SPINE REMOVAL SURGERY AND THE GEOGRAPHY OF SYMPLECTIC FILLINGS

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ABSTRACT. We prove that for any contact 3-manifold supported by a spinal open book decomposition with planar pages, there is a universal bound on the Euler characteristic and signature of its minimal symplectic fillings. The proof is an application of the spine removal surgery operation recently introduced in joint work of the authors with Van Horn-Morris [LVWa].

1. Introduction

It was conjectured in 2002 by Stipsicz [Sti02] that every closed contact 3-manifold admits a universal bound on the signatures and Euler characteristics of its possible Stein fillings. Counterexamples to this conjecture were found a few years ago by Baykur and Van Horn-Morris [BV15], but it was also shown by Kaloti [Kal] that the conjecture holds for the special class of *planar* contact manifolds, i.e. those that are supported (in the sense of Giroux [Gir02]) by a planar open book decomposition. Kaloti's proof was based on relations in the mapping class groups of compact planar surfaces with boundary, using a theorem of the second author [Wen10] that describes fillings of planar contact manifolds in terms of Lefschetz fibrations, together with methods of Plamenevskaya and Van Horn-Morris [PV10] to achieve bounds on numbers of positive factorizations. In this note, we shall use completely different methods to prove the following generalization of Kaloti's result:

Theorem 1. *Suppose (M, ξ) is a closed contact 3-manifold with a supporting spinal open book whose pages are planar. Then there exists a finite list of 4-manifolds $W_1', \dots, W_n'$ with boundary, and a compact 4-manifold X that has $-M$ as a boundary component, such that if (W, ω) is any minimal strong filling of (M, ξ), then $W \cup_M X \cong W_j'$ for some $j \in \{1, \dots, n\}$.*

Corollary 2. *Under the hypotheses of Theorem 1, there exists a number $N \in \mathbb{N}$ such that for all minimal strong fillings (W, ω) of (M, ξ),*

$$|\chi(W)| \leqslant N, \quad \text{and} \quad |\sigma(W)| \leqslant N.$$

Proof. If $W \cup_M X = W_j'$, then $\chi(W) = \chi(W_j') - \chi(X)$ since M has vanishing Euler characteristic, so this gives the bound on $\chi(W)$. A bound on $\sigma(W)$ follows immediately from Novikov additivity; in more elementary terms, one can also see it from the observation that the inclusion $W \hookrightarrow W_j'$ induces an injection of any positive/negative-definite subspace of $H_2(W; \mathbb{Q})$ into $H_2(W_j'; \mathbb{Q})$, implying $b_2^{\pm}(W) \leqslant b_2^{\pm}(W_j')$. □

Spinal open book decompositions were recently introduced in joint work of the authors with Jeremy Van Horn-Morris [LVWa]. The main motivation behind them is that they are the natural structure one obtains on the boundary of any Lefschetz fibration whose fibers and base are both compact oriented surfaces with nonempty boundary. As with ordinary open books, a spinal open book on (M, ξ) with pages of genus zero gives rise to a well-behaved family of pseudoholomorphic curves in the symplectization of (M, ξ), and the followup paper [LVWb] carries out the program of using this technology to classify fillings of (M, ξ). The present paper, however, does not rely on those results: our proof of Theorem 1 is comparatively low-tech. Its main ingredients are the theory of closed J-holomorphic spheres in the spirit of Gromov/McDuff (cf. [McD90]), and a

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surgical operation on spinal open books that was introduced in [LVWa], known as *spine removal surgery*. This operation is a generalization of several previous symplectic cobordism constructions that were inspired by the 2-handle attachment in Eliashberg's symplectic capping argument for contact 3-manifolds [Eli04].

For the convenience of the reader, we recall from [LVWa] the definition of a spinal open book decomposition for a closed 3-manifold. The manifold M is decomposed into two compact regions with matching boundary, the **spine** M_Σ and the **paper** M_P. These are equipped with fibrations $\pi_\Sigma \colon M_\Sigma \to \Sigma$ and $\pi_P \colon M_P \to S^1$, where Σ is a (possibly disconnected) oriented surface, each of whose connected components has non-empty boundary, and the fibers of π_Σ are assumed connected. The connected components of the fibers of π_P are called the **pages**: they also have nonempty boundaries, which are disjoint unions of fibers of π_Σ. The spinal open book decomposition is **planar** if the pages are surfaces of genus 0.

The spinal open book decomposition is then the data

$$\boldsymbol{\pi} := \Big(\pi_\Sigma : M_\Sigma \to \Sigma,\ \pi_P : M_P \to S^1\Big).$$

A contact structure is supported by this spinal open book decomposition if it admits a contact form that restricts to each fiber of π_P as a Liouville form and each fiber of π_Σ is a closed Reeb orbit.

Our conditions give that the components of M_Σ are each S^1-bundles over compact oriented surfaces $\Sigma_1, \dots, \Sigma_N$ with nonempty boundary. We fix a trivialization of each so as to identify the spine with

$$M_\Sigma = \Sigma_1 \times S^1 \amalg \dots \amalg \Sigma_N \times S^1.$$

This choice of trivializations will be referred to in the following as a **framing** of the spinal open book, and several details will depend on this choice, but the important point is that it can be fixed in advance, with no knowledge of the fillings of (M, ξ).

According to the results of [LVWa], if (M, ξ) is supported by a planar spinal open book decomposition, it can only be strongly fillable if $\boldsymbol{\pi}$ satisfies a condition known as *symmetry*, which implies that all pages have the same topology and there exist numbers $k_i \in \mathbb{N}$ for $i = 1, \dots, N$ such that exactly k_i boundary components of each page lie in $\Sigma_i \times S^1$. In the following, we will assume the spinal open book decomposition satisfies the hypotheses of Theorem 1, so, in particular, we may assume that the spinal open book decomposition is symmetric.

Remark 3. A result for planar spinal open books is also stated in [Kal], but its proof is framed in terms of Dehn twist factorizations, thus it needs to assume that the fillings of (M, ξ) are all characterized in terms of Lefschetz fibrations. The latter is not true for all spinal open books with planar pages, but only for a special class, satisfying a technical condition known as *Lefschetz-amenability* (see [LVWa, §1.1]). More generally, a spinal open book may have the property that its monodromy permutes boundary components of the pages, in which case [LVWb] produces on any filling a foliation by J-holomorphic curves that can include finitely many so-called *exotic* fibers, i.e. singularities that are different from Lefschetz singular fibers. The classification problem in these cases requires something more than an understanding of positive factorizations in the mapping class group. The following example exhibits a class of contact manifolds to which our theorem applies, but whose fillings cannot generally be understood in terms of Lefschetz fibrations with fixed boundary.

Example 4. Suppose B is a closed and connected (but not necessarily orientable) surface, and $\Gamma \subset B$ is a nonempty multicurve such that $B \backslash \Gamma$ is orientable. We say that Γ **inverts orientations** if for every sufficiently small open subset $\mathcal{U} \subset B$ that is divided by Γ into two components $\mathcal{U}_+$ and $\mathcal{U}_-$, $\mathcal{U}$ can be given an orientation that matches that of $B \backslash \Gamma$ on $\mathcal{U}_+$ and is the opposite on $\mathcal{U}_-$. Under this condition, a slight generalization of a well-known construction of Lutz [Lut77] (see [LVWa, §1.4]) assigns to any S^1-bundle $M \to B$ with oriented total space a canonical isotopy class of contact structures ξ_Γ, which are tangent to the fibers over Γ and positively transverse to the fibers in $B \backslash \Gamma$ (with respect to their orientations induced by the orientations of M and $B \backslash \Gamma$). As shown in [LVWa, §1.4], (M, ξ_Γ) admits a supporting spinal open book with a family of

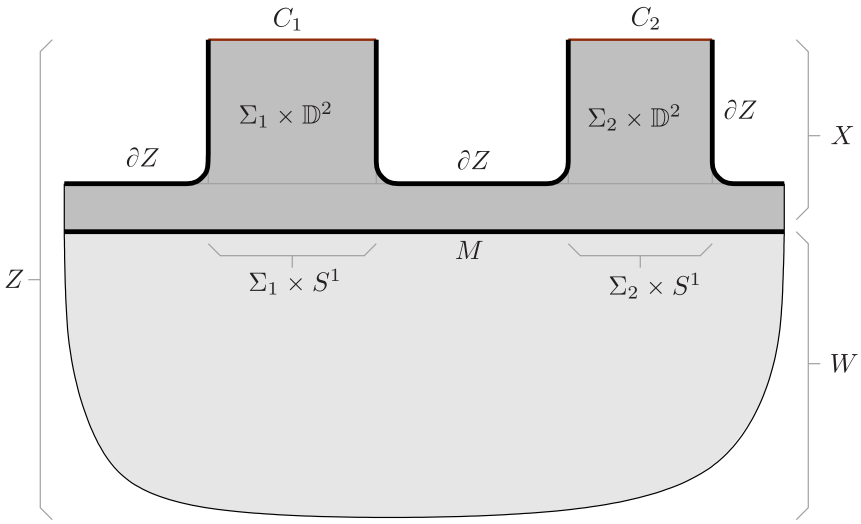

FIGURE 1. The manifold $Z = W \cup_M X$ is constructed by stacking a spine removal cobordism X on top of a given filling W of M, where X is formed by attaching a "handle" $\Sigma_i \times \mathbb{D}^2$ on top of the trivial cobordism $[0,1] \times M$ along every spinal component $\Sigma_i \times S^1 \subset M$. The picture is slightly misleading in that the subsets C_i for $i = 1, 2$ do not belong to ∂Z; rather, they are the *interior* codimension 2 symplectic submanifolds $\Sigma_i \times \{0\} \subset X$, i.e. the co-cores of the handles.

annular pages corresponding to each component of Γ, where the monodromy exchanges boundary components of the annulus for each component of Γ whose normal bundle is nontrivial. It is not hard to cook up concrete examples of this phenomenon where B is the Klein bottle, in which case (M, ξ_Γ) can also be described as a contact parabolic torus bundle (see [LVWb]). Theorem 1 applies to all contact manifolds of this type, and thus gives bounds on the geography of their minimal symplectic fillings.

Before proving Theorem 1 in detail, here is a sketch. We construct the symplectic manifold X as a spine removal cobordism that caps every boundary component of the pages (see Figure 1). Planarity implies that the rest of ∂X is then a disjoint union of symplectic S^2-fibrations over S^1, so for a generic choice of compatible almost complex structure J making these S^2-fibers complex, the usual holomorphic curve methods (as in [McD90]) produce a symplectic Lefschetz fibration

$$Z := W \cup_M X \xrightarrow{\Pi} \Sigma_0$$

over some compact oriented surface Σ_0 with boundary, where the regular fibers are J-holomorphic spheres and the singular fibers each consist of a pair of transversely intersecting J-holomorphic exceptional spheres. The co-cores of the spine removal handles

$$C_1 \cup \ldots \cup C_N \subset X \subset Z$$

define multisections of this Lefschetz fibration with degrees determined by the original spinal open book, and they are also symplectic submanifolds, so they can be arranged to be J-holomorphic. Now since $\partial \Sigma_0 \neq \emptyset$, blowing down an exceptional sphere in each singular fiber produces a trivial fibration

$$\check{Z} \cong \Sigma_0 \times S^2 \xrightarrow{\check{\Pi}} \Sigma_0,$$

implying that the topology of Z is $(\Sigma_0 \times S^2) \# m\overline{\mathbb{C}P}^2$, where m is the number of singular fibers. In order to obtain bounds on both Σ_0 and m, we observe first that each $\Pi|_{C_i} : C_i \to \Sigma_0$ is a branched cover, so that the Riemann-Hurwitz formula implies a lower bound on $\chi(\Sigma_0)$. Finally, we will find a bound on m by blowing down m exceptional spheres and considering the resulting multisections

$$\check{C}_1, \dots, \check{C}_N \subset \Sigma_0 \times S^2.$$

Since the original filling was assumed to be minimal, each exceptional sphere in Z must intersect at least one of the C_i's, thus each blow-down operation makes some positive contribution to the relative first Chern numbers of the normal bundles of $\check{C}_i$. But since the topology of $\Sigma_0 \times S^2$ is quite simple, we will also be able to show that these Chern numbers depend only on the framed link $\coprod_{i=1}^N \partial \check{C}_i \subset \partial\Sigma_0 \times S^2$, which (up to some finite ambiguity due to choices of trivialization) again depends only on the original spinal open book.

The motivating idea in this argument is that the topology of the unknown filling W can be encoded in the arrangement of positively intersecting multisections $\check{C}_1 \cup \ldots \cup \check{C}_N \subset \Sigma_0 \times S^2$, whose pattern of intersections depends on the intersections of the co-cores $C_i \subset Z$ with the singular fibers. Crucially, if we make the right choices in blowing down Z to construct $\check{Z}$, then the relative homology classes of the $\check{C}_i$ are determined by their restrictions to the boundary, and are thus independent of the choice of filling. Figures 2 and 3 show an example of how two slightly different arrangements with the same boundary can correspond to distinct minimal symplectic fillings of the same contact manifold.

Note that while it is convenient in our argument to assume the co-cores $C_i \subset X$ are J-holomorphic, it is not essential—we use this assumption mainly in order to ensure that their intersections are positive and that they have well-defined blow-downs $\check{C}_i$, but we do not need any Fredholm or compactness theory for these curves. That is fortunate, because in most cases, they live in moduli spaces of negative virtual dimension and would thus disappear under any non-trivial deformation of J. There is one exception: if the original spinal open book is an *ordinary* supporting open book in the sense of Giroux [Gir02], then each co-core C_i is a disk that can be completed to a J-holomorphic plane having index 0 and an unobstructed deformation theory. This provides a reason to expect strictly more rigidity in the presence of planar open books, suggesting in particular the following conjecture:

Conjecture. *Every planar contact 3-manifold has at most finitely many distinct deformation classes of minimal symplectic fillings.*

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2. The proof

2.1. **Spine removal.** Assume for the rest of this paper that (M, ξ) satisfies the hypotheses of Theorem 1. In particular, ξ is supported by the symmetric, planar spinal open book decomposition

$$\boldsymbol{\pi} := \Big(\pi_\Sigma : M_\Sigma \to \Sigma,\ \pi_P : M_P \to S^1\Big).$$

Fix a framing of the spinal open book, which then identifies each component of the spine with $\Sigma_i \times S^1$. Since the spinal open book decomposition is symmetric, there are integers k_i that give the incidence of any page with the spine component $\Sigma_i \times S^1$.

Denote by

$$X \cong ([0,1] \times M) \cup_{\{1\}\times M_\Sigma} \coprod_{i=1}^N \Sigma_i \times \mathbb{D}^2$$

the cobordism obtained by performing spine removal surgery on every component of M_Σ, attaching $\Sigma_i \times \mathbb{D}^2$ along its boundary in the obvious way to each spinal component $\Sigma_i \times S^1 \subset \{1\} \times M$ and then smoothing the corners (see Figure 1). According to [LVWa], this cobordism carries a symplectic structure ω_X such that we can write

$$\partial X = (-\partial_- X) \amalg \partial_+ X,$$

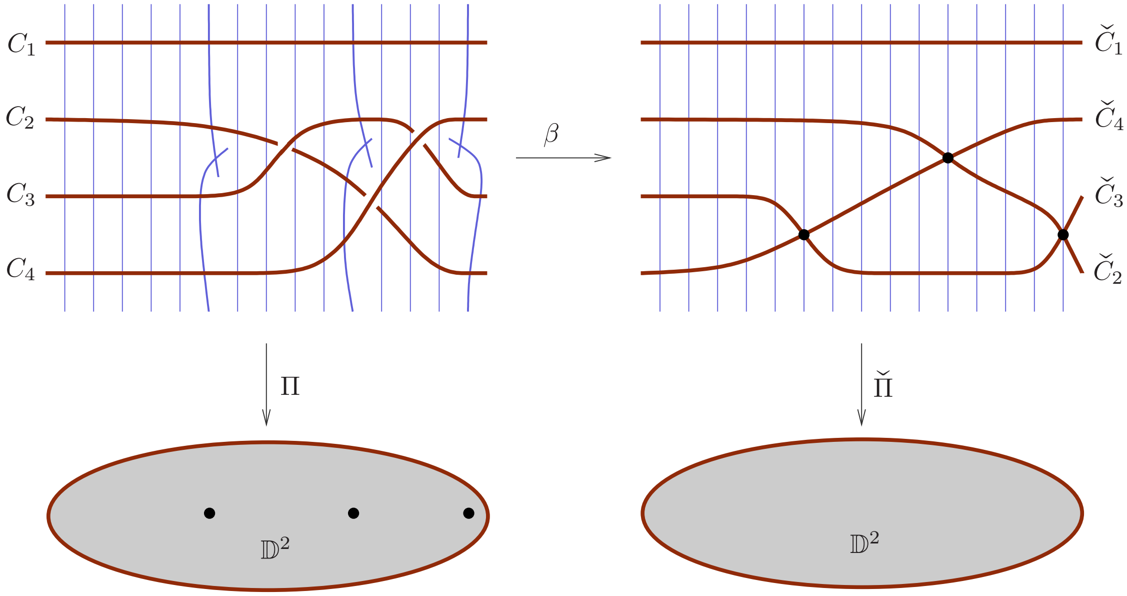

FIGURE 2. An example with $\Sigma_0 = \mathbb{D}^2$ for the Lefschetz fibration $\Pi : Z \to \Sigma_0$ and its blowdown $\breve{\Pi} : \breve{Z} \to \Sigma_0$, with four disjoint sections $C_i \subset Z$ and their (no longer disjoint) blowdowns $\breve{C}_i \subset \breve{Z}$, defined by composing them with the blowdown map $\beta : Z \to \breve{Z}$. Here $\breve{Z}$ is obtained from Z by blowing down all exceptional spheres that are disjoint from C_1.

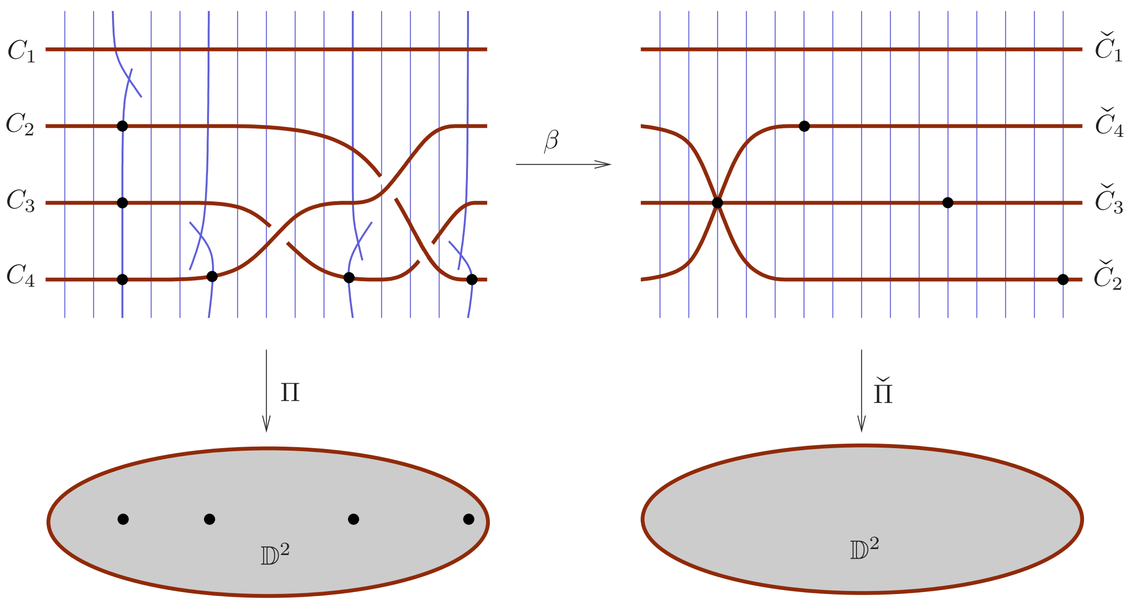

FIGURE 3. Another example of the Lefschetz fibration on $Z = W \cup_M X$ and its blowdown, but with W chosen to be a different minimal filling of the same contact manifold M as in Figure 2. In particular, the blown-down sections $\breve{C}_i \subset \breve{Z}$ are homologous to those in Figure 2 and restrict to the same framed link at the boundary. One can recover Lefschetz fibrations of both fillings by deleting neighborhoods of the sections $C_i \subset Z$ in both figures: in this example, they are related to each other by the lantern relation, so the two pictures represent the two Stein fillings of $L(4,1)$ described originally by McDuff [McD90].

where $\partial_- X = M$ is concave with induced contact structure ξ, and $\partial_+ X$ is a disjoint union of symplectic fibrations over S^1, one for each connected component of M_P, whose fibers are closed surfaces obtained by capping off all boundary components of the pages with disks. By the assumption that the spinal open book is planar, these are spheres. Since $\mathrm{Diff}_+(S^2) \simeq \mathrm{SO}(3)$ is connected, these fibrations are all trivial and are thus diffeomorphic to $S^1 \times S^2$, and each of the fibers has exactly k_i transverse and positive intersections with the symplectic co-cores

$$C_i := \Sigma_i \times \{0\} \subset \Sigma_i \times \mathbb{D}^2 \subset X$$

for $i = 1, \ldots, N$. The boundaries

$$\partial C_i = \partial \Sigma_i \times \{0\} \subset \partial_+ X$$

form a link transverse to the sphere fibers, which comes with a natural framing dependent on our original choice of framing for $\boldsymbol{\pi}$.

2.2. **Lefschetz fibration and multisections.** Now suppose (W, ω_W) is a minimal strong filling of (M, ξ). After a rescaling and deformation of ω_W near ∂W, we glue symplectically to form an enlarged symplectic manifold

$$(Z, \omega) := (W, \omega_W) \cup_M (X, \omega_X)$$

with boundary $\partial Z = \partial_+ X$. Choose an ω-compatible almost complex structure J on Z that makes both the co-cores $C_1, \ldots, C_N$ and the S^2-fibers on $\partial_+ X$ into J-holomorphic curves and is generic everywhere else. Then by standard arguments as in [McD90, Wen18], the compactified moduli space of J-holomorphic spheres homotopic to the fibers on $\partial_+ X$ forms the fibers of a Lefschetz fibration

$$\Pi : Z \to \Sigma_0,$$

where Σ_0 is a compact oriented surface (homeomorphic to the compactified moduli space) whose number of boundary components is the number of connected components of M_P. The singular fibers of $\Pi : Z \to \Sigma_0$ each have two irreducible components, both exceptional spheres which intersect each other once transversely. Let $[F] \in H_2(Z)$ denote the homology class of the fibers and $[C_i] \in H_2(Z, \partial C_i)$ the relative homology class of the co-core $C_i \subset X \subset Z$ for each $i = 1, \ldots, N$; the intersection products $[F] \cdot [C_i] \in \mathbb{Z}$ are then well defined and satisfy

$$[F] \cdot [C_i] = k_i.$$

By positivity of intersections, it follows that C_i intersects each fiber at most k_i times, with equality for the fibers to which it is transverse. A standard genericity argument (cf. [CM07, Prop. 9.1(b)]) implies:

Lemma 5. *The following holds for generic choices of J satisfying the conditions described above. All singular fibers of $\Pi : Z \to \Sigma_0$ are transverse to each co-core C_i, with intersections occurring only at regular points, while the collection of all regular fibers that intersect one of the co-cores nontransversely is finite. Moreover, every such fiber $F \subset Z$ has at most one non-transverse intersection with any of the co-cores, and it has local intersection index 2.* □

We will need a slightly more precise picture of these curves in certain regions.

Lemma 6. *Suppose $E_1, \ldots, E_m \subset Z$ is a collection of pairwise disjoint exceptional spheres that are each irreducible components of singular fibers for Π. Then after smooth and C^0-small deformations of J, the Lefschetz fibration, and the co-cores $C_1, \ldots, C_N$ such that the co-cores and fibers remain J-holomorphic, the following can be assumed without loss of generality:*

1. *For each $i = 1, \ldots, m$, J is integrable on a neighborhood of E_i, which is biholomorphically equivalent to a neighborhood of the zero-section in the tautological line bundle over $\mathbb{CP}^1$.*
2. *For each $i = 1, \ldots, N$ and each point $z_0 \in \Sigma_0$ such that the fiber over z_0 has a tangential intersection with C_i, we can identify a neighborhood $\mathcal{U} \subset \Sigma_0$ of z_0 with $\mathbb{D}^2$ and identify $\Pi^{-1}(\mathcal{U})$ with $\mathbb{D}^2 \times S^2$ where $S^2 := \mathbb{C} \cup \{\infty\}$, such that on this neighborhood, $J = i \oplus i$, $\Pi(z, w) = z$, and $(C_1 \cup \ldots \cup C_N) \cap \Pi^{-1}(\mathcal{U})$ is the disjoint union of one surface of the form*

$$\left\{ (z^2, az) \in \mathbb{D}^2 \times S^2 \;\middle|\; z \in \mathbb{D}^2 \right\}, \qquad a \in \mathbb{C} \backslash \{0\}$$

with a finite collection of other surfaces of the form

$$\{(z,b) \in \mathbb{D}^2 \times S^2 \mid z \in \mathbb{D}^2\} \qquad b \in \mathbb{C}.$$

Proof. Two preliminary remarks: first, the J-holomorphic fibers of the Lefschetz fibration will deform smoothly under any generic deformation of J. Indeed, deformations are unobstructed for all J since the curves satisfy the automatic transversality criterion of [HLS97], and for index reasons, the only danger of new bubbling under these deformations would come from index -1 curves when J becomes nongeneric, but closed holomorphic curves can only have even index (cf. [Wen18, Remark 2.20]). Second, the deformation of J does not need to be generic everywhere, as it suffices to have genericity only in some open region that intersects all of the curves we are concerned about.

Now, denote by $\widetilde{\mathbb{C}}^2 \to \mathbb{CP}^1$ the tautological line bundle, which can be obtained by performing a complex blowup on $\mathbb{C}^2$ at the origin, so let i denote its standard complex structure. Since the co-cores C_i intersect each of the exceptional spheres E_j transversely, we can identify the complex normal bundle of E_j with a J-invariant subbundle of $TZ|_{E_j}$ that is tangent to each C_i, and feeding this into the tubular neighborhood theorem then identifies a neighborhood of E_j with a neighborhood of the zero-section $\mathbb{CP}^1 \subset \widetilde{\mathbb{C}}^2$ such that J matches i along $E_j = \mathbb{CP}^1$ and each C_i is tangent to fibers of $\widetilde{\mathbb{C}}^2 \to \mathbb{CP}^1$ at the zero-section. We can then make a C^1-small deformation of each C_i near E_j so that it matches fibers *precisely* in some smaller neighborhood of E_j, and a corresponding C^0-small deformation of J to make it match i in this smaller neighborhood; since the change in C_i was C^1-small, we can assume it is still symplectic and thus adjust J correspondingly away from E_j to make sure that C_i is J-holomorphic. Notice that E_j remains holomorphic throughout this deforation, which we can also assume is supported in a tubular neighborhood of E_j; then since $[E_j] \cdot [E_j] = -1$, positivity of intersections implies that E_j is always the only closed holomorphic curve that is contained fully in this neighborhood, so that all other fibers (or irreducible components of fibers) of Π pass through regions in which J can still be assumed generic, hence they deform smoothly as indicated in the previous paragraph.

The changes near E_j can be assumed to have no effect on any fiber $F := \Pi^{-1}(z_0)$ that intersects some co-core C_i non-transversely, since F is a regular fiber. To understand the neighborhood of F, identify its (necessarily trivial) complex normal bundle with a J-invariant subbundle of $TZ|_F$ that is tangent to every co-core C_i except at the unique point where one of them is tangent to F, and use this to identify the neighborhood of F with $\mathbb{D}^2 \times S^2$ such that $F = \{0\} \times S^2$ and $J|_F = i \oplus i$. One can also arrange this so that the surfaces $\{z\} \times S^2$ are all fibers of Π, and after suitably reparametrizing S^2, we can assume all transverse intersections of $C_1 \cup \ldots \cup C_N$ with F occur at points $(0,b) \in \mathbb{D}^2 \times S^2$ with $b \neq \infty$, while the tangential intersection occurs at $(0,0)$. In light of our choice of normal subbundle, the surfaces C_j are tangent to surfaces of the form $\mathbb{D}^2 \times \{\text{const}\}$ whenever they intersect F transversely. We can now modify J near F without changing it in directions tangent to the fibers such that, after a C^0-small deformation, $J = i \oplus i$ near F; after a similar C^1-small adjustment to the co-cores C_j near their tranverse intersections with F, these will also have the form $\mathbb{D}^2 \times \{b\}$ near those intersections, and J can still be adjusted outside this small neighborhood to make the modified C_j holomorphic. Near the tangential intersection of C_i and F, Taylor's theorem and the assumption that C_i intersects F with local index 2 implies that C_i can be parametrized by an embedded holomorphic curve of the form

$$u : \mathbb{D}^2 \hookrightarrow \mathbb{D}^2 \times S^2 : z \mapsto (z^2 + R(z), az)$$

for some $a \in \mathbb{C}\backslash\{0\}$ and a smooth remainder function $R(z) \in \mathbb{C}$ satisfying $\lim_{z\to 0} \frac{R(z)}{|z|^2} = 0$. Choosing $\epsilon > 0$ small and a smooth cutoff function $\beta : [0,\infty) \to [0,1]$ with $\beta(s) = 1$ for $s \geqslant 1$ and $\beta(s) = 0$ near $s = 0$, we modify u to an embedding of the form

$$u_\epsilon(z) := (z^2 + \beta(|z|/\epsilon)R(z), az),$$

and modify C_i near $(0,0)$ by defining it as the image of u_ϵ. Modifying J to match $i \oplus i$ near F now makes u_ϵ J-holomorphic near the intersection, and for $\epsilon > 0$ sufficiently small it still traces out a symplectic submanifold C^1-close to the original C_i, thus J can be modified further outside the neighborhood of $(0,0)$ to make the new C_i everywhere J-holomorphic. The latter modification can be assumed to take place in a small neighborhood that contains no closed holomorphic curves, thus we can assume all fibers passing through that neighborhood also pass through regions where J is generic, and they therefore survive the deformation. □

Corollary 7. *For each $i = 1, \dots, N$, the map*

$$\varphi_i := \Pi|_{C_i} : C_i \to \Sigma_0$$

is a branched cover of degree k_i. Moreover, its branch points are all simple (i.e. of order 2), and for any $i, j = 1, \dots, N$, any two distinct branch points of $\varphi_i : C_i \to \Sigma_0$ and $\varphi_j : C_j \to \Sigma_0$ have distinct images in Σ_0, all of which are regular values of $\Pi : Z \to \Sigma_0$. □

We shall refer to the surfaces $C_i \subset Z$ as **multisections** of $\Pi : Z \to \Sigma_0$ with degree k_i. The multisection C_i is an honest section if and only if $k_i = 1$. Let b_i be the number of branch points of the branched covering $\varphi_i \colon C_i \to \Sigma_0$ from Corollary 7. Then, the Riemann-Hurwitz formula gives

$$k_i \chi(\Sigma_0) = \chi(C_i) + b_i$$

since each branch point is simple. We then have

$$\chi(\Sigma_0) \geqslant \frac{1}{k_i} \chi(C_i) \quad \text{for each } i = 1, \dots, N. \tag{2.1}$$

Recall that the number of boundary components of Σ_0 is given by the number of connected components of the paper M_P. This therefore gives an upper bound on the genus of Σ_0 that is determined by the spinal open book $\boldsymbol{\pi}$ and hence is independent of the choice of filling W. The topological type of Σ_0 thus belongs to a finite list of possibilities determined by (M, ξ) and $\boldsymbol{\pi}$.

Let $m \geqslant 0$ denote the number of singular fibers in the Lefschetz fibration $\Pi : Z \to \Sigma_0$. Blowing down one exceptional sphere in each of these fibers then gives a smooth S^2-fibration over Σ_0, which is necessarily trivial since $\partial \Sigma_0 \neq \emptyset$, hence

$$Z \cong (\Sigma_0 \times S^2) \# m \overline{\mathbb{C}P}^2.$$

The remaining task is thus to establish an a priori upper bound on the number of singular fibers m.

Lemma 8. *Every symplectic exceptional sphere in Z intersects at least one of the co-cores $C_1, \dots, C_N$.*

Proof. If not, then one can remove tubular neighborhoods of the co-cores and find a symplectic filling that is symplectic deformation equivalent to (W, ω) but contains an exceptional sphere, contradicting the assumption that (W, ω) is minimal. □

The argument for bounding m is slightly more straightforward if we can assume one of the numbers k_i is 1, so let us consider the case $k_1 = 1$ before tackling the general situation.

2.3. **The case $k_1 = 1$.** Under this assumption, C_1 is a section of $\Pi : Z \to \Sigma_0$, so in particular, every singular fiber consists of exactly one exceptional sphere that intersects C_1 once and one that is disjoint from C_1. We shall denote the spheres of the latter type by

$$E_1, \dots, E_m \subset Z$$

and note that they are in one-to-one correspondence with the singular fibers of $\Pi : Z \to \Sigma_0$. Applying Lemma 6 to deform J so that it becomes integrable near $E_1 \cup \ldots \cup E_m$, define $\check{Z}$ to be the symplectic manifold obtained from Z by performing a blow-down operation on each $E_1, \dots, E_m$. This symplectic manifold inherits a compatible almost complex structure $\check{J}$ such that the blowdown map

$$\beta : (Z, J) \to (\check{Z}, \check{J}),$$

defined as the identity outside of $E_1 \cup \ldots \cup E_m$ but collapsing each E_i to a point, is pseudoholomorphic. Since all singular fibers of $\Pi : Z \to \Sigma_0$ are in the interior, the boundary of $\check{Z}$ is still ∂Z and thus contains the same framed link $\coprod_{i=1}^N \partial C_i \subset \check{Z}$. The Lefschetz fibration $\Pi : Z \to \Sigma_0$ now becomes a smooth symplectic S^2-fibration

$$\check{\Pi} : \check{Z} \to \Sigma_0$$

with $\check{J}$-holomorphic fibers, and the co-cores C_i for $i = 1, \ldots, N$ project through $\beta : Z \to \check{Z}$ to immersed (but possibly non-injective) $\check{J}$-holomorphic curves

$$\check{C}_i \looparrowright \check{Z}$$

with boundary $\partial\check{C}_i = \partial C_i \subset \partial\check{Z}$. These curves are immersed since each C_i is transverse to every singular fiber by Lemma 5, but they have transverse self-intersections whenever C_i intersects one of the E_j more than once. They are again multisections of $\check{\Pi} : \check{Z} \to \Sigma_0$ with degree k_i. Notice that since C_i and C_j might both intersect some E_k, $\check{C}_i$ and $\check{C}_j$ might intersect. The co-cores C_i and C_j intersect E_k in different points, however, so the intersection of $\check{C}_i$ and $\check{C}_j$ will be transverse and positive.

If τ denotes the framing of $\partial C_i \subset \partial Z$ we fixed previously, then by construction, τ extends to a global trivialization of the complex normal bundle $N_{C_i} \subset TZ|_{C_i}$ of C_i, so that its relative first Chern number is $c_1^\tau(N_{C_i}) = 0$. After blowing down, each of these Chern numbers will in general be larger, as can be measured by counting the intersections of C_i with a small perturbation of itself: a positive transverse intersection in the blowdown is forced wherever C_i and its perturbation pass through one of the E_j, proving

$$c_1^\tau(N_{\check{C}_i}) = \sum_{j=1}^{m} [C_i] \cdot [E_j] \geqslant 0. \tag{2.2}$$

Since we chose $E_1, \ldots, E_m$ to be disjoint from C_1, we have $c_1^\tau(N_{\check{C}_1}) = 0$. Note that $\check{C}_1$ also does not intersect any of $\check{C}_2, \ldots, \check{C}_N$.

We claim that for each $i = 1, \ldots, N$, the left hand side of (2.2) satisfies a bound that is independent of the choice of minimal filling for (M, ξ). To see this, note first that we can write

$$c_1^\tau(N_{\check{C}_i}) = c_1^\tau(\check{C}_i) - \chi(C_i),$$

where $c_1^\tau(\check{C}_i)$ is an abbreviation for the relative first Chern number of the complex vector bundle $T\check{Z}$ pulled back along the immersion $\check{C}_i \looparrowright \check{Z}$, relative to the obvious trivialization induced by τ at the boundary. In particular, $\chi(C_i)$ is determined by the spinal open book, and $c_1^\tau(\check{C}_i)$ depends only on the relative homology class $[\check{C}_i] \in H_2(\check{Z}, \partial C_i)$.

To understand the latter, observe first that there are finitely many homotopy classes of trivializations of the oriented S^2-bundle

$$\partial Z = \partial\check{Z} \xrightarrow{\check{\Pi}} \partial\Sigma_0,$$

corresponding to choices of elements in $\pi_1(\mathrm{Diff}_+(S^2)) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ for each component of $\partial\Sigma_0$. Each of these trivializations identifies the framed link $\coprod_{i=1}^N \partial\check{C}_i \subset \partial\check{Z} = \partial Z$ with some isotopy class of framed links in $\partial\Sigma_0 \times S^2$ transverse to the S^2-fibers, producing a finite list of such framed links that depends only on the original framed spinal open book and not on the filling W.

Note that Σ_0 is a surface with boundary and retracts to its 1-skeleton. The fibration $\check{\Pi} : \check{Z} \to \Sigma_0$ is thus globally trivializable. After choosing such a trivialization, we may identify

$$\check{Z} = \Sigma_0 \times S^2 \xrightarrow{\check{\Pi}} \Sigma_0 : (z, w) \mapsto z,$$

where the isotopy class of the framed link $\coprod_{i=1}^N \partial\check{C}_i \subset \partial\Sigma_0 \times S^2$ must belong to the aforementioned finite list, which is independent of the filling.

We now claim that the isotopy class of this framed link uniquely determines the relative homology classes $[\check{C}_i]$ for $i = 1, \ldots, N$. Indeed, let $[\check{C}_i] \cdot_\tau [\check{C}_i] \in \mathbb{Z}$ denote the **relative self-intersection number** of $[\check{C}_i]$, defined by counting the signed intersections of $\check{C}_i$ with a small generic perturbation of itself that is pushed in the direction of τ at the boundary (cf. [Hut02, Sie11]). Then

$$[\check{C}_1] \cdot_\tau [\check{C}_1] = [C_1] \cdot_\tau [C_1] = 0$$

by construction. If $C' \looparrowright \check{Z}$ is then another properly immersed surface with $\partial C' = \partial\check{C}_1$, the relative homology classes of C' and $\check{C}_1$ differ by some absolute class in $H_2(\Sigma_0 \times S^2) \cong \mathbb{Z}$, which is generated by the fiber class $[F]$, thus $[C'] = [\check{C}_1] + \ell[F]$ for some $\ell \in \mathbb{Z}$, and we have

$$[C'] \cdot_\tau [C'] = [\check{C}_1] \cdot_\tau [\check{C}_1] + 2\ell[\check{C}_1] \cdot [F] + \ell^2[F] \cdot [F] = 2\ell$$

since $[\check{C}_1] \cdot [F] = k_1 = 1$ and $[F] \cdot [F] = 0$. It follows that C' can only have relative self-intersection zero if it is homologous to $\check{C}_1$. Thus $[\check{C}_1] \in H_2(\Sigma_0 \times S^2, \partial C_1)$ is uniquely determined. For $\check{C}_i$ with $i = 2, \ldots, N$, the fact that

$$[\check{C}_1] \cdot [\check{C}_i] = 0$$

and $[\check{C}_1] \cdot [F] \neq 0$ similarly implies that $[\check{C}_i] \in H_2(\Sigma_0 \times S^2, \partial C_i)$ is uniquely determined.

From this, it follows that for $i = 1, \ldots, N$, there exist integers $m_i \in \mathbb{Z}$ depending only on the framed spinal open book $\boldsymbol{\pi}$ such that

$$\sum_{j=1}^{m} [C_i] \cdot [E_j] \leqslant m_i.$$

At the same time, Lemma 8 implies

$$\sum_{i=1}^{N} [C_i] \cdot [E_j] \geqslant 1$$

for each $j = 1, \ldots, m$, thus

$$m \leqslant \sum_{j=1}^{m} \sum_{i=1}^{N} [C_i] \cdot [E_j] \leqslant \sum_{i=1}^{N} m_i,$$

and this bound depends only on $\boldsymbol{\pi}$ with its chosen framing. Since $Z \cong (\Sigma_0 \times S^2) \# m\overline{\mathbb{C}P}^2$, this concludes the proof for the case $k_1 = 1$.

2.4. **The general case.** The above strategy fails if none of the $k_i = [C_i] \cdot [F]$ equal 1 since it could then happen that for every singular fiber of $\Pi : Z \to \Sigma_0$, both exceptional spheres intersect every co-core C_i. We can deal with this by replacing Z by a branched cover, defined essentially as the pullback of the Lefschetz fibration $\Pi : Z \to \Sigma_0$ via the map $\varphi_1 : C_1 \to \Sigma_0$.

Let us begin with a construction at the boundary that requires no knowledge of the filling. Recall that ∂Z and $\partial \Sigma_0$ depend only on the spinal open book decomposition $\boldsymbol{\pi}$, and do not depend on the filling (though Z and Σ_0 do). Since ∂C_1 is transverse to the fibers of $\partial Z \xrightarrow{\Pi} \partial \Sigma_0$, the map $\partial C_1 \xrightarrow{\varphi_1} \partial \Sigma_0$ is a smooth k_1-fold covering map and there exists a smooth closed 3-manifold[1]

$$\partial Z' := \left\{ (z, x) \in \partial C_1 \times \partial Z \;\middle|\; \varphi_1(z) = \Pi(x) \right\},$$

which admits a smooth oriented S^2-fibration

$$\Pi' : \partial Z' \to \partial C_1 : (z, x) \mapsto z$$

and a k_1-fold covering map

$$\Phi : \partial Z' \to \partial Z : (z, x) \mapsto x$$

such that $\varphi_1 \circ \Pi' = \Pi \circ \Phi$. The framed link $\coprod_{i=1}^N \partial C_i \subset \partial Z$ then gives rise to a new framed link consisting of the disjoint union of

$$\partial C_i' := \Phi^{-1}(\partial C_i) \subset \partial Z', \qquad i = 1, \ldots, N.$$

These links are similarly transverse to the S^2-fibers, and $\partial C_1'$ contains a distinguished component given by the *tautological section*

$$\partial \sigma_1 := \{(z, z) \in \partial Z' \mid z \in \partial C_1\}$$

which is a lift of $\partial C_1 \subset \partial Z$ to the cover. All of this depends only on the framed link $\coprod_{i=1}^N \partial C_i$ in ∂Z, and thus on the original framed spinal open book, but not on the filling W.

Next we extend the covering map $\partial Z' \to \partial Z$ to a branched cover $Z' \to Z$. The construction is completely analogous to that of the previous paragraph, except that it does depend on the filling:

[1]The notation $\partial Z'$ is chosen because it will turn out to be the boundary of something constructed further below, but the definition of $\partial Z'$ itself does not require this knowledge.

recall first that by Corollary 7, the branch points of $\varphi_1 : C_1 \to \Sigma_0$ occur in regular fibers, implying that φ_1 is transverse to Π and the set

$$Z' := \left\{ (z,x) \in C_1 \times Z \;\middle|\; \varphi_1(z) = \Pi(x) \right\}$$

is therefore a smooth compact 4-manifold with boundary $\partial Z'$, while

$$\Pi' : Z' \to C_1 : (z,x) \mapsto z$$

is a Lefschetz fibration and

$$\Phi : Z' \to Z : (z,x) \mapsto x$$

is a k_1-fold branched cover satisfying $\varphi_1 \circ \Pi' = \Pi \circ \Phi$. The regular fibers of $\Pi' : Z' \to C_1$ are again spheres, and Φ is locally 2-to-1 near a branching locus that consists of a finite collection of regular fibers corresponding to the branch points of φ_1. Since singular fibers are disjoint from the branching locus, $\Pi' : Z' \to C_1$ has exactly $k_1 m$ singular fibers.

For each $i = 1, \dots, N$, let

$$C_i' := \Phi^{-1}(C_i) \subset Z'.$$

For $i = 2, \dots, N$, $C_i' \subset Z'$ is a submanifold since (by Corollary 7) the branch points of $\varphi_i : C_i \to \Sigma_0$ occur outside the branching locus of Φ, hence Φ is transverse to C_i. The situation is slightly different for $i = 1$ since Φ is not transverse to C_1, and $\Phi^{-1}(C_1)$ is the set of all pairs $(z,x) \in C_1 \times C_1$ such that z and x belong to the same fiber of Π. This contains the tautological section of $\Pi' : Z' \to C_1$ defined by

$$\sigma_1 := \{(z,z) \in Z' \mid z \in C_1\}.$$

Lemma 9. *We have $C_1' = \sigma_1 \cup C_1''$, where C_1'' is a (possibly disconnected) smooth submanifold of Z' that has finitely many intersections with σ_1, all transverse and positive, occurring at each of the points $(z,z) \in \sigma_1$ for the branch points $z \in C_1$ of $\varphi_1 : C_1 \to \Sigma_0$. Moreover, the map*

$$\varphi_1' := \Pi'|_{C_1''} : C_1'' \to C_1$$

is a branched cover of degree $k_1 - 1$.

Proof. As a set, we define C_1'' to be the closure of $C_1' \backslash \sigma_1$:

$$C_1'' := \overline{C_1' \backslash \sigma_1} \subset Z'.$$

Recall that Z' was defined as a smooth submanifold of $C_1 \times Z$, and C_1' is the intersection of this with the submanifold $C_1 \times C_1 \subset C_1 \times Z$. We claim that for any $(z,x) \in Z' \cap (C_1 \times C_1)$ with the property that z and x are not both branch points of $\varphi_1 : C_1 \to \Sigma_0$, this intersection of submanifolds in $C_1 \times Z$ is transverse. This is equivalent to the claim that the tangent space

$$T_{(z,x)} Z' = \left\{ (X,Y) \in T_z C_1 \oplus T_x Z \;\middle|\; T\varphi_1(X) - T\Pi(Y) = 0 \right\}$$

contains elements of the form (X,Y) for arbitrary vectors Y spanning some subspace of $T_x Z$ transverse to $T_x C_1$. If z is a regular point of φ_1, then $T_z \varphi_1 : T_z C_1 \to T_{\varphi_1(x)} \Sigma_0$ is an isomorphism, so for any $Y \in T_x Z$ there is a unique $X \in T_z C_1$ satisfying $T\varphi_1(X) = T\Pi(Y)$, which gives $(X,Y) \in T_{(z,x)} Z'$. The alternative is that $T_z \varphi_1$ vanishes but x is a regular point of φ_1, so $T_{(z,x)} Z' = T_z C_1 \oplus \ker T_x \Pi$, where $\ker T_x \Pi$ is the tangent space to the fiber at x. The regularity of φ_1 at x then implies that $\ker T_x \Pi$ is transverse to $T_x C_1$ and thus proves the claim.

Observe that by Corollary 7, any $(z,x) \in Z' \cap (C_1 \times C_1)$ with $z \neq x$ has the property that z and x cannot both be branch points of φ_1, since $\varphi_1(z) = \varphi_1(x)$. We have thus proved that the only possible singularities of C_1' are at points $(z,z) \in \sigma_1$ such that z is a branch point of φ_1.

We now analyze a neighborhood of any such point, using the local model from Lemma 6. Locally, we can assume $\Sigma_0 = \mathbb{D}^2 \subset \mathbb{C}$, $Z = \mathbb{D}^2 \times S^2$, $\Pi(\zeta, w) = \zeta$ and $C_1 \subset Z$ is parametrized by an embedding of the form

$$\mathbb{D}^2 \hookrightarrow \mathbb{D}^2 \times S^2 : \zeta \mapsto (\zeta^2, a\zeta) \tag{2.3}$$

for some $a \in \mathbb{C} \backslash \{0\}$. Since there are no singular fibers in this local picture, $\Pi' : Z' \to C_1$ is actually just the pullback of $\Pi : Z \to \Sigma_0$ through $\varphi_1 : C_1 \to \Sigma_0$, and the parametrization for C_1 above then provides a smooth trivialization of Π' identifying $((\zeta^2, a\zeta), (\zeta^2, x)) \in C_1 \times Z$ with $(\zeta, x) \in \mathbb{D}^2 \times S^2$. Under this identification, we can write

$$Z' = \mathbb{D}^2 \times S^2,$$

with $\Phi(\zeta, x) = (\zeta^2, x)$ and $\Pi'(\zeta, x) = (\zeta^2, a\zeta)$, and the tautological section is now parametrized by

$$\sigma_1 : \mathbb{D}^2 \hookrightarrow \mathbb{D}^2 \times S^1 : \zeta \mapsto (\zeta, a\zeta).$$

We then have

$$C_1' = \left\{(\zeta, x) \in \mathbb{D}^2 \times S^2 \;\middle|\; (\zeta^2, x) \in C_1\right\} = \left\{(\pm\zeta, a\zeta) \in \mathbb{D}^2 \times S^2 \;\middle|\; \zeta \in \mathbb{D}^2\right\},$$

which can be written as the union of two transversely and positively intersecting submanifolds

$$\{(\zeta, a\zeta)\} \cup \{(\zeta, -a\zeta)\},$$

where the first is the tautological section σ_1. It thus follows that C_1'' is a smooth submanifold of Z'.

Finally, the coordinate description above implies that the map $\varphi_1' : C_1'' \to C_1$ is a local diffeomorphism near $C_1'' \cap \sigma_1$, and it is clearly also a local diffeomorphism near all points of the form $(z, x) \in C_1''$ such that z and x are regular points of φ_1. If on the other hand $(z, x) \in C_1''$ where x is a branch point of φ_1 and z is not, then we can choose coordinates as above so that locally $\Sigma_0 = \mathbb{D}^2$ and $Z = \mathbb{D}^2 \times S^2$, with $\Pi(\zeta, w) = \zeta$ and a neighborhood of x in C_1 parametrized by a map in the form of (2.3) with $x = (0, 0)$, while a neighborhood of z in C_1 is a straightforward section

$$\mathbb{D}^2 \hookrightarrow \mathbb{D}^2 \times S^2 : \zeta \mapsto (\zeta, b)$$

with $z = (0, b)$ for some constant $b \in \mathbb{C}\backslash\{0\}$. The neighborhood of (z, x) in C_1' is thus parametrized by the embedding

$$h : \mathbb{D}^2 \hookrightarrow C_1 \times \mathbb{Z} : \zeta \mapsto \left((\zeta^2, b), (\zeta^2, a\zeta)\right),$$

which satisfies $\varphi_1'(h(z)) = (\zeta^2, b)$, showing that (z, x) is a simple branch point of φ_1'. The degree of φ_1' can be deduced by counting $\varphi_1^{-1}(z)$ for a generic point $z \in C_1$: it is the set of all pairs (z, x) such that $x \in C_1$ as in the same fiber as z but is not equal to it, so outside of the finitely many fibers that are not transverse to C_1, the number of points in this set will be exactly $k_1 - 1$, and they are all regular points by the discussion above. □

The next lemma follows from Lemma 8:

Lemma 10. *For each singular fiber of $\Pi' : Z' \to C_1$, each of the two irreducible components intersects $C_1' \cup \ldots \cup C_N'$, and the intersections are transverse and positive.* □

Lemma 11. *Each of the surfaces $C_1'', C_2', \ldots, C_N' \subset Z'$ has topological type belonging to a finite list of possibilities that depend on the spinal open book $\boldsymbol{\pi}$ and its chosen framing, but not on the filling W.*

Proof. For $i = 2, \ldots, N$, C_i' is a (possibly disconnected) multisection of $\Pi' : Z' \to C_1$ with degree k_i, and

$$\psi_i := \Phi|_{C_i'} : C_i' \to C_i$$

is a k_1-fold branched cover with exactly k_i simple branch points in every component of the branching locus of Φ. Then, using again the notation introduced following Corollary 7, where b_i is the number of branch points of $\varphi_i = \Pi|_{C_i} : C_i \to \Sigma_0$, this gives that $\psi_i : C_i' \to C_i$, is a k_1-fold branched cover with $k_i b_1$ simple branch points. Riemann-Hurwitz then implies

$$\chi(C_i') = k_1 \chi(C_i) - k_i b_1.$$

Recall from Corollary 7 and the subsequent discussion that Riemann-Hurwitz also gives us

$$\chi(C_i) = k_i \chi(\Sigma_0) - b_i.$$

It follows then

$$\begin{aligned}\chi(C_i') &= k_1\chi(C_i) - k_i b_1 \\ &= k_1\chi(C_i) - k_i(k_1\chi(\Sigma_0) - \chi(C_1)) \\ &= k_1\chi(C_i) + k_i\chi(C_1) - k_1 k_i \chi(\Sigma_0).\end{aligned}$$

Recall that $\chi(\Sigma_0) \leqslant 1$ since Σ_0 is connected (and has boundary). This then gives a lower bound

$$k_1\chi(C_i) + k_i\chi(C_1) - k_1 k_i \leqslant \chi(C_i'),$$

which only depends on the spinal open book decomposition. We also immediately obtain the upper bound $\chi(C_i') \leqslant k_1\chi(C_i)$, which also depends only on the spinal open book decomposition.

We now consider $C_1' = \sigma_1 \cup C_1''$. Observe that the number of intersections of σ_1 with C_1'' is precisely b_1, the number of branch points of φ_1. Applying the Riemann-Hurwitz formula and again using $\chi(\Sigma_0) \leqslant 1$, we obtain the following bound that depends only on $\boldsymbol{\pi}$:

$$0 \leqslant [\sigma_1] \cdot [C_1''] = -\chi(C_1) + k_1\chi(\Sigma_0) \leqslant k_1 - \chi(C_1). \tag{2.4}$$

The topologies of σ_1 and C_1'' satisfy similar bounds in terms of $\boldsymbol{\pi}$: σ_1 is diffeomorphic to C_1 since it is a section, and the topology of C_1'' can be bounded via the fact that $\varphi_1' : C_1'' \to C_1$ is a $(k_1 - 1)$-fold branched cover with $k_1 - 2$ simple branch points in each connected component of the branching locus of Φ, i.e. it has $(k_1 - 2)b_1$ branch points. Again by Riemann-Hurwitz,

$$-\chi(C_1'') + (k_1 - 1)\chi(C_1) = (k_1 - 2)b_1 = (k_1 - 2)(-\chi(C_1) + k_1\chi(\Sigma_0)).$$

Hence,

$$\chi(C_1'') = (2k_1 - 3)\chi(C_1) - k_1(k_1 - 2)\chi(\Sigma_0).$$

This is bounded below by $(2k_1 - 3)\chi(C_1) - k_1(k_1 - 2)$ since $\chi(\Sigma_0) \leqslant 1$, and above by $(2k_1 - 3)\chi(C_1) - (k_1 - 2)\chi(C_1) = (k_1 - 1)\chi(C_1)$ due to (2.1). □

We now proceed by adapting the blow-down argument that worked in the case $k_1 = 1$, but with Z replaced by Z' and the tautological section σ_1 playing the role previously played by C_1. Let τ denote the framing of the link $\coprod_{i=1}^N \partial C_i' \subset \partial Z'$ defined by lifting the framing of $\coprod_{i=1}^N \partial C_i \subset \partial Z$. In light of the bounds we have already established on the topology of Σ_0, the next lemma implies both lower and upper bounds on the value of $[\sigma_1] \cdot_\tau [\sigma_1]$.

Lemma 12. $[\sigma_1] \cdot_\tau [\sigma_1] = \chi(C_1) - k_1\chi(\Sigma_0)$.

Proof. We construct a perturbed section $\sigma_1^\epsilon \subset Z'$ as follows. Each branch point of $\varphi_1 : C_1 \to \Sigma_0$ corresponds to a point at which C_1 touches a fiber tangentially, and the Riemann-Hurwitz formula dictates that the number of such points is $-\chi(C_1) + k_1\chi(\Sigma_0)$. Now consider a coordinate neighborhood of such a point as in Lemma 6: the local model near this intersection is

$$C_1 = \left\{(z^2, az) \in \mathbb{D}^2 \times S^2 \;\middle|\; z \in \mathbb{D}^2\right\}$$

for some $a \in \mathbb{C}\backslash\{0\}$. For $\epsilon > 0$ small, define the perturbed surface

$$C_1^\epsilon := \left\{(z^2, az + \epsilon) \in \mathbb{D}^2 \times S^2 \;\middle|\; z \in \mathbb{D}^2\right\},$$

which has a single transverse and positive intersection with C_1 in this neighborhood. Now extend C_1^ϵ globally to a small perturbation of C_1 that is pushed in the direction of the framing τ at the boundary, and assume it is generic so that all other intersections between C_1 and C_1^ϵ are transverse and occur in fibers that are transverse to C_1. Since $[C_1] \cdot_\tau [C_1] = 0$, the positive intersections we see in the coordinate neighborhoods above must be canceled by further intersections, the total signed count of which is therefore $\chi(C_1) - k_1\chi(\Sigma_0)$. But C_1^ϵ can also be lifted to a section σ_1^ϵ of $\Pi' : Z' \to C_1$ that will be disjoint from σ_1 near the branching locus, while each of the other intersections of C_1 with C_1^ϵ in Z lifts to an intersection of σ_1 with σ_1^ϵ in Z', giving

$$[\sigma_1] \cdot_\tau [\sigma_1] = [\sigma_1] \cdot [\sigma_1^\epsilon] = \chi(C_1) - k_1\chi(\Sigma_0).$$

□

The rest of the argument is completely analogous to the $k_1 = 1$ case, so a sketch should now suffice. Recall that the goal is to establish a bound on the number $m \geqslant 0$ of singular fibers in $\Pi : Z \to \Sigma_0$. Using the local model in Lemma 6 to understand the branching locus of $\Phi : Z' \to Z$, there is a well-defined almost complex structure $J' := \Phi^* J$ on Z' for which the multisections $\sigma_1, C_1'', C_2', \dots, C_N'$ are all J'-holomorphic. Each of the singular fibers of $\Pi' : Z' \to C_1$ contains a unique exceptional sphere disjoint from σ_1, so denote these by $E_1, \dots, E_{k_1 m}$ and blow them down to produce a smooth S^2-fibration

$$\breve{\Pi}' : \breve{Z}' \to C_1$$

with $\breve{J}'$-holomorphic fibers for some almost complex structure such that the blowdown map $(Z', J') \to (\breve{Z}', \breve{J}')$ is pseudoholomorphic. Composing the latter with our multisections in Z' gives rise to immersed $\breve{J}'$-holomorphic multisections

$$\breve{\sigma}_1, \breve{C}_1'', \breve{C}_2', \dots, \breve{C}_N' \looparrowright \breve{Z}'.$$

Trivializing the fibration $\breve{\Pi}' : \breve{Z}' \to C_1$ then identifies $\breve{Z}'$ with $C_1 \times S^2$ so that the framed link $\coprod_{i=1}^N \partial \breve{C}_i' \subset \partial \breve{Z}'$ belongs to one of a finite collection of isotopy classes of framed links in $\partial C_1 \times S^2$ that are independent of the choice of filling. Using the established bounds on $[\breve{\sigma}_1] \cdot_\tau [\breve{\sigma}_1] = [\sigma_1] \cdot_\tau [\sigma_1]$ and the fact that $[\breve{\sigma}_1] \cdot [F] = 1$ for the fiber class $[F] \in H_2(C_1 \times S^2)$, this framed link determines $[\breve{\sigma}_1] \in H_2(C_1 \times S^2, \partial \breve{\sigma}_1)$ up to a finite ambiguity. Similarly, σ_1 has exactly $-\chi(C_1) + k_1 \chi(\Sigma_0)$ positive and transverse intersections with C_1'', which does not change after blowing down, nor does the fact that σ_1 and C_i' are disjoint for all $i = 2, \dots, N$, hence these relations also determine the relative homology classes $[\breve{C}_1'']$ and $[\breve{C}_i']$ for $i = 2, \dots, N$ up to finite ambiguity. These relative homology classes together with the respective Euler characteristics determine the relative first Chern numbers of the normal bundles of these surfaces, which are then used to bound the numbers

$$m_1 := \sum_{j=1}^{k_1 m} [C_1''] \cdot [E_j], \qquad \text{and} \qquad m_i := \sum_{j=1}^{k_1 m} [C_i'] \cdot [E_j] \text{ for } i = 2, \dots, N.$$

By Lemma 10, we also have

$$[C_1''] \cdot [E_j] + \sum_{i=2}^{N} [C_i'] \cdot [E_j] \geqslant 1$$

for each $j = 1, \dots, k_1 m$, and thus conclude

$$k_1 m \leqslant \sum_{j=1}^{k_1 m} \left([C_1''] \cdot [E_j] + \sum_{i=2}^{N} [C_i'] \cdot [E_j] \right) = \sum_{i=1}^{N} m_i,$$

giving an upper bound on m that depends only on the framed spinal open book $\boldsymbol{\pi}$. This concludes the proof in the general case.

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