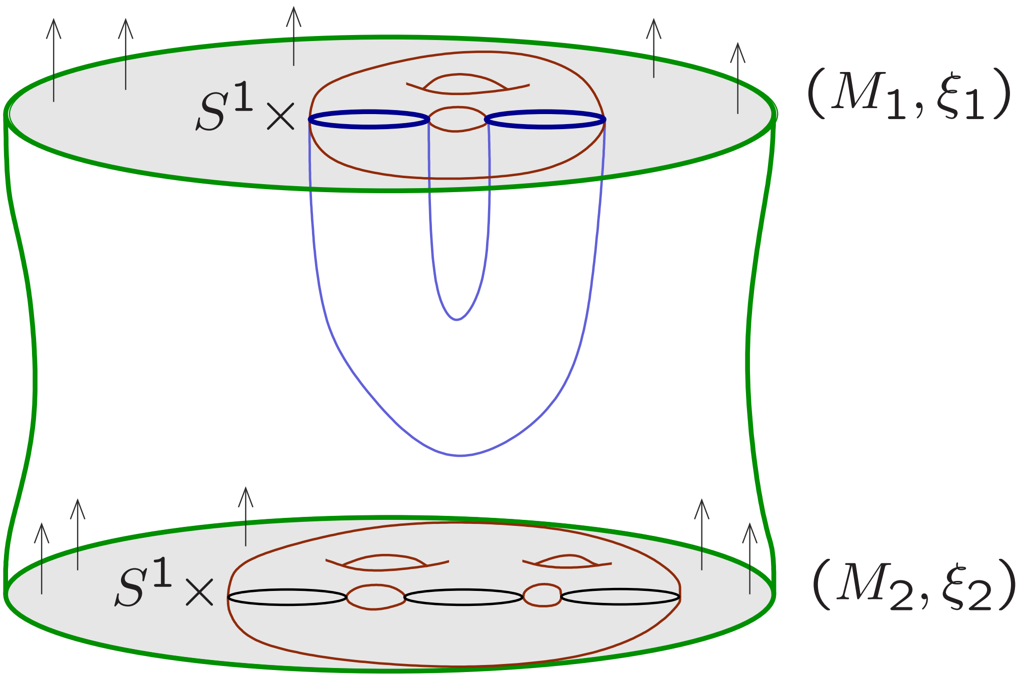
Some Tight Contact Manifolds Are Tighter Than Others



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(includes joint work with J. Latschev, P. Massot and K. Niederkrüger)

Slides available at:

`http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks`

Warmup: Hamiltonian dynamics

(W^{2n}, ω) symplectic: $\omega^n > 0$ and $d\omega = 0$

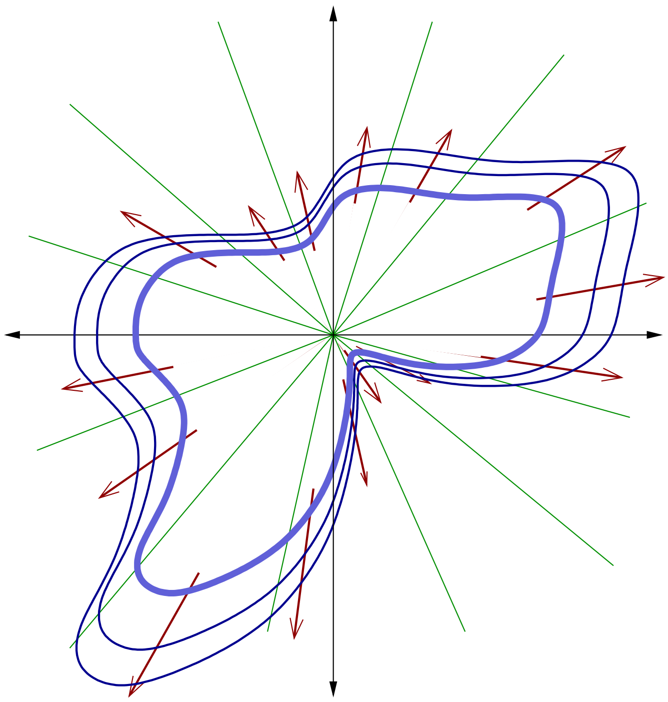
$H : M \to \mathbb{R} \rightsquigarrow$ Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH$$

Flow of X_H preserves level sets $\Sigma_c := H^{-1}(c)$.

Question:
Given $c \in \mathbb{R}$, is there a periodic orbit in Σ_c?

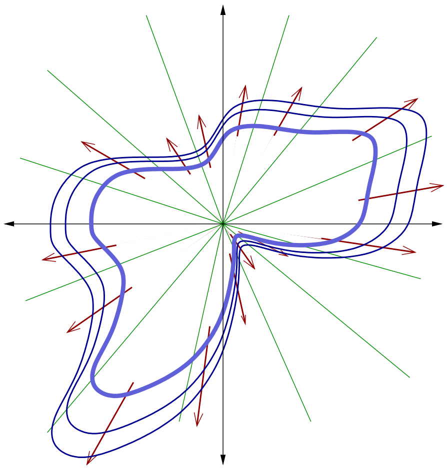
Theorem (Rabinowitz-Weinstein '78)**.**
In $(\mathbb{R}^{2n}, \omega_{\text{std}})$, *every* **star-shaped** *hypersurface admits a periodic orbit.*



Convexity and contact structures

Assume (W, ω) compact, $\partial W =: M \neq \emptyset$.
The boundary is **convex** if it is transverse to an outward pointing *Liouville vector field* Z:

$$\mathcal{L}_Z \omega = \omega$$



What structure does ω induce on ∂W?

$Z \pitchfork M \Rightarrow \alpha := \omega(Z, \cdot)|_{TM}$ is a *contact form*:

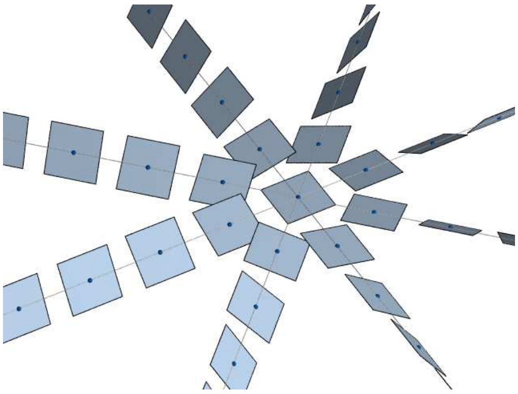
$$\alpha \wedge (d\alpha)^{n-1} > 0.$$

Up to isotopy, the **contact structure** defined by $\xi := \ker \alpha$ is *independent of choices.*

We say (W, ω) is a **symplectic filling** of (M, ξ):

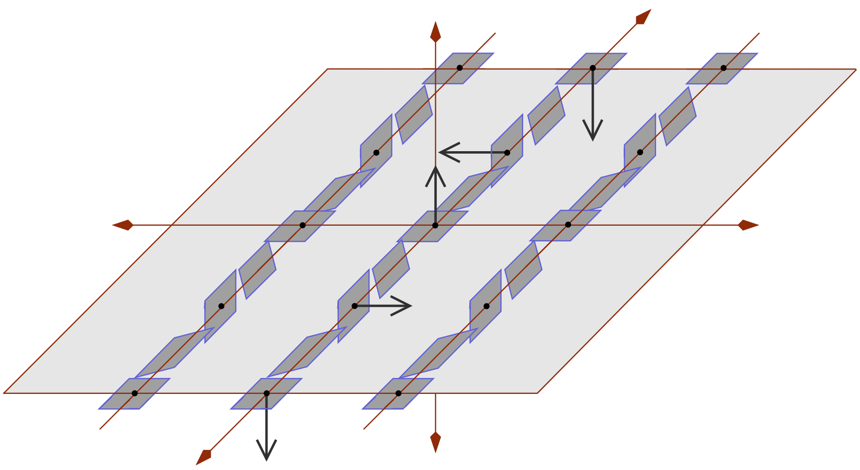
$$\text{“}\partial(W, \omega) = (M, \xi)\text{”}$$

(M^{2n-1}, ξ) contact manifold $\Rightarrow$
the hyperplane field $\xi \subset TM$ is *"maximally nonintegrable"*



and transverse to a **Reeb** (i.e. Hamiltonian) vector field.

Examples: $\mathbb{T}^3 = S^1 \times S^1 \times S^1 \ni (s, \phi, \theta)$. For $k \in \mathbb{N}$, let $\xi_k := \ker\left[\cos(2\pi k s)\, d\theta + \sin(2\pi k s)\, d\phi\right]$



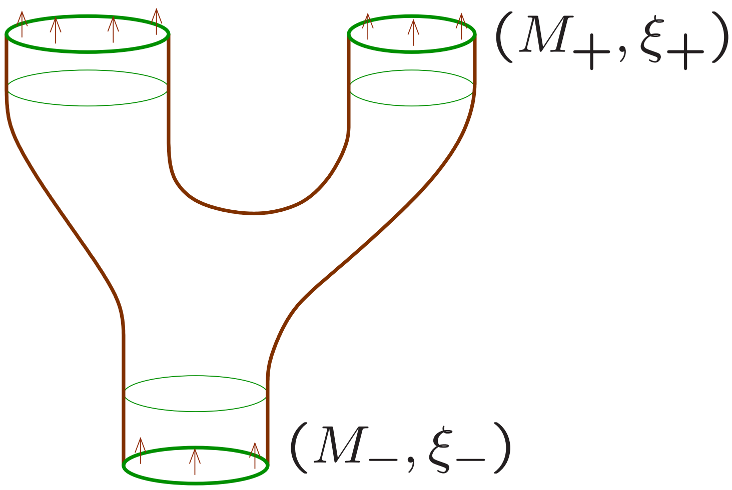
Then $(\mathbb{T}^3, \xi_1) = \partial\left(\mathbb{D}(T^*T^2), \omega_{\text{std}}\right)$.

Some hard problems in contact topology

1. *Classification of contact structures:* given ξ_1, ξ_2 on M, is there a diffeomorphism $\varphi : M \to M$ with $\varphi_* \xi_1 = \xi_2$?

2. *Weinstein conjecture:* Every Reeb vector field on every closed contact manifold has a periodic orbit?

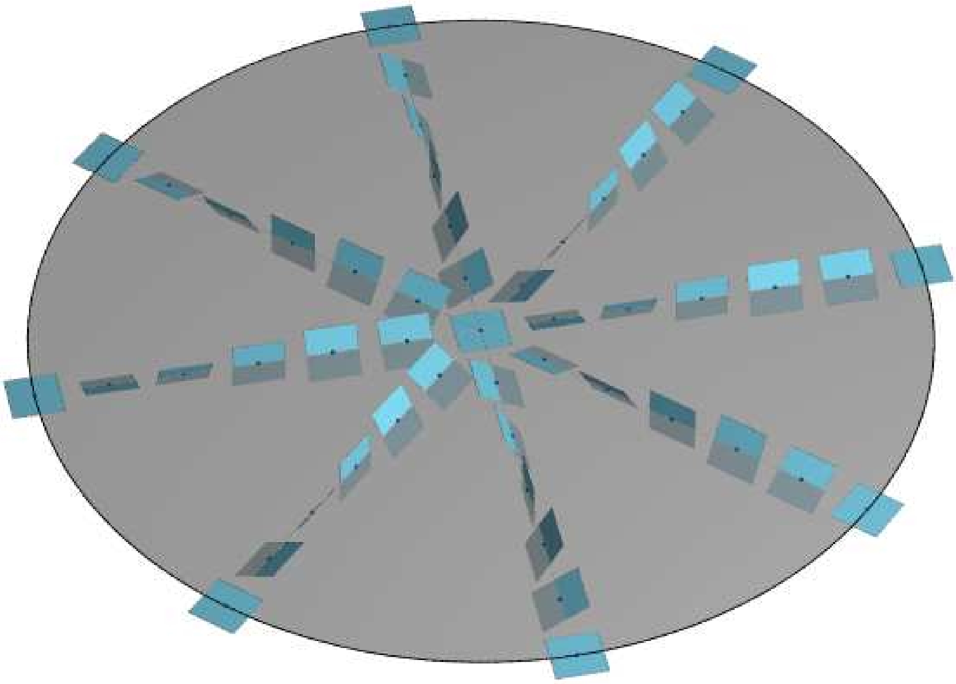
3. *Partial orders:* say $(M_-, \xi_-) \prec (M_+, \xi_+)$ if there is a (symplectic/Liouville/Stein) cobordism between them.



When is $(M_-, \xi_-) \prec (M_+, \xi_+)$?
When is $\emptyset \prec (M, \xi)$? (Is it *fillable*?)

Dimension 3: Overtwisted vs. Tight

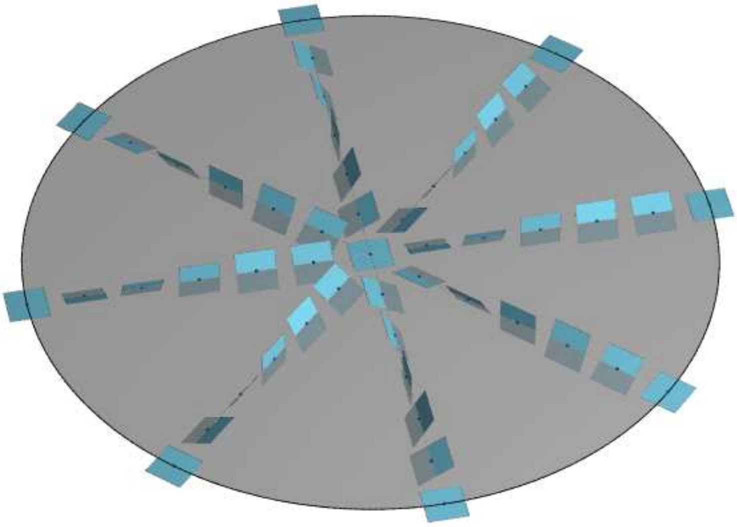
(M^3, ξ) is **overtwisted** if there exists a disk $D \hookrightarrow M$ with $T(\partial D) \subset \xi$ and $TD \pitchfork \xi$ at ∂D.



Non-overtwisted contact structures are called **“tight”**.

They are harder to understand.

The remarkable properties of ξ_{ot}:



1. Flexibility: $(M, \xi_{\mathrm{ot}}) \overset{\text{isotopic}}{\cong} (M, \xi'_{\mathrm{ot}}) \Leftrightarrow \xi_{\mathrm{ot}}$ and ξ'_{ot} are homotopic. *(Eliashberg '89)*

2. Vanishing: All "interesting" contact invariants vanish for ξ_{ot}.

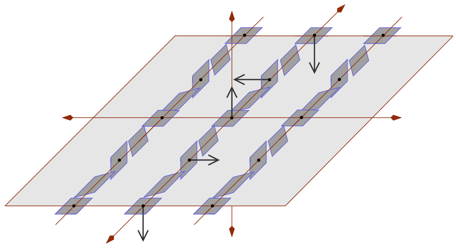
3. Weinstein conjecture: ξ_{ot} always admits a *contractible* Reeb orbit. *(Hofer '93)*

4. Not fillable: $\emptyset \not\prec (M, \xi_{\mathrm{ot}})$.
   (Gromov '85 + Eliashberg '89)

Contrast: the tight 3-tori $(\mathbb{T}^3, \xi_k)$:

1. Not flexible: All ξ_k are homotopic for all $k \in \mathbb{N}$, but $(\mathbb{T}^3, \xi_k) \not\cong (\mathbb{T}^3, \xi_\ell)$ for $k \neq \ell$.
2. Nonvanishing: Contact homology distinguishes ξ_k for different $k \in \mathbb{N}$.
3. Hypertight: ξ_k always has a closed Reeb orbit, but sometimes none are contractible.
4. Usually not fillable: $\emptyset \prec (\mathbb{T}^3, \xi_k)$ iff $k = 1$.

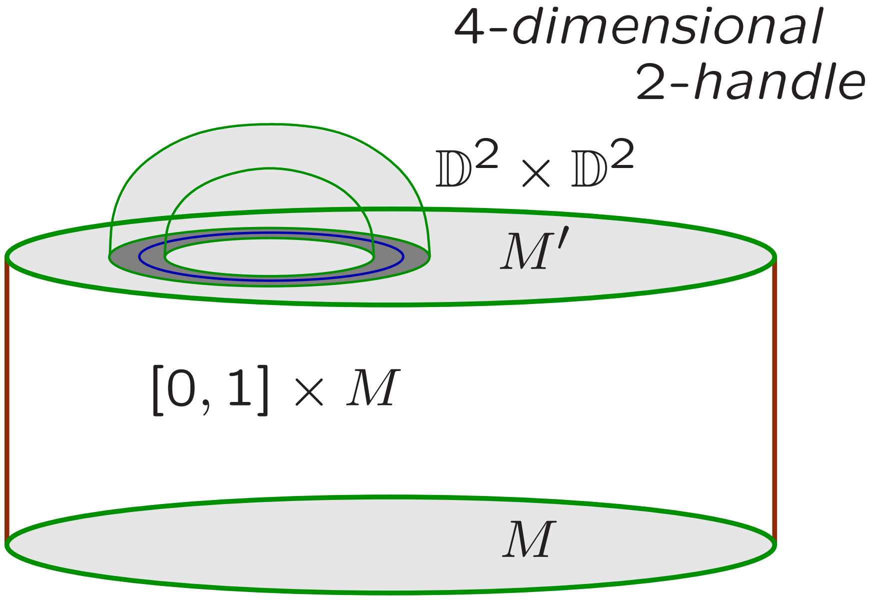
We say (M^3, ξ) has **Giroux torsion** if $([0,1] \times \mathbb{T}^2, \xi_1) \hookrightarrow (M, \xi)$.



- Overtwisted $\Rightarrow$ Giroux torsion
- $(\mathbb{T}^3, \xi_k)$ has Giroux torsion for $k \geq 2$
- Giroux torsion $\Rightarrow$ not fillable *(Gay '06)*

Conjecture.
Suppose $(M,\xi) \xrightarrow{\textit{contact surgery}} (M',\xi')$*.*
Then (M,ξ) *tight* $\Rightarrow$ (M',ξ') *tight.*

Surgery $\rightsquigarrow$ handle attaching cobordism:



$$\partial(([0,1]\times M)\cup(\mathbb{D}^2\times\mathbb{D}^2)) = -M \sqcup M'$$

Cobordism is exact symplectic: $(M,\xi) \prec (M',\xi')$.

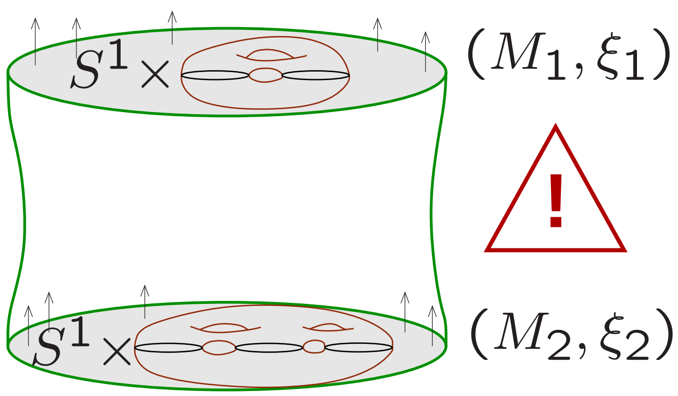
Conjecture.
Overtwistedness is minimal with respect to the relation "$\prec$" (exact symplectic cobordisms).

"There are degrees of tightness"

Theorem (Latschev-W. '10).
There exists a numerical contact invariant $\mathrm{AT}(M,\xi) \in \mathbb{N} \cup \{0,\infty\}$ *such that:*

- $(M_-,\xi_-) \prec (M_+,\xi_+) \;\Rightarrow$
  $\mathrm{AT}(M_-,\xi_-) \leq \mathrm{AT}(M_+,\xi_+)$
- $\mathrm{AT}(\emptyset) = \infty$
  Hence: $\mathrm{AT}(M,\xi) < \infty \Rightarrow$ *non-fillable*
- *Overtwisted* $\Rightarrow \mathrm{AT}(M,\xi) = 0$
- *Giroux torsion* $\Rightarrow \mathrm{AT}(M,\xi) \leq 1$
- $\forall k$, $\exists (M_k^3,\xi_k)$ *with* $\mathrm{AT}(M_k,\xi_k) = k$.

Corollary:
$(M_k,\xi_k) \xrightarrow{\text{contact surgery}} (M_\ell,\xi_\ell) \Rightarrow \ell \geq k$.



Symplectic Field Theory

(Eliashberg-Givental-Hofer '00)

SFT is a (still partly conjectural) Floer-type theory for contact manifolds and symplectic cobordisms.

Data: (M^{2n-1}, ξ) with choice of

- Contact form α ($\rightsquigarrow$ Reeb vector field)
- Admissible $\mathbb{R}$-invariant almost complex structure J on the symplectisation $(\mathbb{R}\times M, d(e^t\alpha))$

To each Reeb orbit γ, associate a formal variable q_γ with degree

$$|q_\gamma| := n - 3 + \mu_{\mathrm{CZ}}(\gamma) \in \mathbb{Z}_2.$$

and a formal differential operator $p_\gamma := \hbar \dfrac{\partial}{\partial q_\gamma}$.

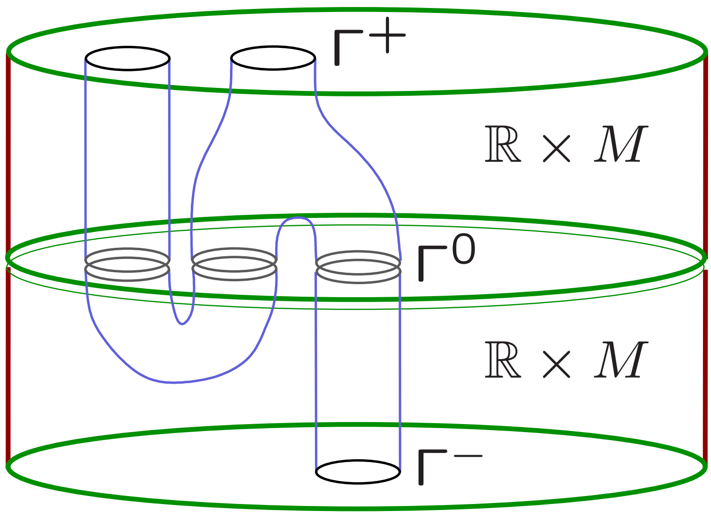
$\mathcal{A}$:= graded commutative unital $\mathbb{R}$-algebra with generators q_γ.

We define an operator

$$\mathcal{H} : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$$

by counting rigid J-holomorphic curves in $\mathbb{R} \times M$ of arbitrary genus $g \geq 0$ with positive/negative cylindrical ends asymptotic to sets of Reeb orbits $\Gamma^{\pm} = (\gamma_1^{\pm}, \ldots, \gamma_{k_{\pm}}^{\pm})$:

$$\mathcal{H} := \sum_{g,\Gamma^+,\Gamma^-} \# \left(\mathcal{M}_g(\Gamma^+, \Gamma^-)/\mathbb{R}\right) \hbar^{g-1} q^{\Gamma^-} p^{\Gamma^+}$$



Compactness/gluing theory $\Rightarrow \mathcal{H}^2 = 0$, and

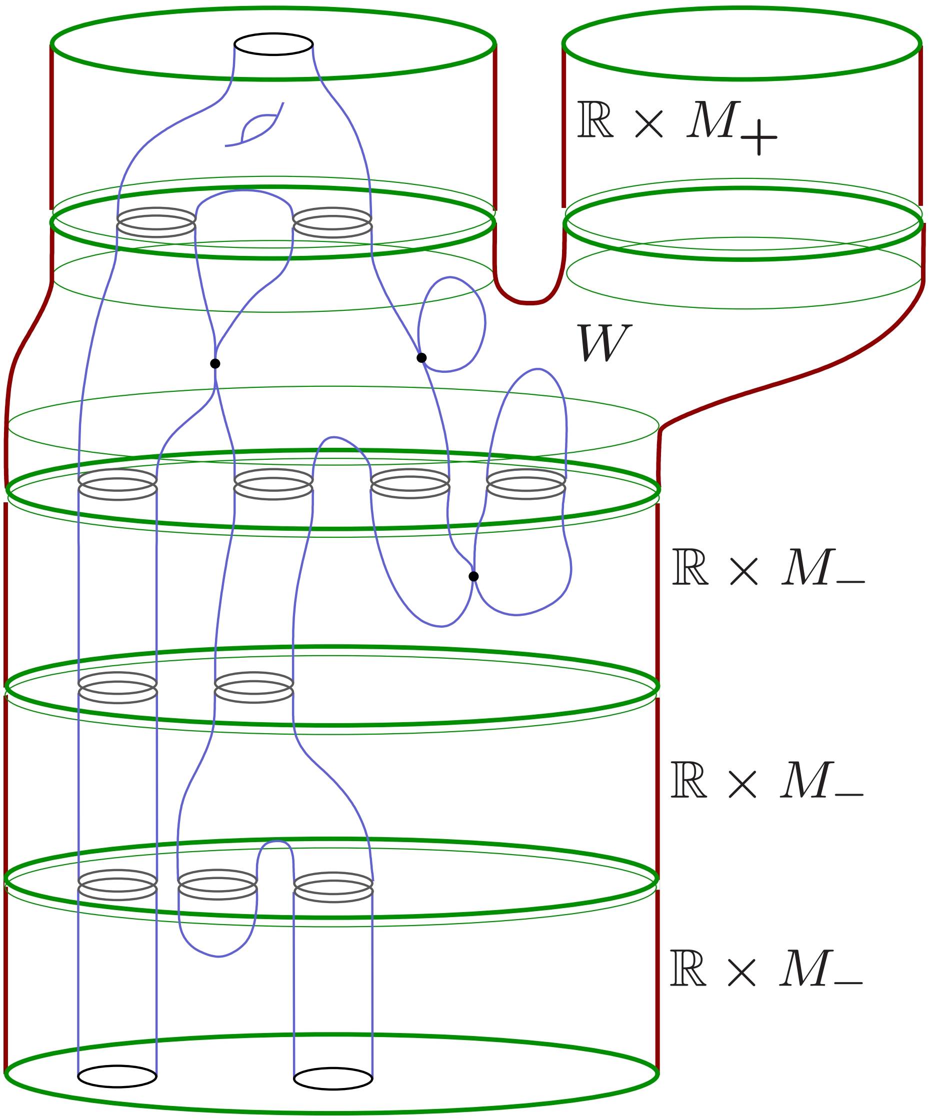
$$H_*^{\mathsf{SFT}}(M, \xi) := H_*(\mathcal{A}[[\hbar]], \mathcal{H})$$

is a **contact invariant**.

Symplectic cobordism $(M_-, \xi_-) \prec (M_+, \xi_+)$
$\Rightarrow$ **natural map**

$$H_*^{\mathsf{SFT}}(M_+, \xi_+) \to H_*^{\mathsf{SFT}}(M_-, \xi_-)$$

preserving elements of $\mathbb{R}[[\hbar]]$.

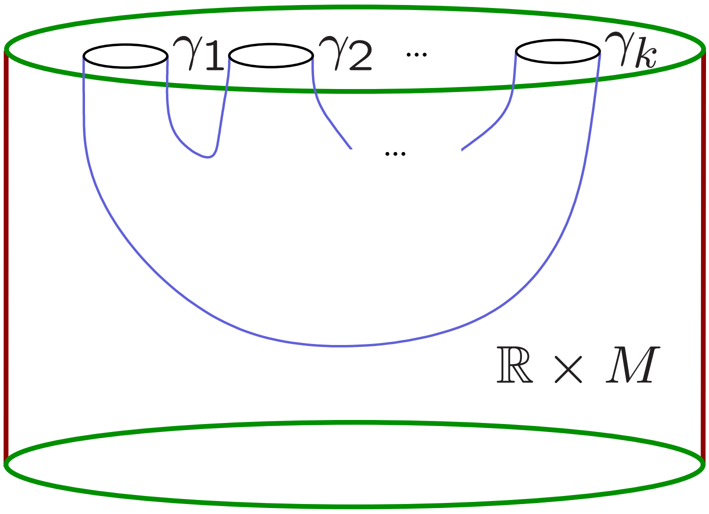


Example 1

If no periodic orbits, then $H_*^{\mathrm{SFT}}(M,\xi) = \mathbb{R}[[\hbar]]$.

Example 2

Suppose $\mathbb{R}\times M$ has exactly one rigid J-holomorphic curve, with genus 0, no negative ends, and positive ends at orbits $\gamma_1,\ldots,\gamma_k$.



Then

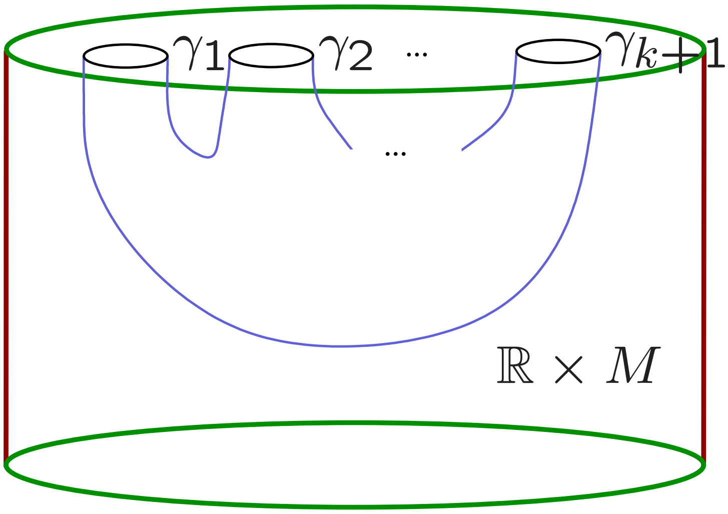
$$\mathcal{H} = \hbar^{-1} p_{\gamma_1} \ldots p_{\gamma_k}.$$

Substituting $p_{\gamma_i} = \hbar \dfrac{\partial}{\partial q_{\gamma_i}}$ gives

$$\mathcal{H}\left(q_{\gamma_1} \ldots q_{\gamma_k}\right) = \hbar^{k-1}$$

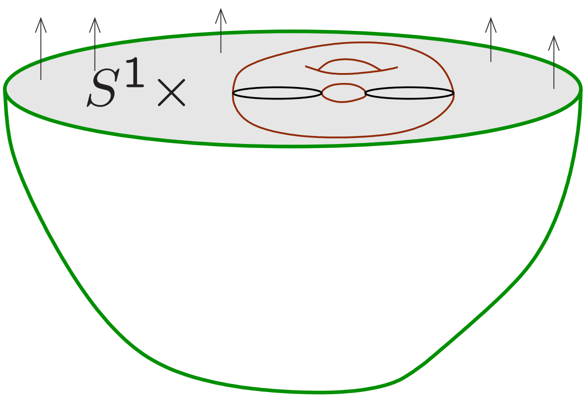
$\Rightarrow [\hbar^{k-1}] = 0 \in H_*^{\mathrm{SFT}}(M,\xi)$

Definition.
We say (M, ξ) has **algebraic k-torsion** if
$[\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi)$.



$$\boxed{\mathrm{AT}(M, \xi) := \sup \left\{ k \;\middle|\; [\hbar^{k-1}] \neq 0 \in H_*^{\mathrm{SFT}}(M, \xi) \right\}}$$

Theorem. *Algebraic k-torsion* $\Rightarrow$ *not fillable.*



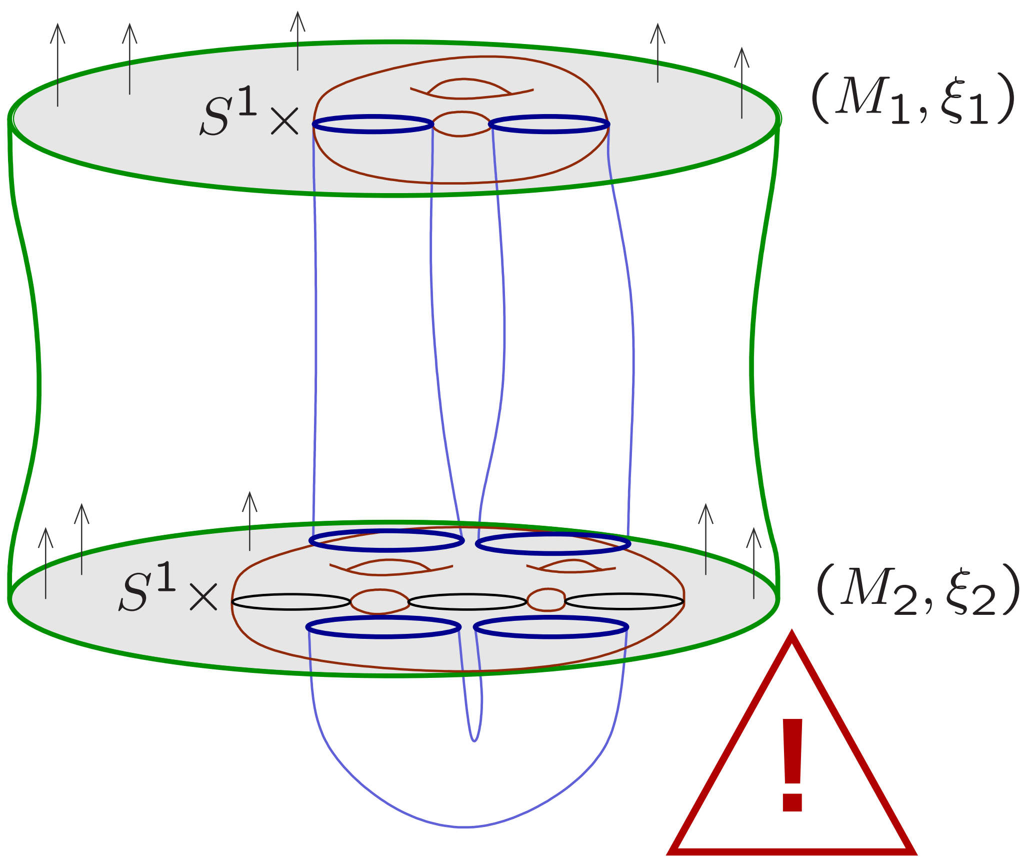
$$\begin{array}{c} [\hbar^k] = 0 \in H_*^{\mathrm{SFT}}(M, \xi) \\ \big\downarrow \\ [\hbar^k] \neq 0 \in H_*^{\mathrm{SFT}}(\emptyset) \end{array}$$

Similarly:

Theorem. $M \prec M' \Rightarrow \mathsf{AT}(M) \leq \mathsf{AT}(M')$.

What does this mean geometrically?

Example: $(M_2, \xi_2) \not\prec (M_1, \xi_1)$



What about dim > 3**?**

- (M, ξ) "PS-overtwisted" $\Rightarrow$ AT$(M, \xi) = 0$
  (Bourgeois-Niederkrüger '07)
- Examples with $1 \le$ AT$(M, \xi) < \infty$???

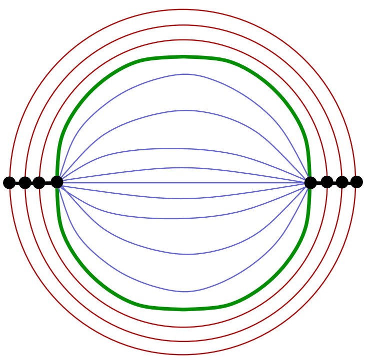
Recall: $(\mathbb{T}^3, \xi_k) \cong \Big((\mathbb{R}/2\pi k\mathbb{Z}) \times \mathbb{T}^2, \ker \alpha_{\mathrm{gt}} \Big)$,

$$\alpha_{\mathrm{gt}} := \frac{\cos s + 1}{2}\, d\theta + \frac{\cos s - 1}{2}(-d\theta) + (\sin s)\, d\phi.$$

$(\mathbb{T}^3, \xi_1) = \partial$(a trivial symplectic fibration):

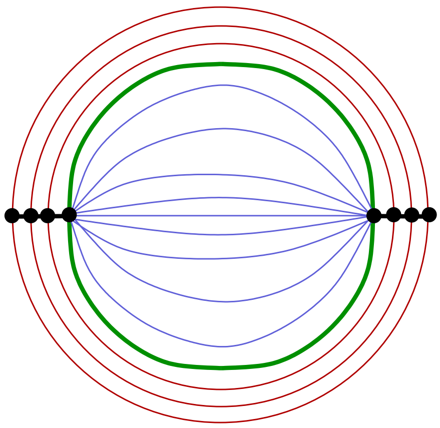
$$T^*\mathbb{T}^2 = \mathbb{R}^2 \times \mathbb{T}^2 \cong (\mathbb{R} \times S^1) \times (\mathbb{R} \times S^1)$$

where $\mathbb{R} \times S^1$ carries the exact symplectic structure $d\left(e^s\, d\theta + e^{-s}(-d\theta)\right)$.

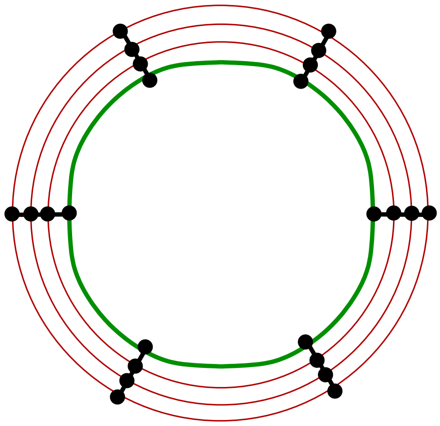


$\Rightarrow$ can foliate $T^*\mathbb{T}^2$ by holomorphic cylinders.

$\Rightarrow$ Symplectisation of $(\mathbb{T}^3, \xi_1)$ is foliated by *two* families of holomorphic cylinders, each with a "twin" that cancels it in SFT.



However, $(\mathbb{T}^3, \xi_k)$ = a k-fold cover of $(\mathbb{T}^3, \xi_1)$:



$k > 1 \Rightarrow$ non-cancelling cylinders!
$\Rightarrow [\hbar] = 0 \in H_*^{\mathsf{SFT}}(\mathbb{T}^3, \xi_k)$.

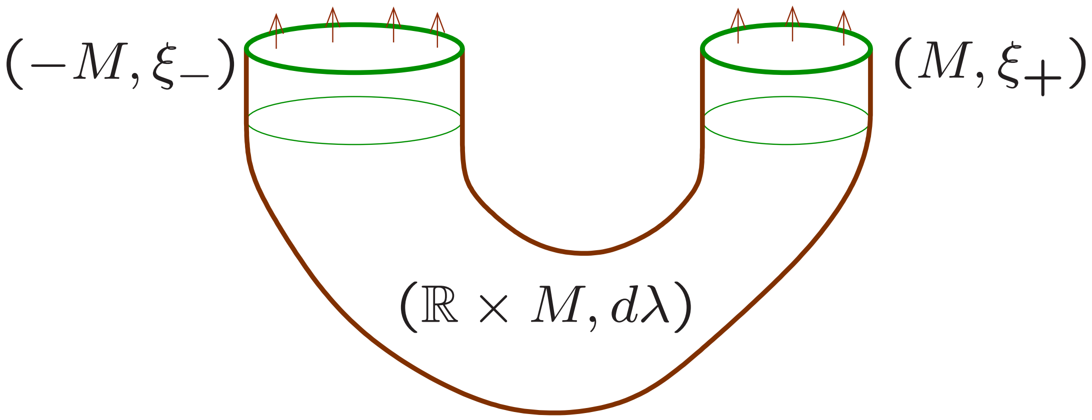
Idea: *Symplectic in dimension* $2n$
$\rightsquigarrow$ *contact in dimension* $2n+1$

Consider a trivial symplectic cylinder bundle

$$(\mathbb{R}\times M)\times(\mathbb{R}\times S^1)\to\mathbb{R}\times M,$$

where $\mathbb{R}\times M$ is *exact convex symplectic* with boundary $(M,\xi_+)\sqcup(-M,\xi_-)$.

($\exists$ examples in dim $=4,6$ by McDuff '91, Geiges, Mitsumatsu '95)



The bundle has boundary $\cong \mathbb{T}^2\times M$.

Theorem (Massot-Niederkrüger-W. '11).
For all $n\in\mathbb{N}$*, there exist closed manifolds* M^{2n-1} *with positive/negative pairs of contact forms* (α_+,α_-) *such that*

$$\left(\mathbb{R}\times M, d(e^s\alpha_+ + e^{-s}\alpha_-)\right)$$

is symplectic.

Theorem (Massot-Niederkrüger-W. '11).
In all odd dimensions, one can choose $(M, \alpha_\pm)$ *as above such that*

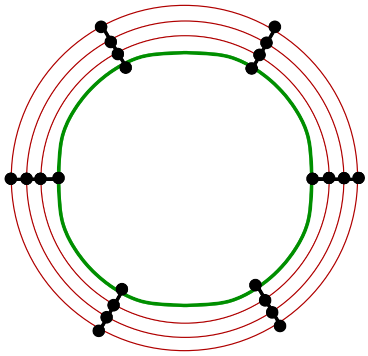
$$\alpha_{\mathrm{gt}} := \frac{\cos s + 1}{2}\,\alpha_+ + \frac{\cos s - 1}{2}\,\alpha_- + (\sin s)\, d\phi$$

defines a contact form on $\mathbb{R} \times S^1 \times M$ *with no contractible Reeb orbits. Moreover,*

$$(\mathbb{T}^2 \times M, \xi_k) := \Big((\mathbb{R}/2\pi k\mathbb{Z}) \times S^1 \times M, \ker \alpha_{\mathrm{gt}} \Big)$$

then have the following properties:

1. *All* ξ_k *are homotopic as almost contact structures but not diffeomorphic for different* $k \in \mathbb{N}$*.*
2. $(\mathbb{T}^2 \times M, \xi_k)$ *is fillable iff* $k = 1$*.*



Theorem (in progress).
For $k \geq 2$, $\mathrm{AT}(\mathbb{T}^2 \times M, \xi_k) = 1$.

Acknowledgment

Contact structure illustrations by Patrick Massot:
`http://www.math.u-psud.fr/~pmassot/`