HIGHER STRUCTURE: EXERCISE 3

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We will have Exercise session this Friday.

Exercise 1: Let (C, m) and (E, n) be A_∞ algebras, and let $F, \tilde{F} : (C, m) \to (E, n)$ be A_∞ morphisms. Let $D := \{D_k : C^{\otimes k} \to E\}_{k \geq 1}$, where D_k is linear and of degree $\deg D - k$. Define d_{NT} by $d_{NT} D := \{(d_{NT} D)_k\}_{k \geq 1}$, where

$$\begin{aligned}(d_{NT}D)_k = &\sum_{r, 1 \leq i \leq r} \sum_{s_1, \cdots, s_r, \sum_{i=1}^r s_i = k} (-1)^a n_r \circ (F_{s_1} \otimes \cdots \otimes F_{s_{i-1}} \otimes D_{s_i} \otimes \tilde{F}_{s_{i+1}} \otimes \cdots \otimes \tilde{F}_{s_r}) \\ &- \sum_{p,q} (-1)^b D_{k+1-p} \circ_{q+1} m_p,\end{aligned}$$

$a = (\sum_{j=1}^{s_1 + \cdots + s_{i-1}} (\deg x_j - 1))(\deg D - 1)$ and $b = \sum_{i=1}^q (\deg x_i - 1) + \deg D - 1$. The notation of NT refers to that such a D with $d_{NT} D = 0$ is a good notion of morphism[1] between A_∞-morphisms F and $\tilde{F}$.

(a) Show that $d_{NT} \circ d_{NT} = 0$. (Actually, $d_{NT} = \mathbf{d}_1$ is part of A_∞ structures $\mathbf{d} := \{\mathbf{d}_k\}_{k \geq 1}$ for target-composing two such D's.)

(b) Show that $F - \tilde{F} = d_{NT} T$ is equivalent to $\Delta \circ \hat{T} = \hat{F} \otimes \hat{T} + \hat{T} \otimes \hat{\tilde{F}}$ and $\hat{F} - \hat{\tilde{F}} = \hat{d}_n \circ \hat{T} + \hat{T} \circ \hat{d}_m$.

Exercise 2: Let (C, m) be an A_∞ algebra. Let $\Phi = \{\Phi_k : C^{\otimes k} \to C\}_{k \geq 1}$ be a formal diffeomorphism, namely, a sequence of linear map Φ_k with Φ_1 isomorphism (of graded module/vector space). On C, construct the A_∞ algebra structure $\Phi_* m := \{(\Phi_* m)_k\}_{k \geq 1}$ inductively so that $\Phi : (C, m) \to (C, \Phi_* m)$ is an A_∞ morphism (actually an isomorphism). (Hint: Denote $\tilde{m}_k := (\Phi_* m)_k$. $\tilde{m}_k$ is determined by $\tilde{m}_k \circ (\Phi_1)^{\otimes k}$, which is determined by lower order terms using A_∞ morphism identity, for the inductive step.) Find the two sided-inverse of Φ under the composition $(\Phi^1, \Phi^2) \mapsto \Phi^1 \circ \Phi^2$ where $(\Phi^1 \circ \Phi^2)_k := \sum_{s_1, \cdots, s_r, s_1 + \cdots + s_r = k} \Phi^1_r \circ (\Phi^2_{s_1} \otimes \cdots \otimes \Phi^2_{s_r})$.

Exercise 3: Using homological perturbation lemma (and possibly also using Exercise 2), show that quasi-isomorphisms between A_∞ algebras are invertible up to homotopy (here up to homotopy means that the identity holds with an error of the form $d_{NT} T$ for some T of degree -1).

Exercise 4: (Guide exercise and also good preparation for Friday's lecture) Show that for a cohomologically unital (C, m) there exists a formal diffeomorphism Φ such that $\Phi_* m$ is strictly unital.

(Hint: Suppose (C, m, e) is an A_∞ algebra with a (d, n)-strict unit, namely, $m_2(e, x) = (-1)^{\deg x} m_2(x, e) = x$ and $m_i \circ_{j+1} e = 0$ for all (i, j) with $i < d$ and (d, j) with $j < n$. Define $\Phi = (\Phi_k)_{k \geq 1}$ with non-trivial terms $\Phi_1 = \mathrm{Id}$, $\Phi_{d-1} = (-1)^{\sum_{i=1}^n (\deg x_i - 1)} m_d \circ_{n+1} e$ and $\Phi_d := (-1)^{\sum_{i=1}^n (\deg x_i - 1)} m_{d+1} \circ_{n+1} e$.

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[1] A natural transformation if we regard an A_∞ algebra as an A_∞ category with one object $*$ and then $C = \mathrm{Mor}(*, *)$.

Show that $(C, \Phi_* m, e)$ is $(d, n+1)$-strictly unital, where one notes that $(d, 0)$-strict unitality is same as $(d-1, d-1)$-strict unitality.)

Exercise 5: Show that (C^+, m^+) appeared in the definition of homotopy unital A_∞ algebra is quasi-isomorphic to (C, m).

Exercise 6: Let X be a space and $x_0 \in X$. $(X, m : X \times X \to X, x_0)$ is called an H-space, if $m(x_0, \cdot) = m(\cdot, x_0) = \mathrm{Id}_X$. A relaxed version is called homotopy H-space, where $m(x_0, \cdot)$ and $m(\cdot, x_0)$ are only required to be homotopic to Id_X by homotopy $H_1, H_2 : X \times [0,1] \to X$ respectively, here $H_i(\cdot, 1) = \mathrm{Id}_X$. Then define $X^+ := \big(X \sqcup [0,1]\big)/x_0 \sim 0$, define the multiplication m^+ on X^+ by $m^+|_{X^2} = m$, $m^+(s,t) = st$ for $s, t \in [0,1]$ and $m^+(x,t) = H_2(x,t)$ and $m^+(s,y) = H_1(y,s)$.

Note that $(X^+, m^+, 1)$ is an H-space and X^+ is homotopy equivalent to X.

(C^+, m^+) being a strictly unital A_∞ algebra constructed from a homotopy unital A_∞ algebra from (C, m) is an algebraic analogue of the above. Try to make connection more precise.

Bonus: Recall in lecture 2, we introduced the based loop space ΩN which is a homotopy H-space. Suppose $C_*(\Omega N), m$ is a chain model with Pontryagin product up to homotopy, and denote $C^* = C_{-*}(\Omega N)$ to convert it to a cochain with the same A_∞ algebra structure m. Try to construct a quasi-isomorphic cochain model with A_∞ structure and a strict unit.