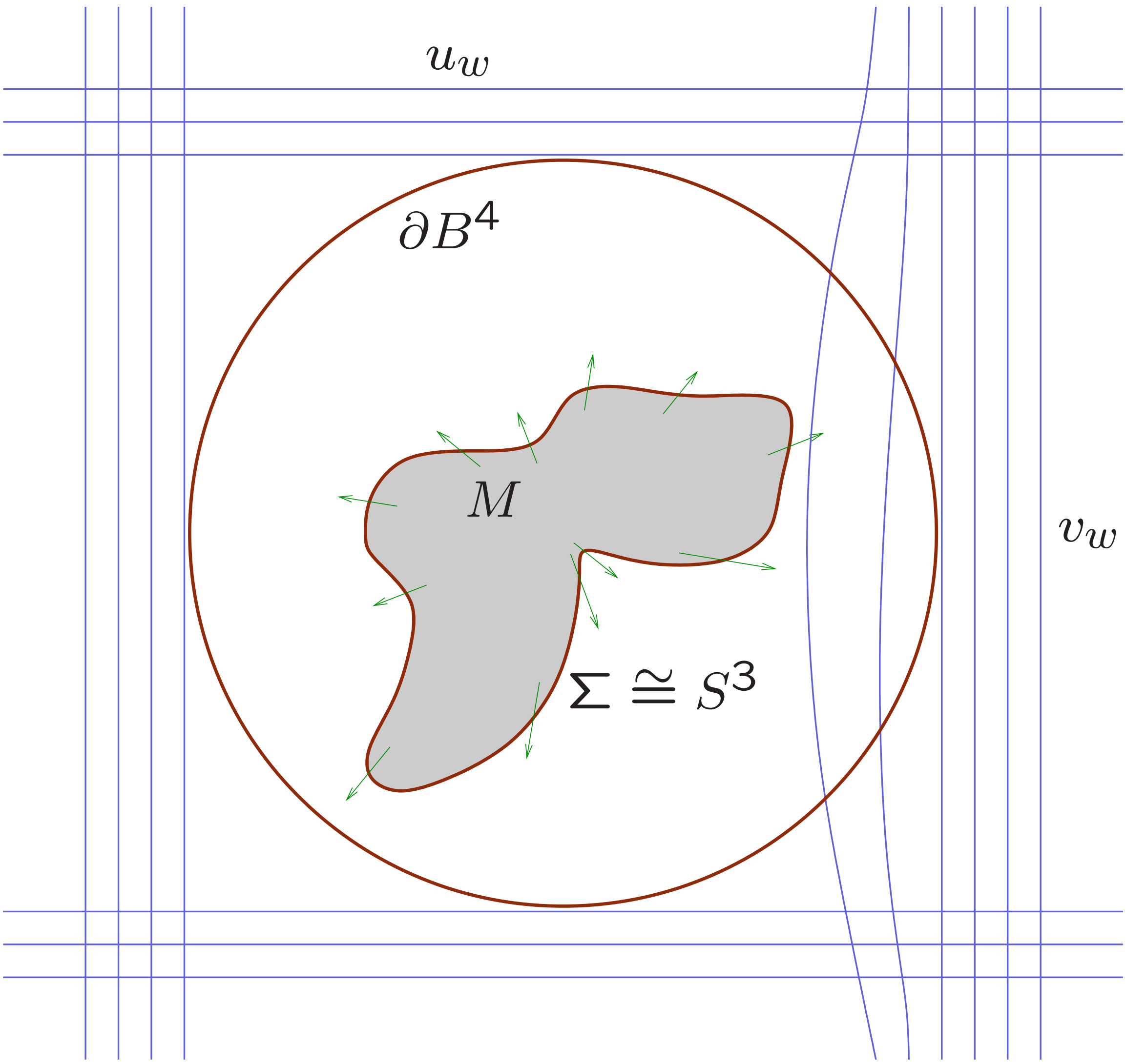

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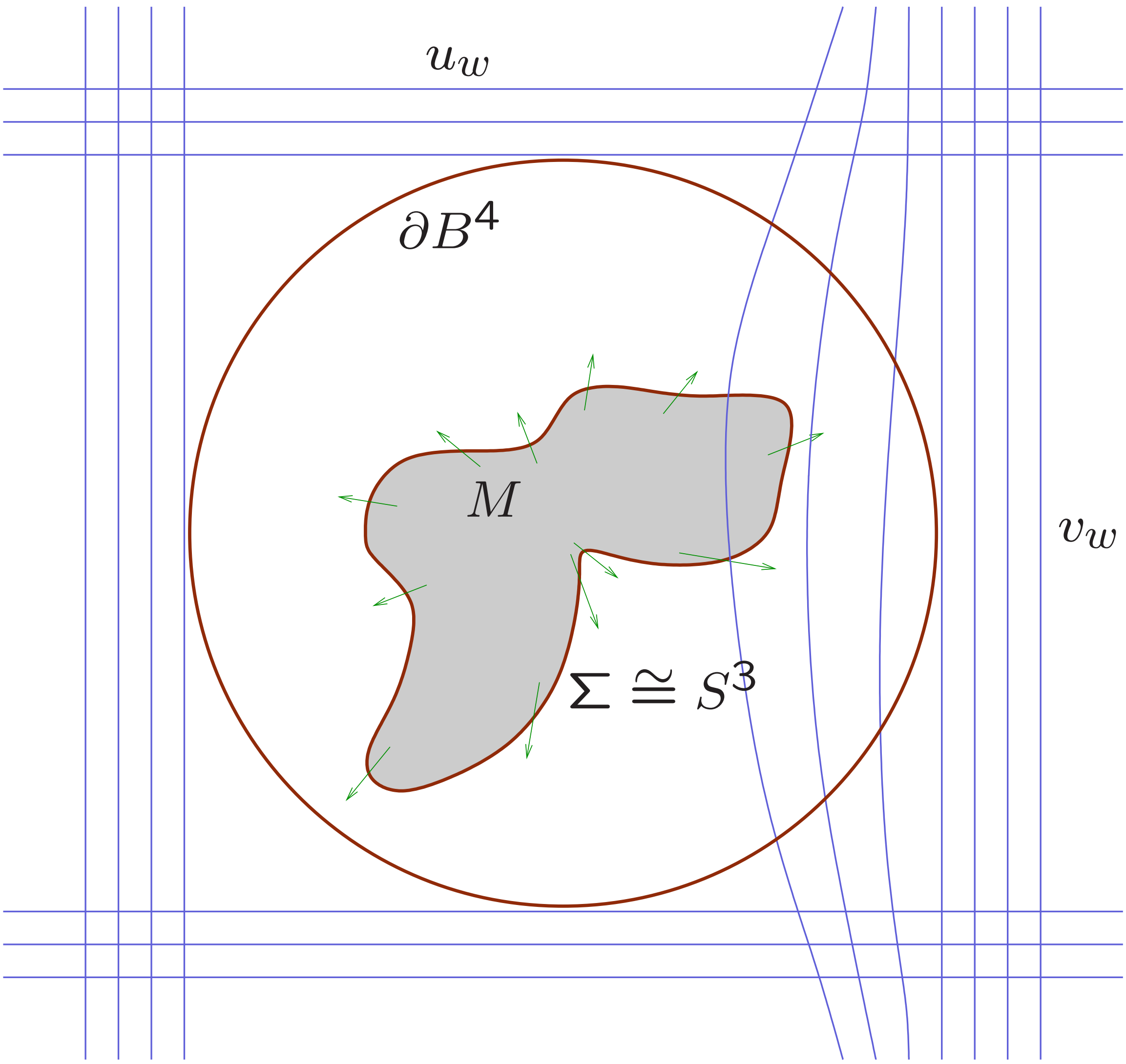

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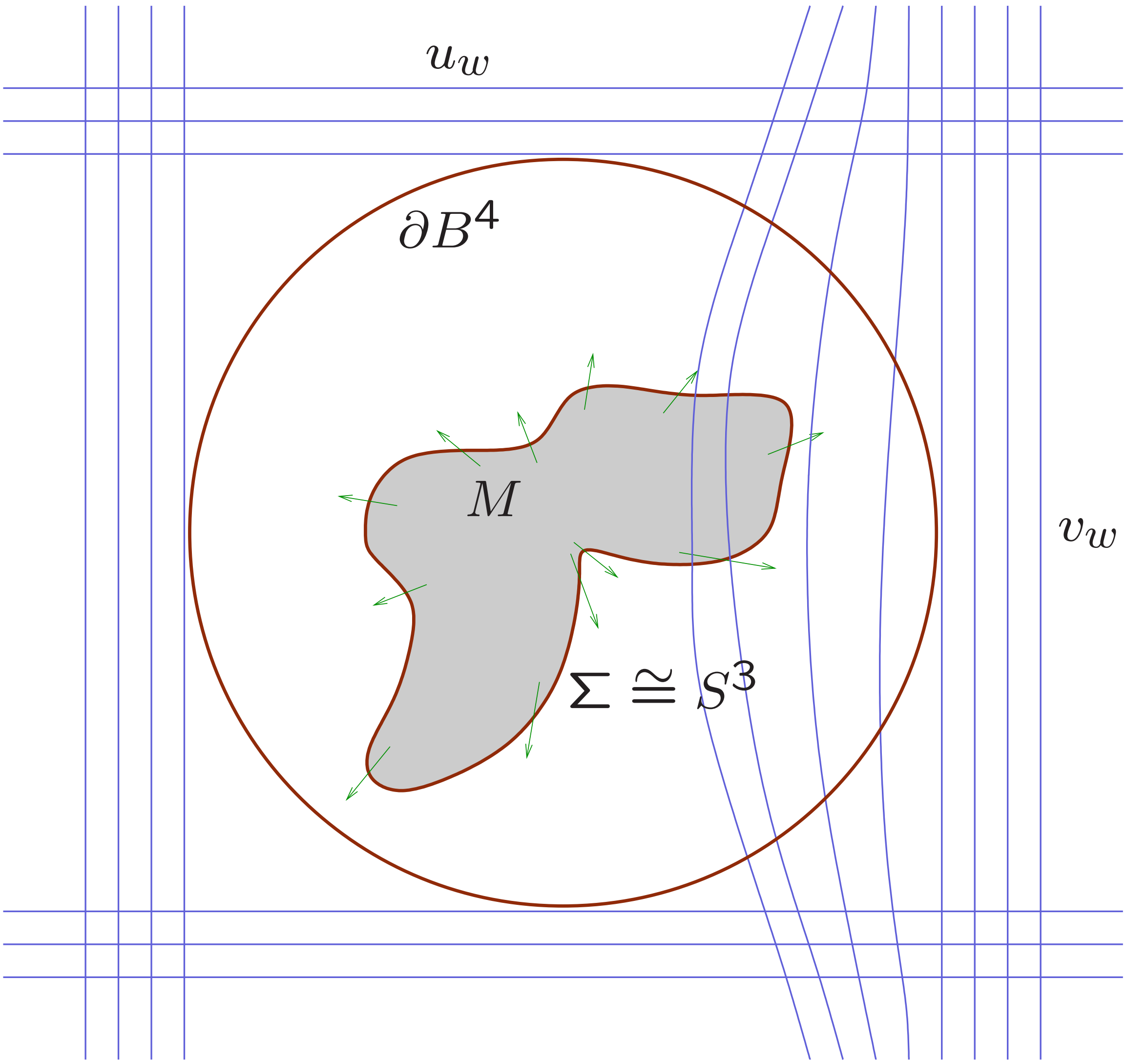

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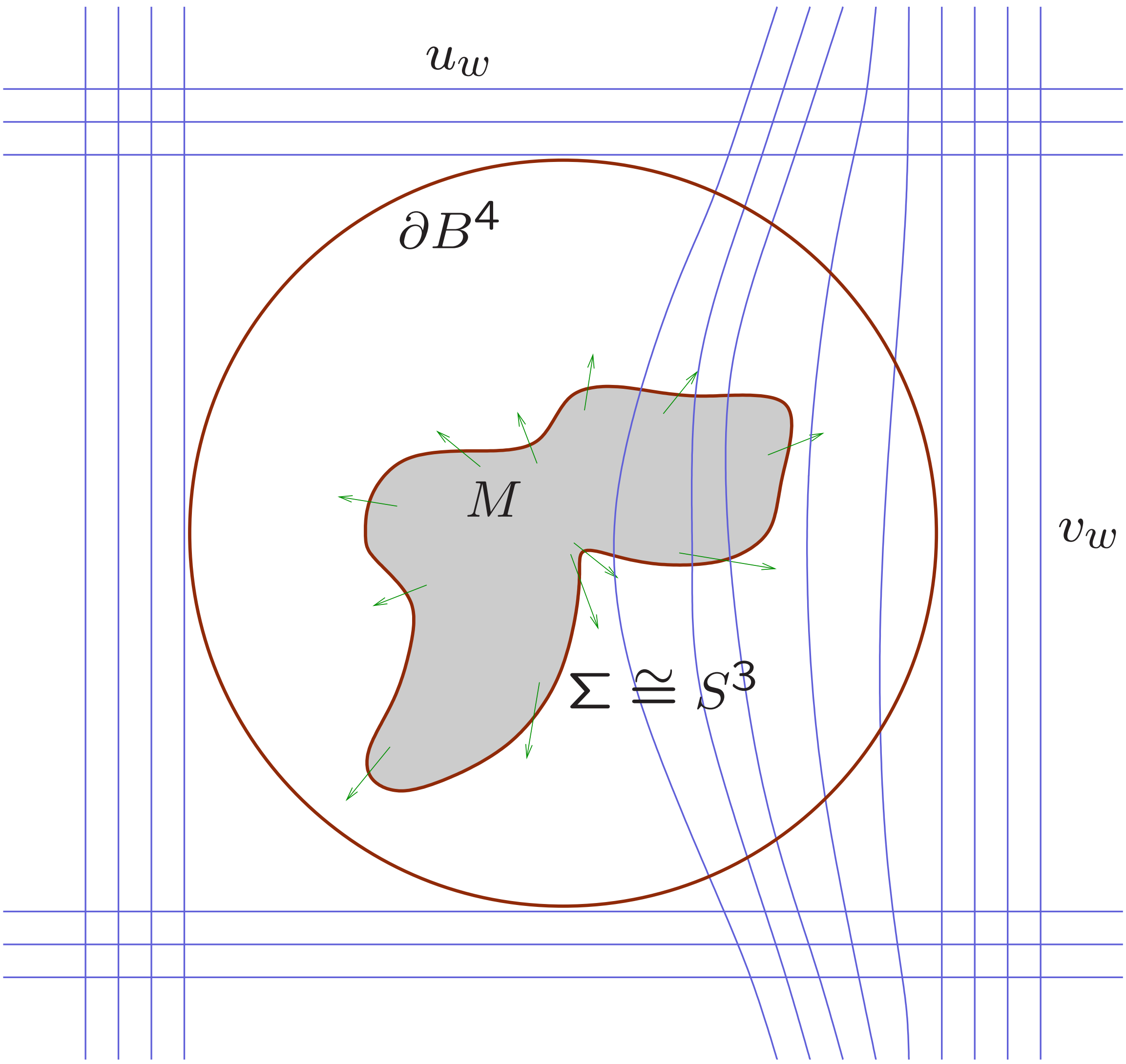
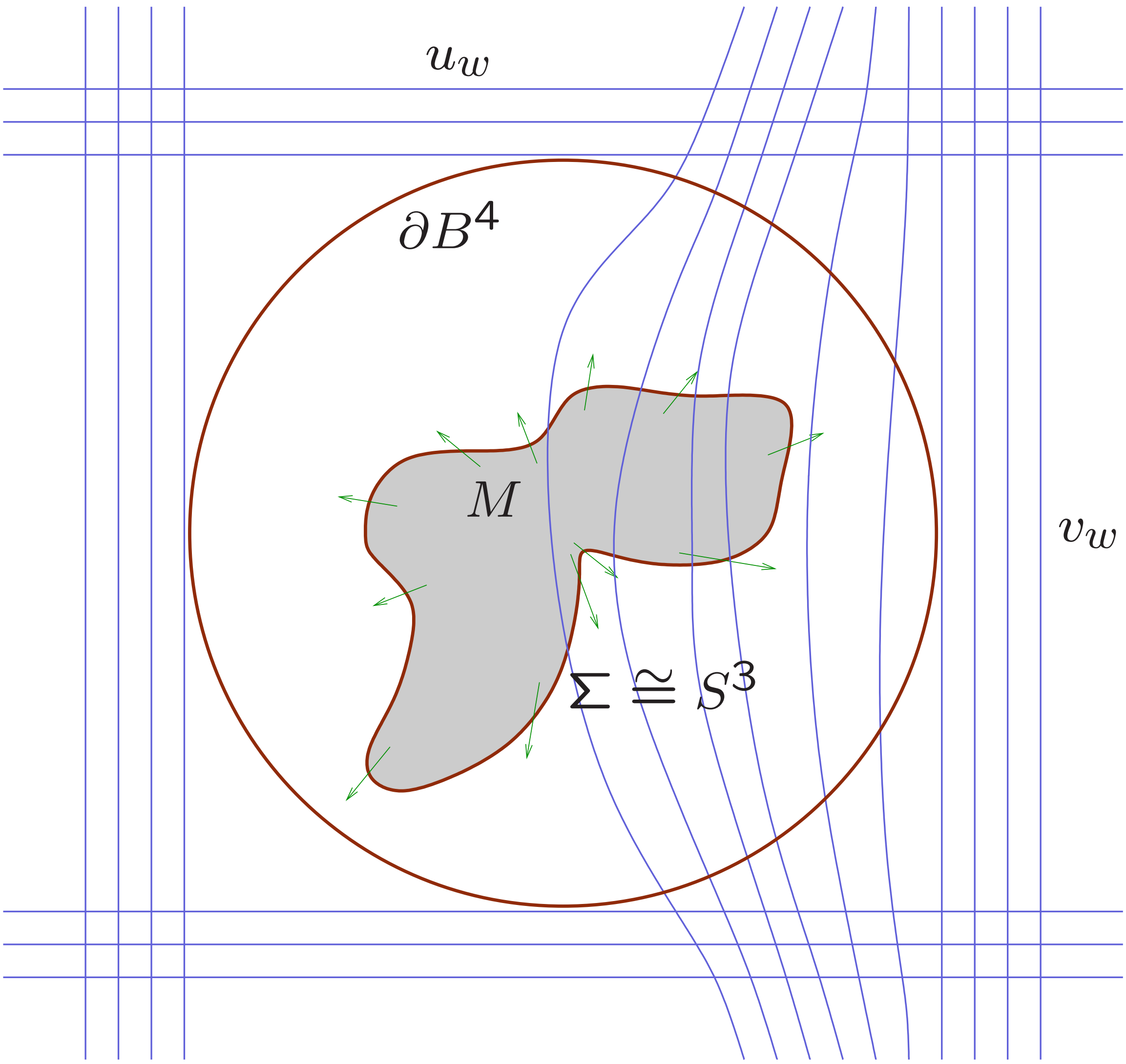
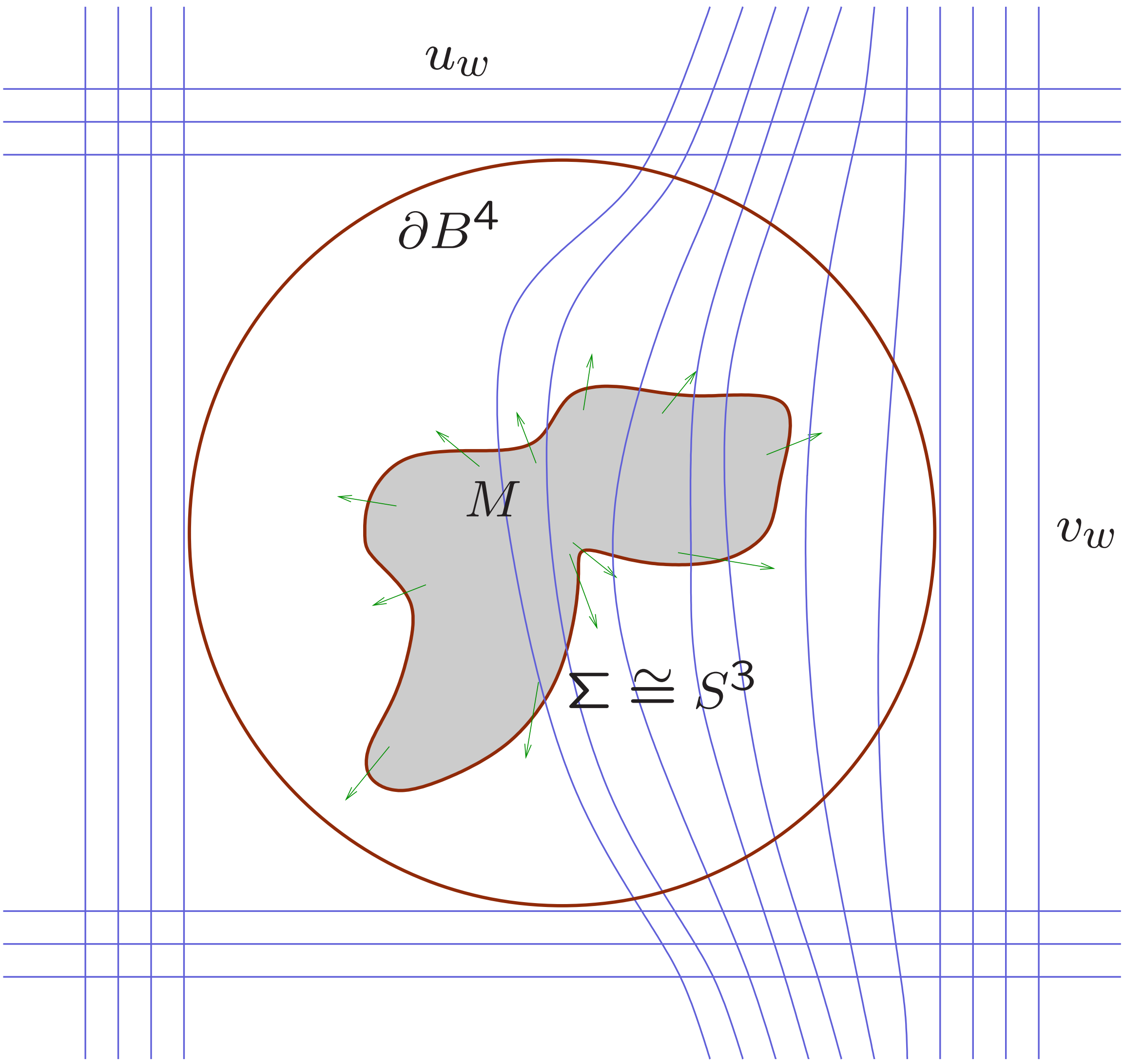


Final step: "turn on the machine. . ."



Final step: "turn on the machine..."



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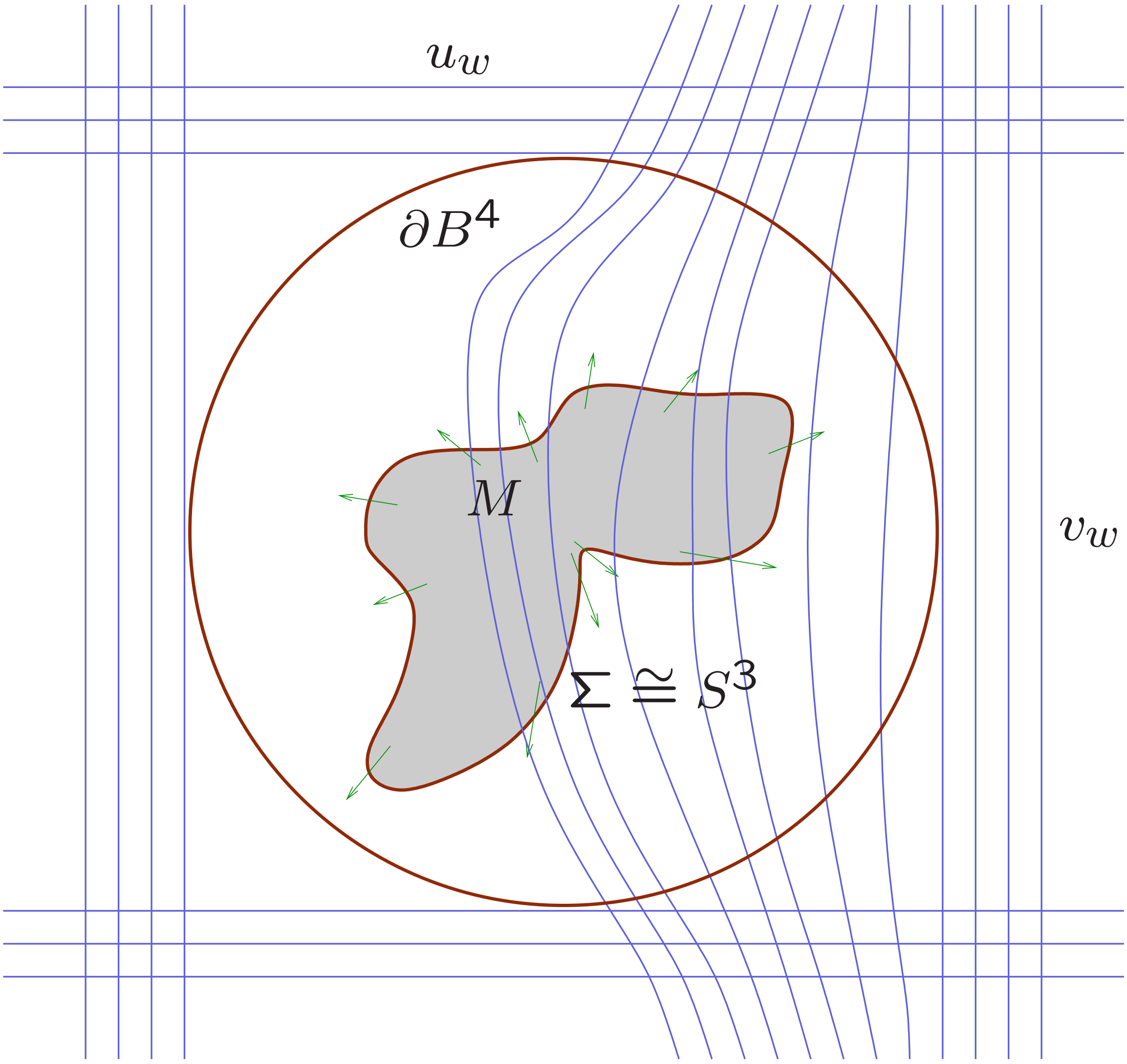

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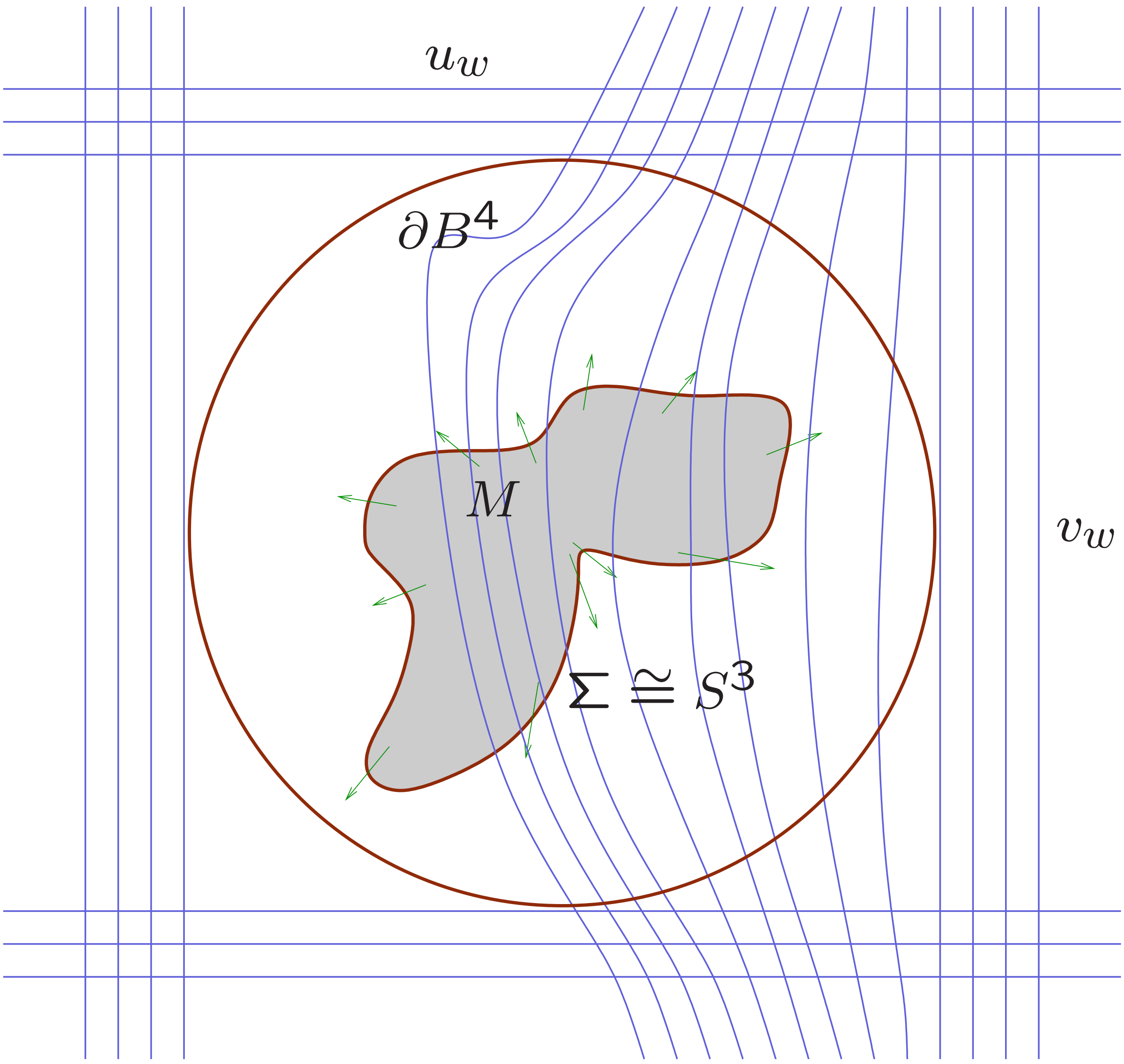

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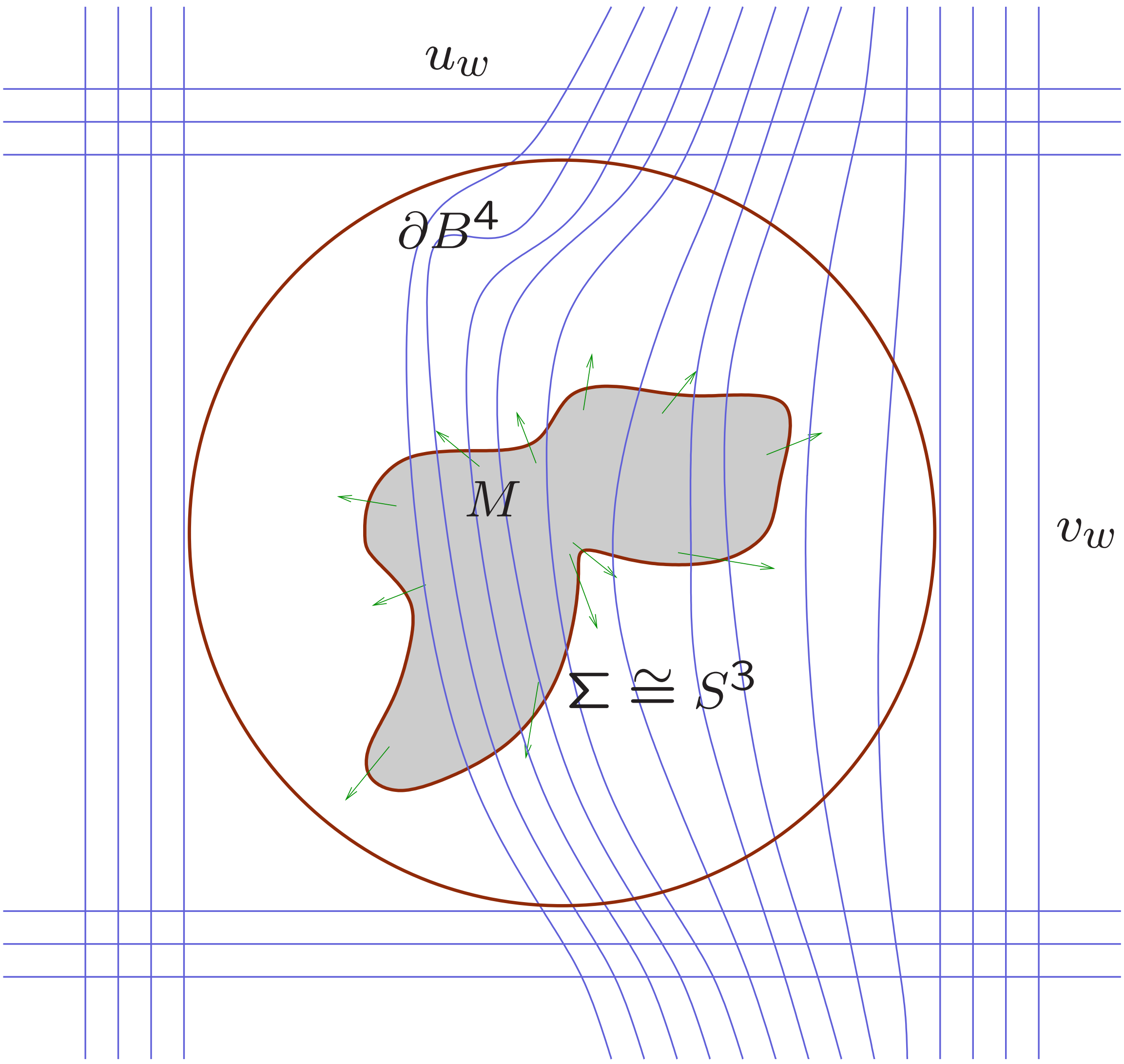
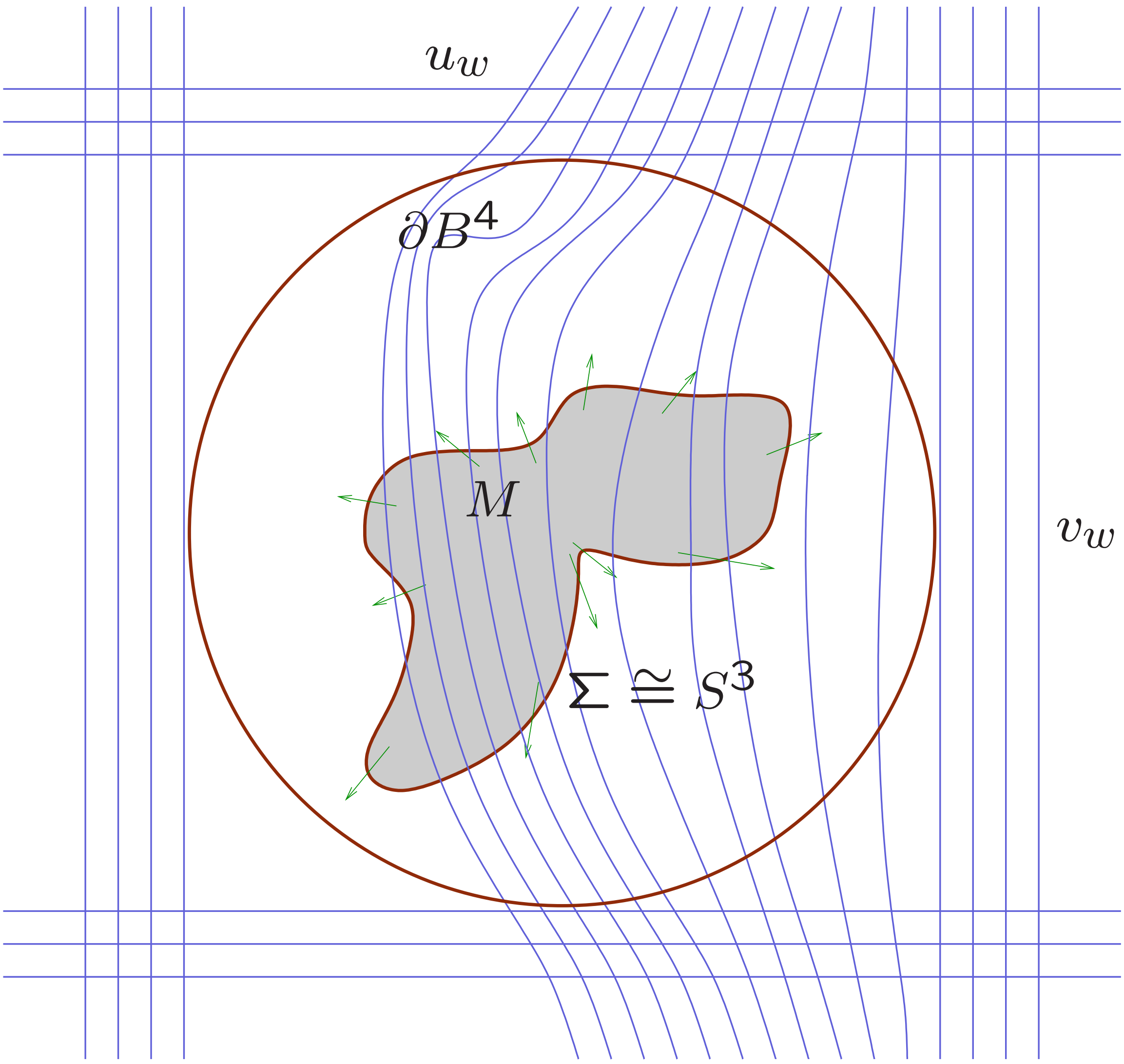


Final step: "turn on the machine. . . "



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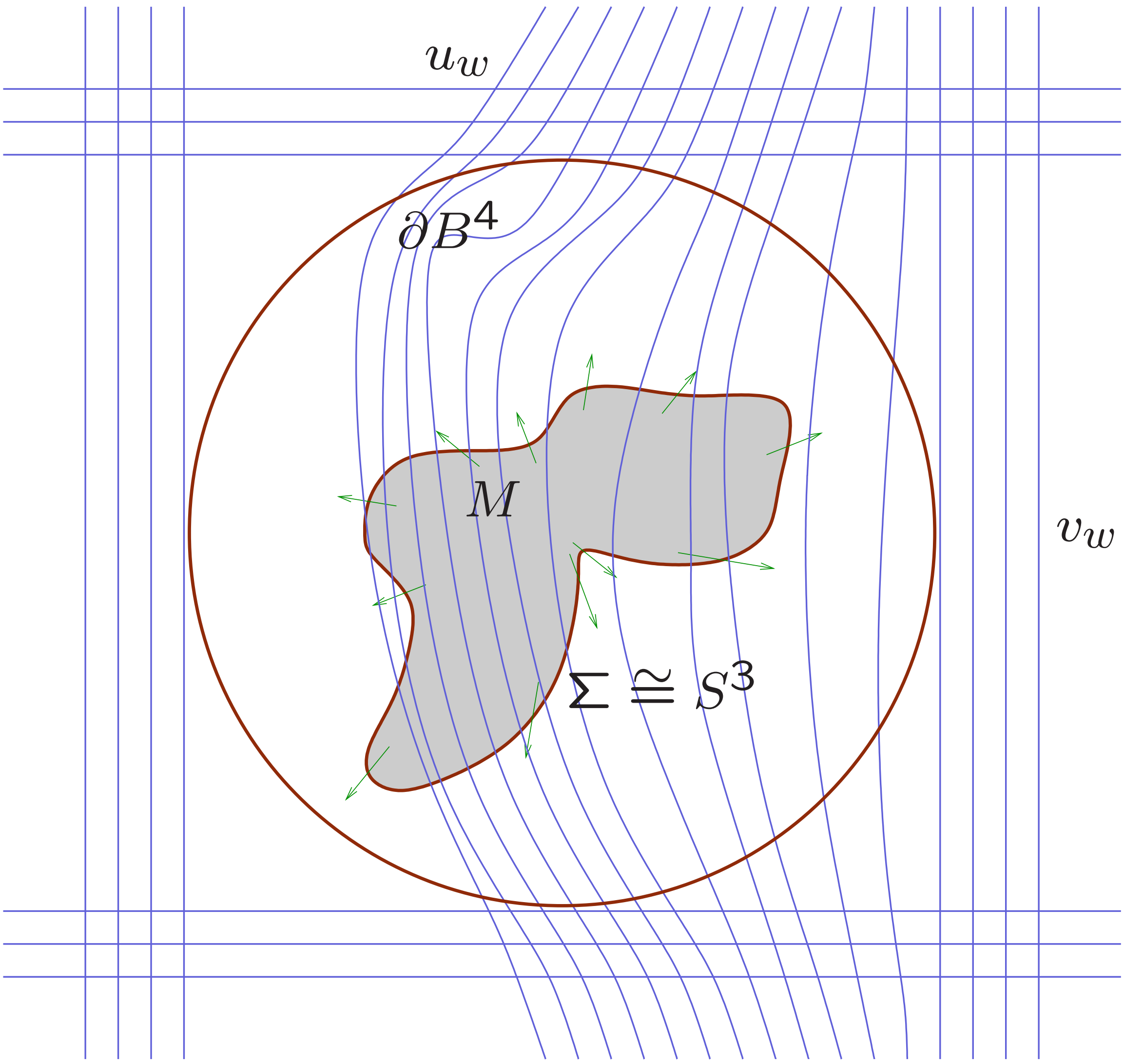

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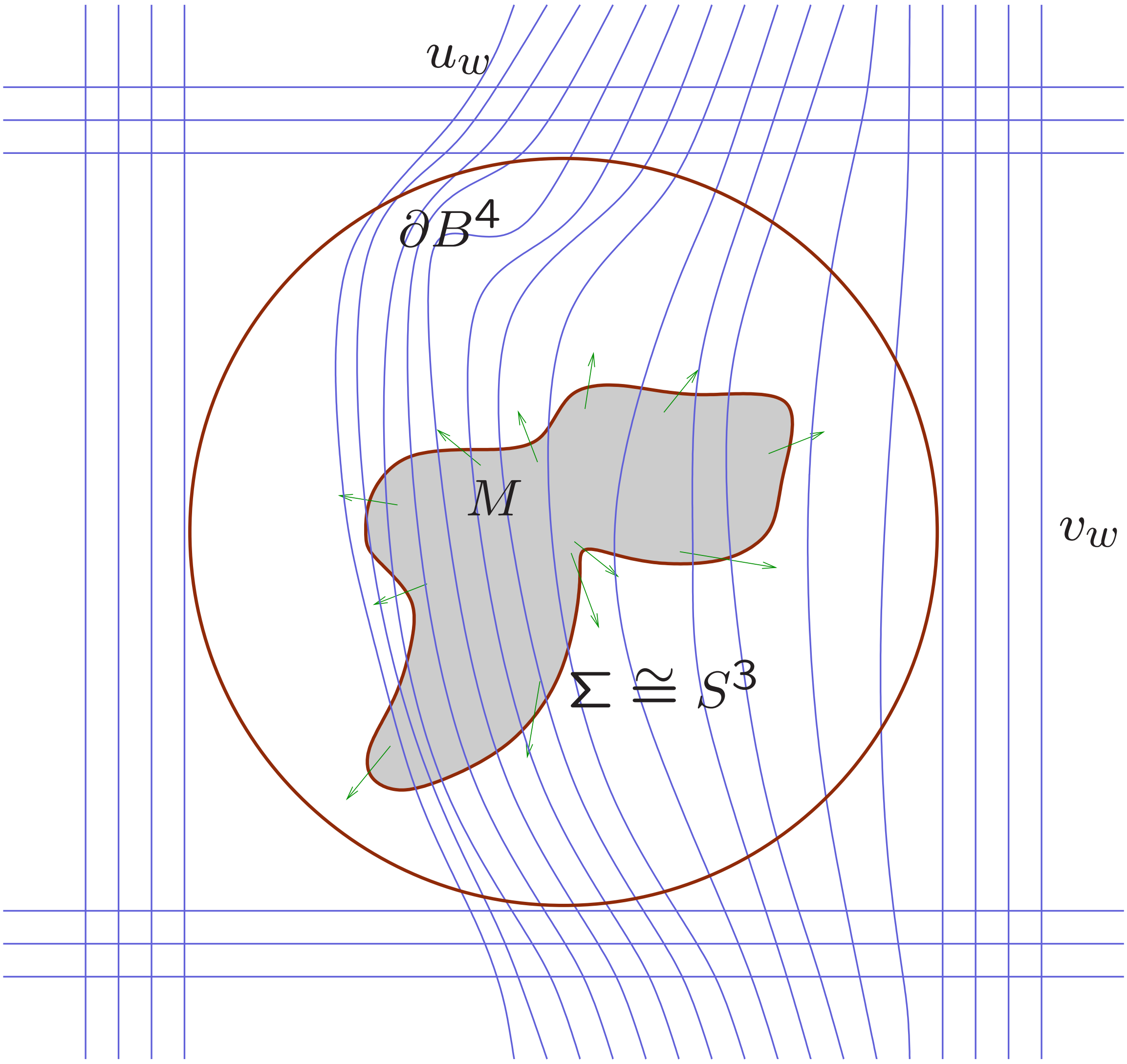
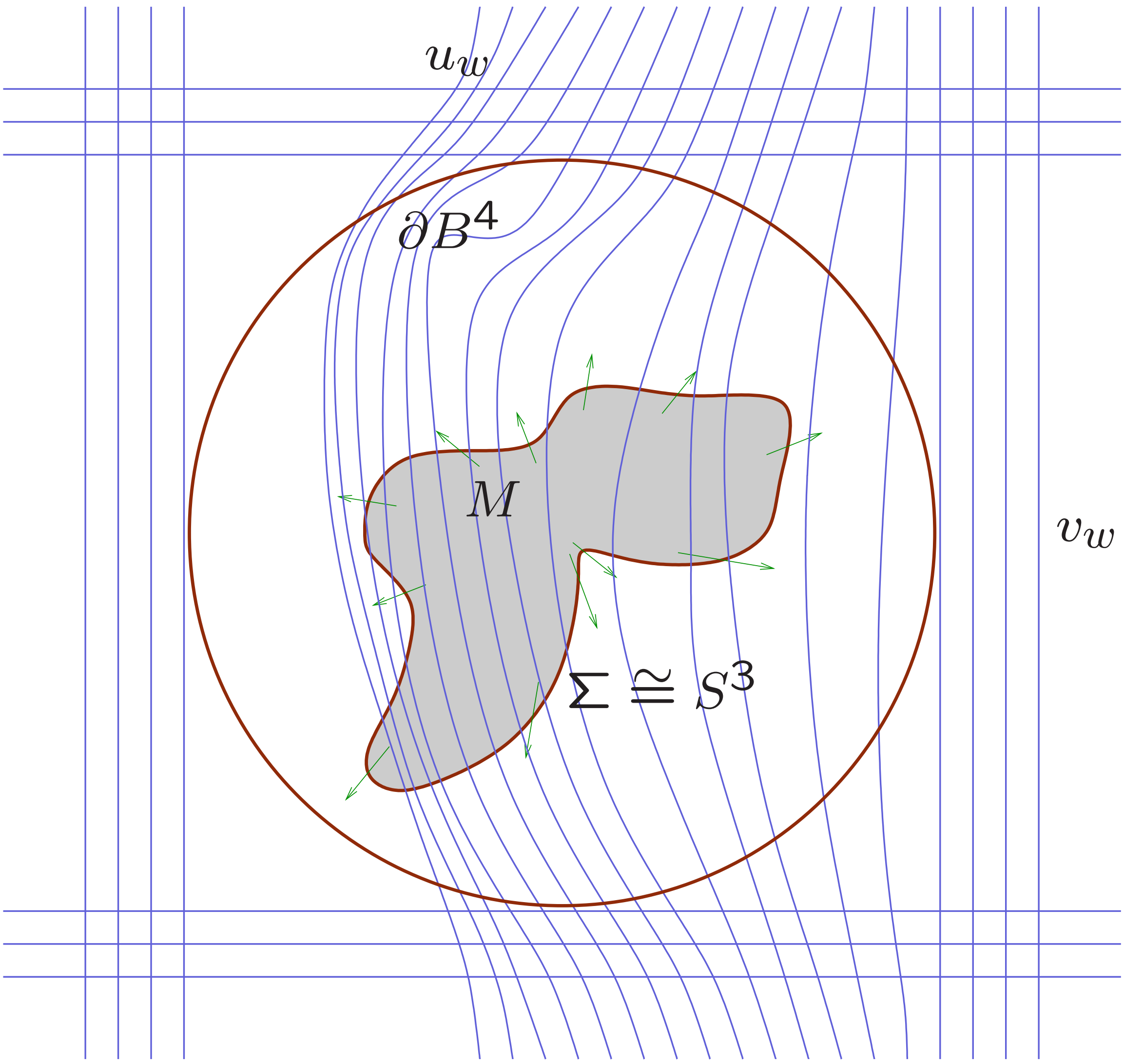


Final step: "turn on the machine. . ."



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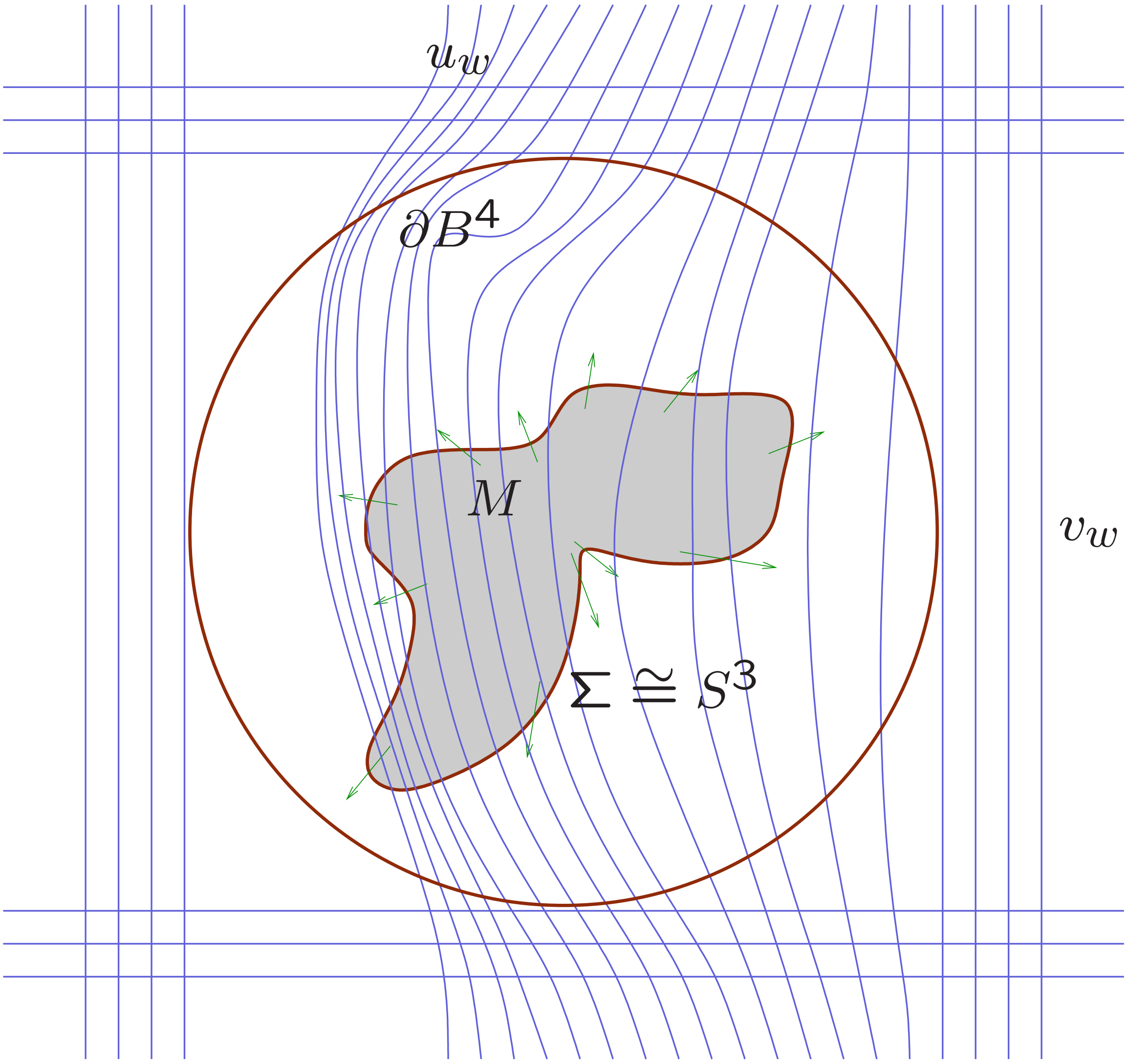
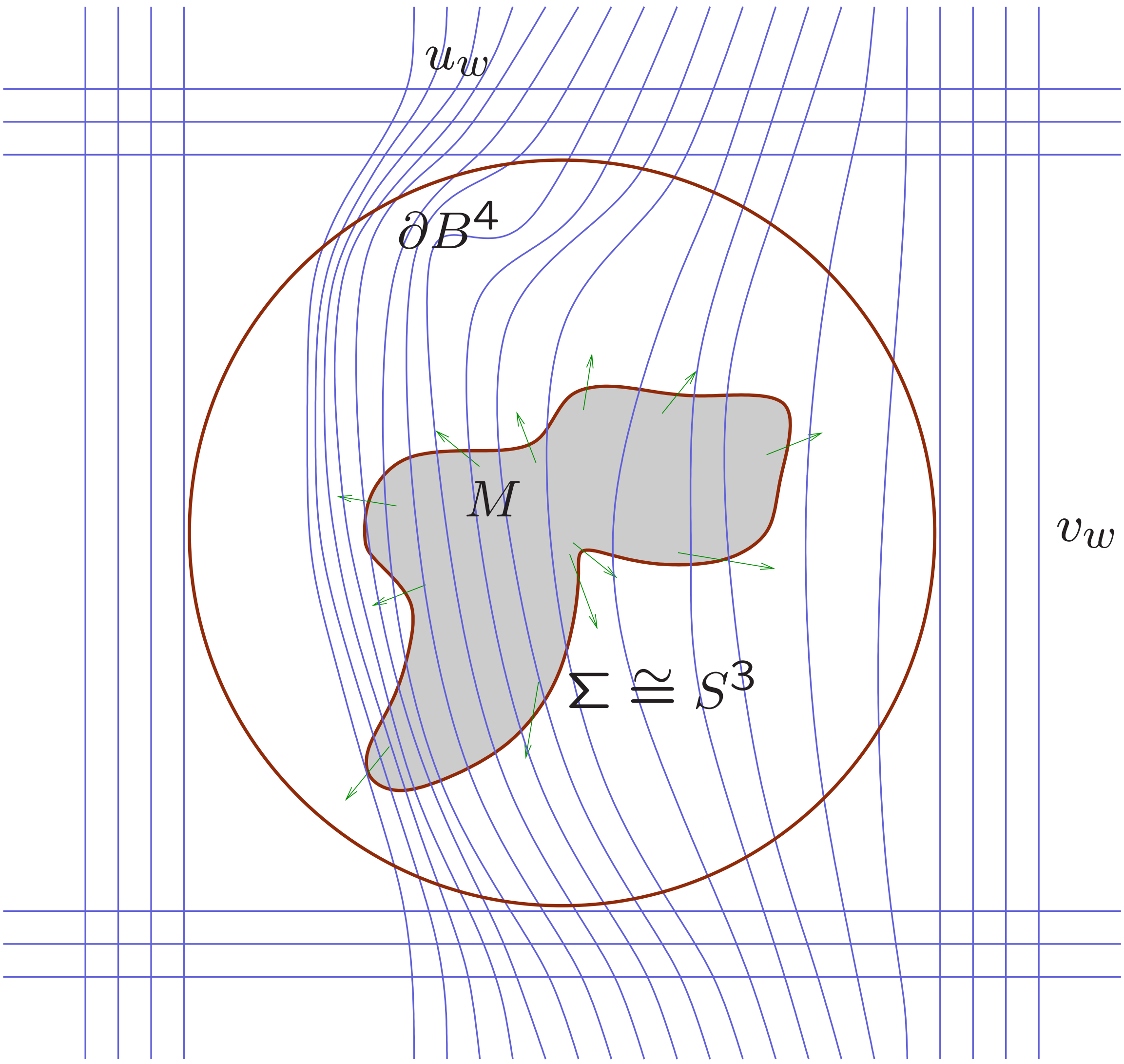


Final step: “turn on the machine. . . ”



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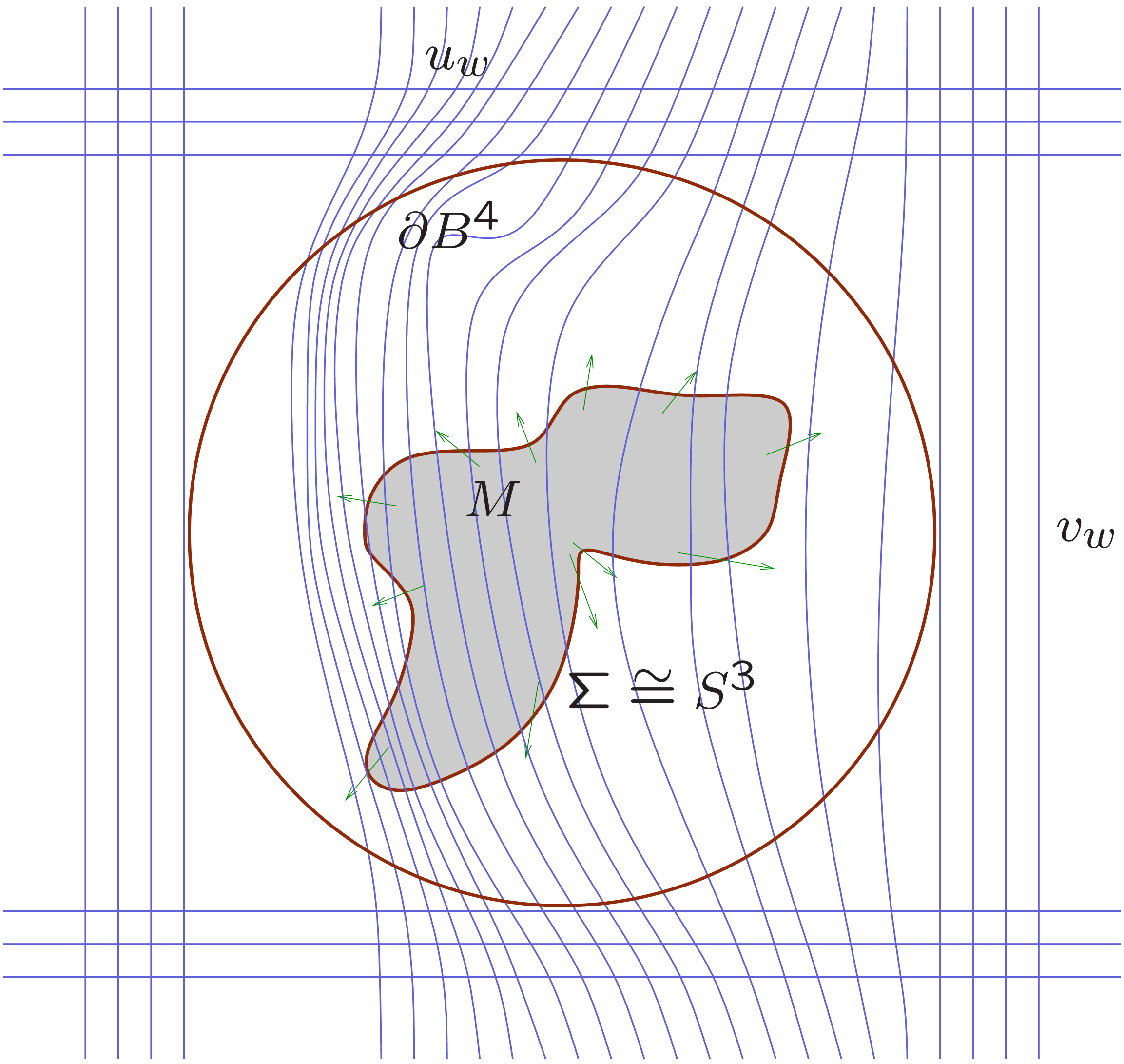
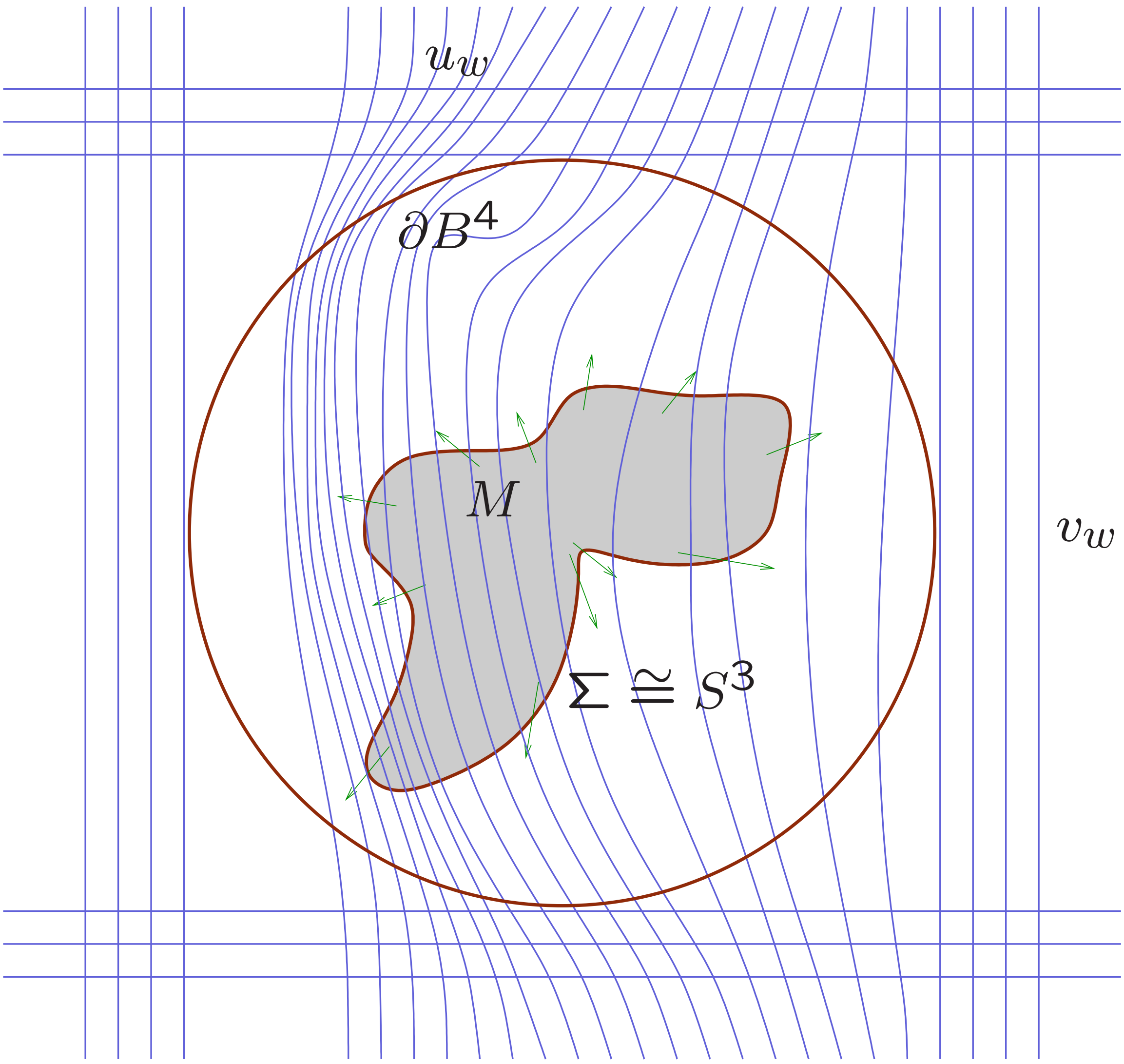


Final step: "turn on the machine. . . "



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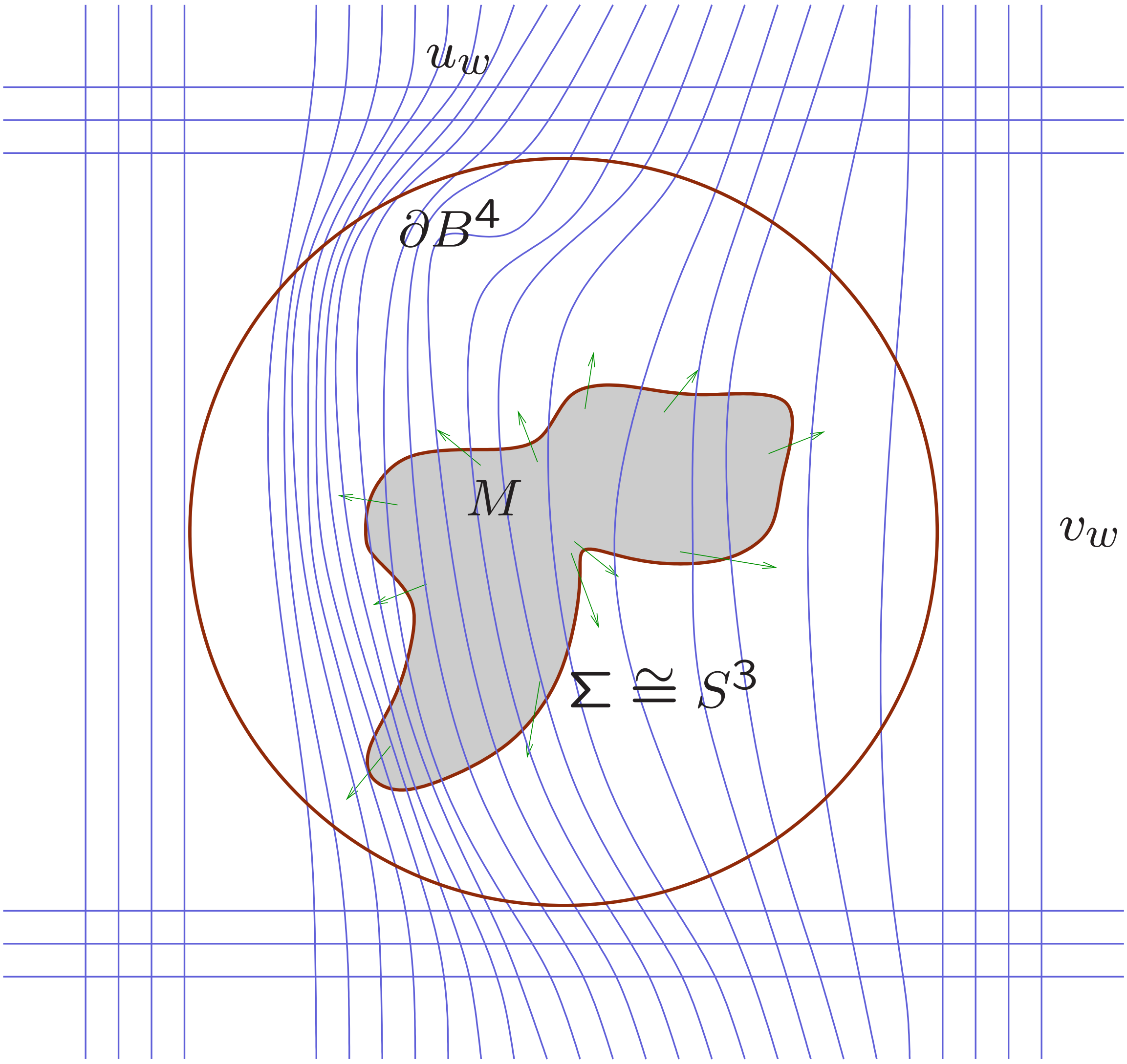
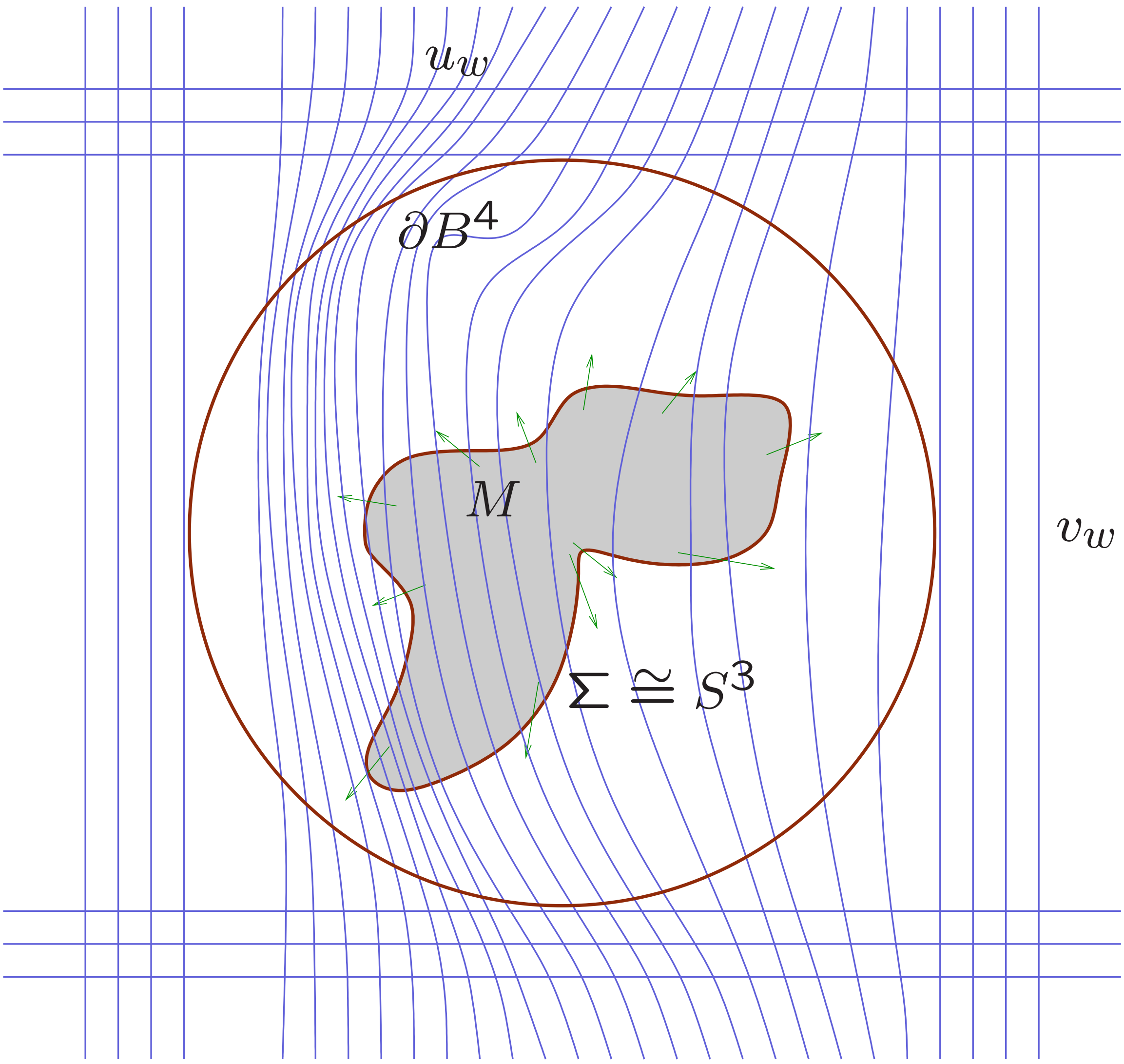
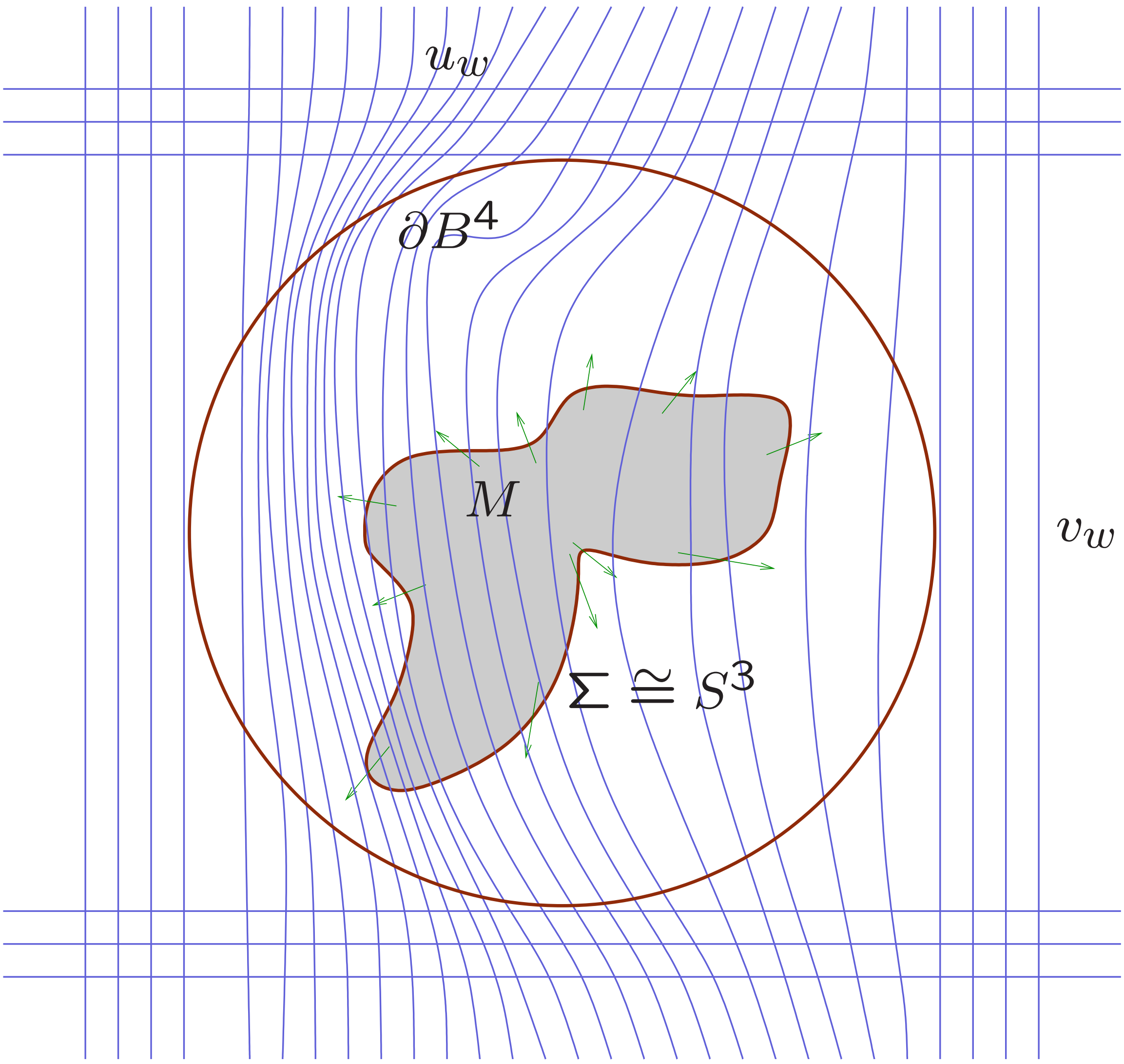


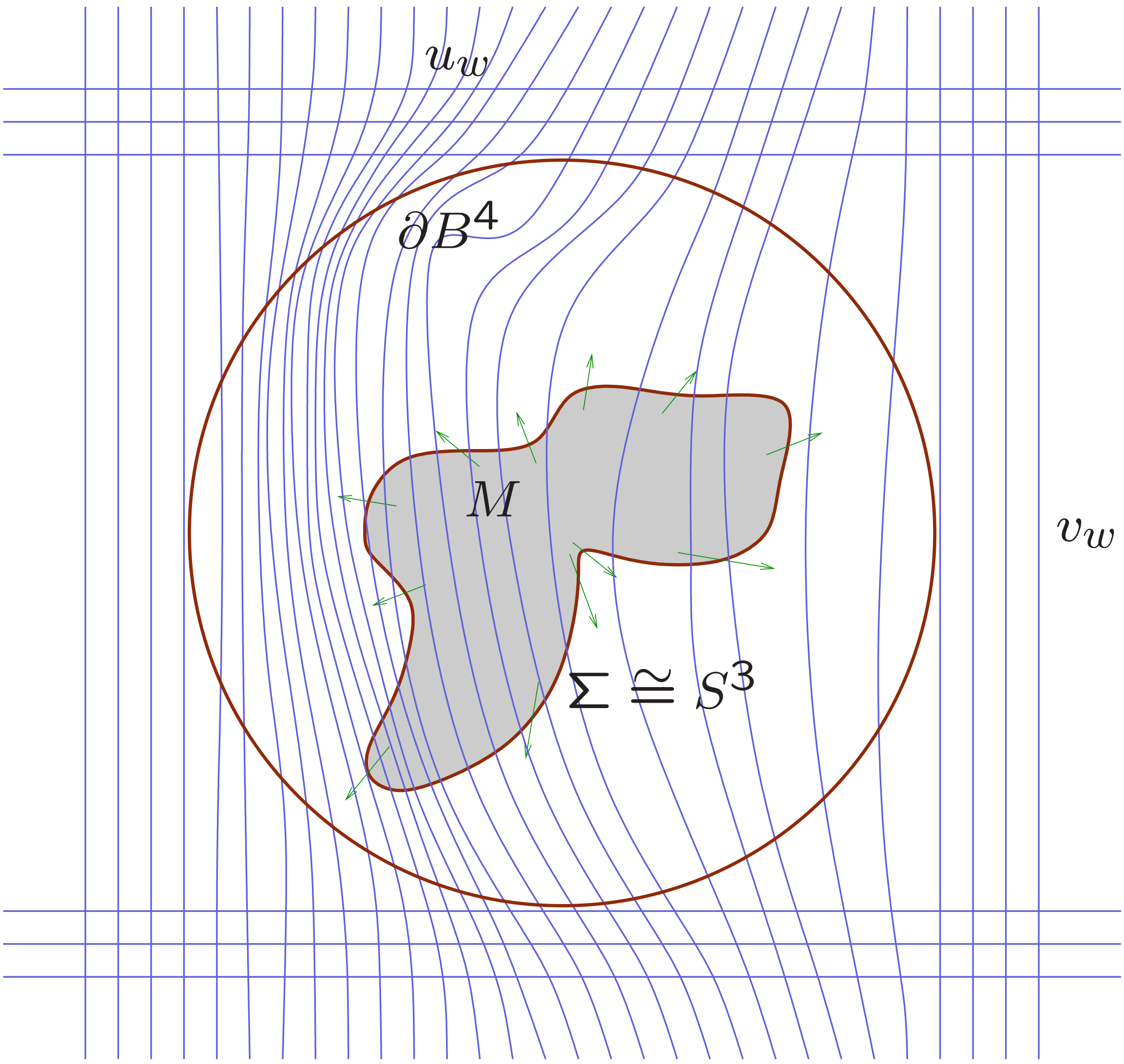
Final step: "turn on the machine. . . "



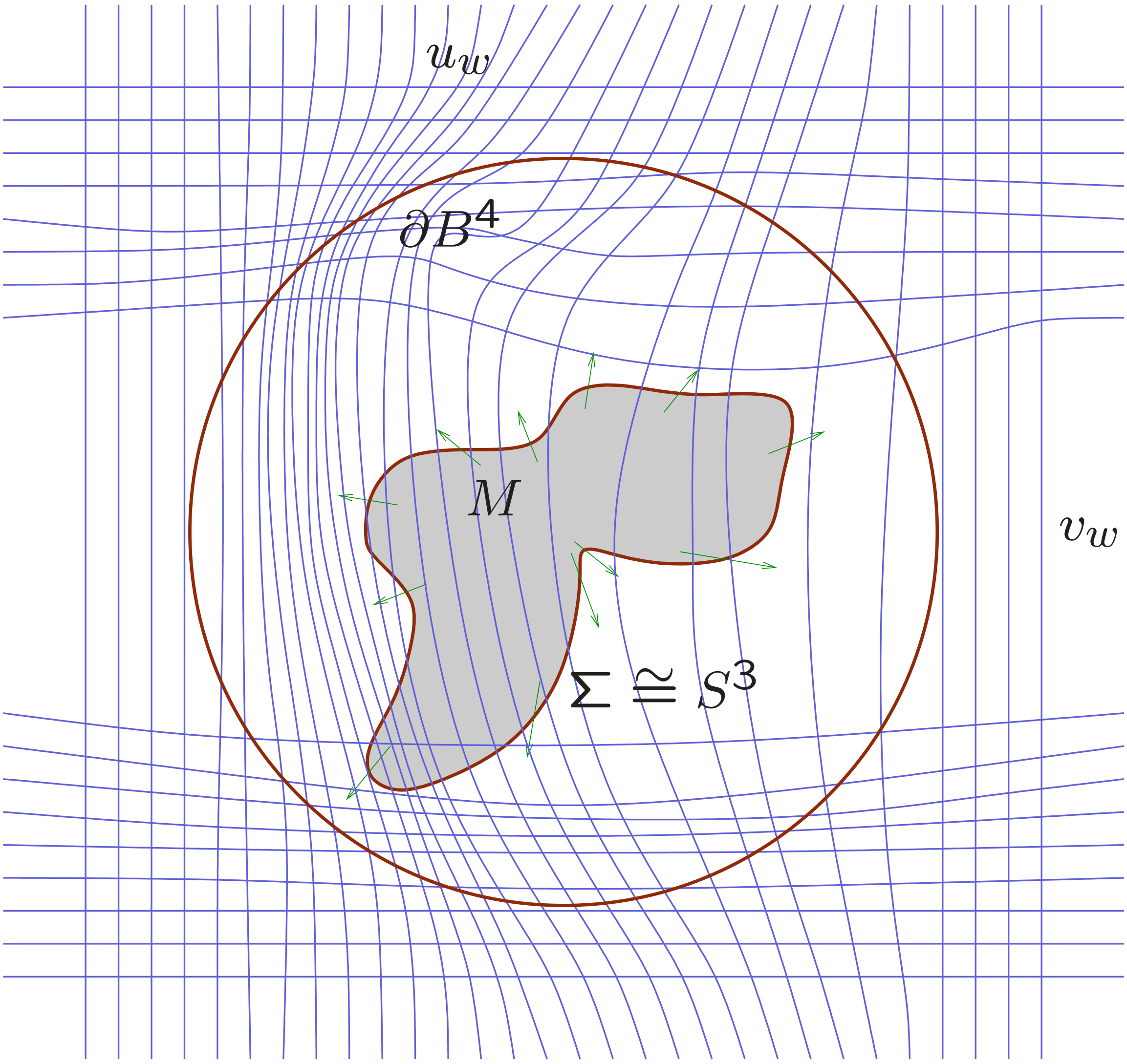
Final step: "turn on the machine. . ."



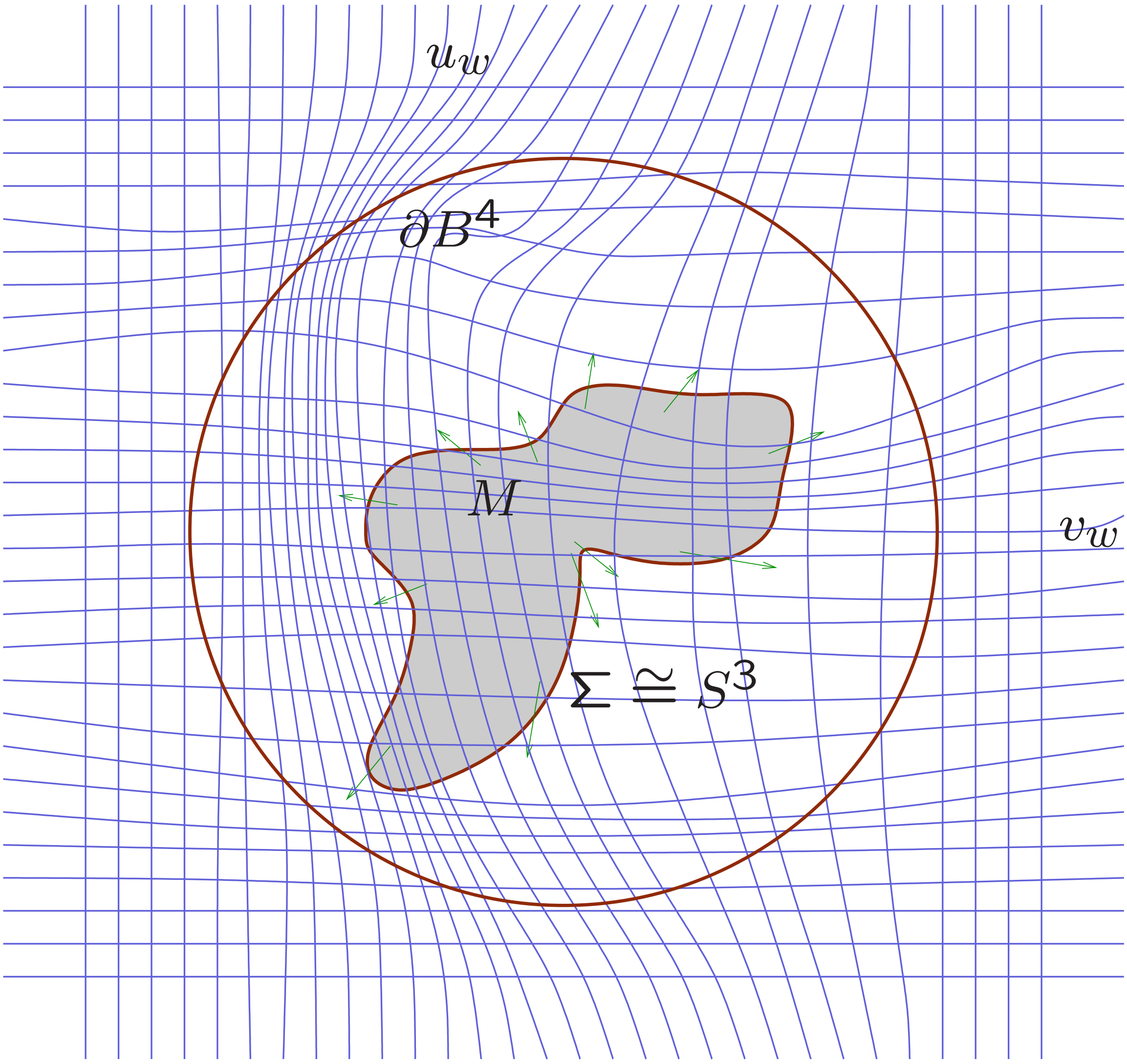
Final step: "turn on the machine. . . "



Final step: "turn on the machine. . ."



Final step: "turn on the machine..."



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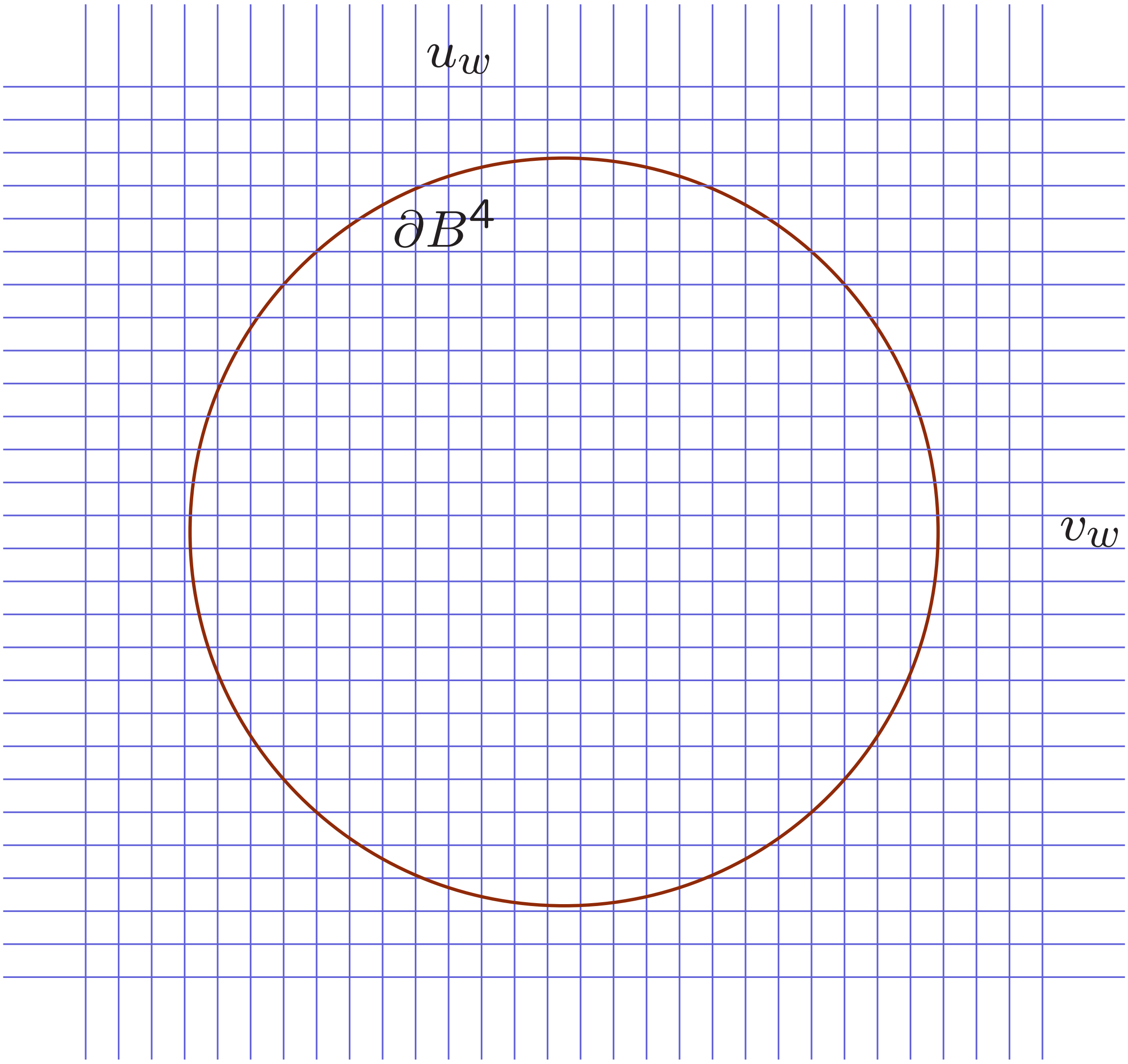

$\Rightarrow \quad W \cong \mathbb{C}^2.$ $\square$

That was nearly 30 years ago.

Here is a more recent but similar result. . .

Theorem (W. 2010)
The only exact symplectic fillings of a 3-dimensional torus

$$\mathbb{T}^3 := S^1 \times S^1 \times S^1$$

are star-shaped domains in the cotangent bundle of $\mathbb{T}^2$.

Question:
For a surface Σ of genus $g \geq 2$, does the unit cotangent bundle have more than one exact symplectic filling?

No one has any idea.