HIGHER STRUCTURE: EXERCISE 2

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Exercise 1 (Transversality): Let U and V be compacted embedded submanifolds in $\mathbb{R}^N$. Denote shift $T_r : x \mapsto x + r$ on $\mathbb{R}^N$. Show that for generic $r \in \mathbb{R}^N$, $T_r(U)$ and V are transverse in $\mathbb{R}^N$.

Exercise 2: Show coproduct Δ for the tensor coalgebra $T^c_+(C[1]) := \bigoplus_{k\geq 1}(C[1])^{\otimes k}$ is coassociative: $(\Delta \times \mathrm{Id}) \circ \Delta = (\mathrm{Id} \times \Delta) \circ \Delta$.

Exercise 3: Recall for an A_∞ structure with trivial $m_i = 0$ for $i \geq 3$, we have

$$\begin{aligned} m_1 \circ m_1 &= 0, \\ m_1 \circ m_2 + m_2 \circ_1 m_1 + (-1)^{\deg x_i - 1} m_2 \circ_2 m_1 &= 0, \\ m_2 \circ_1 m_2 + (-1)^{\deg x_1 - 1} m_2 \circ_2 m_2 &= 0, \end{aligned}$$

where by convention x_1 denotes the first entry of the homogeneous element. Now, for any for some $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z}^2 \to \mathbb{Z}$, define operations $d := (-1)^{f(\deg x_1)} m_1$ and $\wedge := (-1)^{g(\deg x_1, \deg x_2)} m_2$, where we suppress the dependence on f and g in the notations. Find the general form of f and g such that the associated $(C, d, \wedge)$ is a DGA. Namely, $d \circ \wedge = \wedge \circ (d \times \mathrm{Id}) + (-1)^{\deg x_1} \wedge \circ (\mathrm{Id} \times d)$ and $\wedge \circ (\wedge \times \mathrm{Id}) = \wedge \circ (\mathrm{Id} \times \wedge)$.

Exercise 4: Show $\hat{d}$ is a coderivation, namely, $\hat{d}$ satisfies co-Leibniz (without assuming that $\{m_k\}_{k\geq 1}$ satisfies the A_∞ relation), namely, using the Sweedler notation $\Delta(x) := \sum_{(x)} x_{(1)} \otimes x_{(2)}$, $\Delta(\hat{d}x) = \sum_{(x)} \left(\hat{d}x_{(1)} \otimes x_{(2)} + (-1)^{\deg x_{(1)}} x_{(1)} \otimes \hat{d}x_{(2)}\right)$.

Exercise 5: Show that $\hat{d} \circ \hat{d} = 0$ if and only if $\{m_k\}_{k\geq 1}$ satisfies the A_∞ relation.

Exercise 6: Define the coassociative coalgebra $(\mathcal{F}^c(C), \Delta)$ equipped with linear map $\mathcal{F}^c(C) \to C$ as the cofree coassociative coalgebra generated by a vector space C by the following universal property: Let (V, μ) be a coassociative coalgebra that is nilpotent [1] and let $\varphi : V \to C$ be any linear map, then there exists a unique coalgebra map $\tilde{\varphi} : V \to \mathcal{F}^c(C)$ (that is, $(\tilde{\varphi} \otimes \tilde{\varphi}) \circ \mu = \Delta \circ \tilde{\varphi}$) such that $pr \circ \tilde{\varphi} = \varphi$.

Show that cofree coassociative coalgebra generated by V in the above sense exists, namely $T^c_+(C)$ with the projection pr onto C.

Let $\hat{d}$ be a coderivation for the cofree $(T^c_+(C[1]), \Delta)$. Then it is determined by $d := pr \circ \hat{d} : T^c_+(C[1]) \to C[1]$; and $(C, \{m_n := S^{-1} \circ d \circ \iota_n \circ (S^{\oplus n})\}_{n\geq 1})$ is an A_∞ algebra, where $\iota_n : C[1]^{\otimes n} \to T^c_+(C[1])$ is the inclusion and $S : C \to C[1]$ is the degree shift.

(Hint: Denote $\Delta^{n-1}x := \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(n)}$, then because of cofreeness,

$$pr_n(d(x)) = \sum_{i=1}^{n} \sum_{(x)} pr(x_{(1)}) \otimes \cdots \otimes d(x_{(i)}) \cdots \otimes pr(x_{(n)}),$$

where $pr_n : T^c_+(C[1]) \to C[1]^{\otimes n}$ is the projection.)

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[1] Being nilpotent means that for any $x \in V$, there exists $n = n(x)$ such that $\mu^n(x) = 0$, where the iterated composition is defined to be $\bigcirc_{i=1}^{n}(\mu \times \mathrm{Id}^{n-i}) := (\mu \times \mathrm{Id}^{n-1}) \circ \cdots \circ (\mu \times \mathrm{Id}) \circ \mu$.