Legendrian knots in surgery diagrams and the knot complement problem

MARC KEGEL

INAUGURAL - DISSERTATION
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

Köln, Januar 2017

Legendrian knots in surgery diagrams and the knot complement problem

Inaugural - Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

Marc Kegel

aus Köln

Deutschland

Köln, 2017

Berichterstatter:	Prof. Hansjörg Geiges, Ph.D.
	Prof. Silvia Sabatini, Ph.D.
Vorsitzender:	Prof. Dr. Bernd Kawohl

Tag der mündlichen Prüfung: 11.01.2017

Abstract

The famous knot complement theorem by Gordon and Luecke states that two knots in the 3-sphere are equivalent if and only if their complements are homeomorphic. By contrast, this does not hold for links in the 3-sphere or for knots in general 3-manifolds.

In this thesis the same questions are studied for Legendrian and transverse knots and links. The main results are that Legendrian as well as transverse knots in the 3-sphere with the unique tight contact structure are equivalent if and only if their exteriors are contactomorphic, where the exteriors of the transverse knots have to be sufficiently large. I will also present Legendrian links in the tight contact 3-sphere and Legendrian knots in general contact manifolds that are not determined by their exteriors.

It will turn out that these questions are closely related to the existence of cosmetic contact surgeries, i.e. contact surgeries that do not change the contact manifold. These are studied with new formulas for computing the classical invariants of Legendrian and transverse knots in the complements of surgery links.

Another application of cosmetic contact surgeries is that one can switch the crossing type of a given Legendrian or transverse knot by contact surgery without changing the contact manifold. It follows that every Legendrian or transverse knot admits a contact surgery description to an unknot. This yields connections of the classical invariants of Legendrian and transverse knots to the unknotting information of the underlying topological knot type.

Zusammenfassung

Der berühmte Satz über das Knotenkomplement von Gordon und Luecke besagt, dass zwei Knoten in der 3-Sphäre genau dann äquivalent sind, wenn ihre Komplemente homöomorph sind. Im Gegensatz dazu gilt dies nicht für Verschlingungen in der 3-Sphäre und für Knoten in allgemeinen 3-Mannigfaltigkeiten.
In dieser Doktorarbeit werden die gleichen Fragen für Legendre- und transversale Knoten und Verschlingungen untersucht. Die Hauptresultate sind, dass sowohl Legendre- als auch transversale Knoten in der 3-Sphäre mit ihrer eindeutigen straffen Kontaktstruktur genau dann äquivalent sind, wenn ihre Außenräume kontaktomorph sind, wobei die Außenräume der transversalen Knoten hinreichend groß sein müssen. Außerdem werde ich Legendre-Verschlingungen in der straffen Kontakt-3-Sphäre und Legendre-Knoten in allgemeinen Kontaktmannigfaltigkeiten präsentieren, die nicht durch ihre Außenräume bestimmt sind.
Es wird sich herausstellen, dass diese Fragen eng verknüpft sind mit der Existenz von kosmetischen Kontaktchirurgien, d.h. Kontaktchirurgien, welche die Kontaktmannigfaltigkeit nicht ändern. Um diese zu untersuchen, werden neue Formeln hergeleitet, um die klassischen Invarianten von Legendre- und transversalen Knoten in den Komplementen von Chirurgie-Verschlingungen zu berechnen.
Eine weitere Anwendung von kosmetischen Kontaktchirurgien ist, dass man mit diesen den Kreuzungstyp eines gegebenen Legendre- oder transversalen Knotens wechseln kann, ohne dabei die Kontaktmannigfaltigkeit zu ändern. Daraus folgt, dass jeder Legendre- oder transversale Knoten eine Kontaktchirurgiebeschreibung zu einem Unknoten besitzt. Dies führt zu Zusammenhängen zwischen den klassischen Invarianten von Legendre- und transversalen Knoten und den Entknotungsinformationen des zugrunde liegenden topologischen Knotentyps.

Contents

Introduction

The main goals in 3-dimensional topology are to understand 3-manifolds and surfaces and knots in these 3-manifolds. One way to do this for orientable closed connected 3-manifolds is via *Dehn surgery*. Dehn surgery is an easy and effective method to construct new 3-manifolds out of old ones. One takes a knot in a given 3-manifold, cuts out a whole tubular neighborhood of this knot and glues it, in a different way, back to the manifold. A fundamental result due to Lickorish–Wallace says that one can get every oriented closed connected 3-manifold by a sequence of Dehn surgeries from the 3-sphere S^3. This gives a nice connection between 3-manifolds and knot theory in S^3.

One aspect of this is given by the so-called *knot complement problem*. The statement of the knot complement conjecture is that two given knots in S^3 are equivalent if and only if their complements are homeomorphic. It will turn out that this conjecture is closely related to the existence of *cosmetic Dehn surgeries*, i.e. Dehn surgeries along a knot in S^3 that yield again S^3. A groundbreaking result of Gordon and Luecke says that such cosmetic surgeries are only possible along the unknot. Using this fact, it is easy to deduce that a given knot in S^3 is determined by its complement or equivalently its exterior.

This fact turns out to have a big influence on knot theory in S^3 because knot exteriors are Haken manifolds, and for Haken manifolds there exist algorithms to check if two such manifolds are homeomorphic or not.

In Chapter 1 I will recall Dehn surgery and the knot complement problem in more detail.

The main goal of this thesis is to generalize the knot complement theorem by Gordon–Luecke to Legendrian and transverse knots in the unique tight contact structure of S^3.

A *contact structure* is a completely non-integrable tangential 2-plane field on a 3-manifold. In particular, it follows that there exists no surface (no matter if closed or open) embedded in this 3-manifold which is everywhere tangent to the 2-plane field. In this sense a contact structure is the opposite of a codimension one foliation, because a codimension one foliation gives, by considering the tangent spaces of the leaves, a 2-plane field, such that at every point of the manifold there exists a surface through this point that is everywhere tangent to this 2-plane field.

Already from the definition it becomes clear that contact structures are interesting geometric objects, which is confirmed by the fact that there are a lot of natural applications of contact geometry in low-dimensional topology and classical mechanics, where examples of contact manifolds appear as energy hypersurfaces in phase spaces.

Much of the contact geometry is actually encoded in so-called Legendrian and transverse knots. *Legendrian knots* are knots that are always tangent to the contact structure, while a knot is called *transverse* if it is always transverse to the contact structure.

In Chapter 2 definitions and some basic results about contact geometry and Legendrian and transverse knots are given.

An obvious question is if the knot exterior problem holds true also for Legendrian and transverse knots in contact manifolds, i.e. is a Legendrian or a transverse knot determined by the contactomorphism type of its exterior? The main results in this thesis are that Legendrian knots in the unique tight contact structure ξ_{st} of S^3 are determined by their exteriors (see Chapter 5) and that the same holds true for transverse knots, as long as the exterior of the transverse knot is sufficiently big (see Chapter 6). In particular, this implies that all invariants of transverse or Legendrian knots in the tight contact structure of S^3 are in fact invariants of the contactomorphism type of their exteriors. On the way I will also present examples to see that this fact is not true for Legendrian links in the tight contact structure of S^3 and for Legendrian knots in general contact manifolds.

The proof strategy is a similar one as in the topological case. First one translates the knot exterior problem into the problem of excluding *cosmetic contact Dehn surgeries* along Legendrian or transverse knots. Therefore, in Chapter 3 I will explain the classical constructions how to do Dehn surgery along Legendrian or transverse knots compatible with contact structures.

Then, the strategy to exclude cosmetic contact surgeries is first to use the topological result due to Gordon and Luecke about cosmetic contact surgeries to conclude that every cosmetic contact surgery along a Legendrian knot has to be along a Legendrian unknot. The next step is to exclude cosmetic contact surgeries along Legendrian unknots which are stabilized positively and negatively. For that one looks at a Legendrian knot in the exterior of the surgery unknot. Then this Legendrian knot represents also a Legendrian knot in the new surgered manifold. By showing that the classical invariants of this knot violate the Bennequin inequality in this new contact manifold it follows directly that this new contact manifold cannot be the tight contact S^3. The idea to exclude cosmetic contact surgeries along transverse knots is the same.

Therefore, I will present in Chapter 4 new formulas (partly obtained in joint work with Sebastian Durst and Mirko Klukas) for computing the *classical invariants*, i.e. the Thurston–Bennequin invariant and the rotation number for Legendrian knots and the self-linking number for transverse knots, in contact surgery diagrams.

On the way I will classify all cosmetic surgeries along Legendrian knots in (S^3, ξ_{st}) that result again in (S^3, ξ_{st}) (see Section 5.4). As an application it follows that a given crossing of a Legendrian or transverse knot can be switched by inserting a contact surgery that does not change the manifold. By doing this for finitely many crossings of the given Legendrian or transverse knot one gets an unknot; such a contact surgery diagram yielding again (S^3, ξ_{st}) but changing a given knot to an unknot is called *contact surgery description* of the Legendrian or transverse knot.

By analyzing how the classical invariants of the given Legendrian or transverse knot change, one gets connections of the classical invariants with the unknotting information of the underlying topological knot type. This is explained in Chapter 7.

1
Dehn surgery and the knot complement problem

In this chapter I want to present the known results about the topological knot complement problem. In Section 1.1 the link complement problem for links in S^3 is discussed. First I give examples due to Whitehead of non-equivalent links in S^3 with homeomorphic complements and then present the fundamental result of Gordon and Luecke that the knot complement problem for knots in S^3 is true.

In Section 1.2 I will introduce Dehn surgery and show how to relate the knot complement problem to a problem of excluding cosmetic Dehn surgeries.

Finally, in Section 1.3 I will give well-known counterexamples to the knot complement problem in general manifolds and present a few known results in general manifolds.

The main part of this chapter is based on and very similar to Sections 2, 3, 4 and 11 of my paper [54].

This chapter differs from the rest of this thesis in the way that I work here in the topological category, i.e. I assume all maps, surfaces and 3-manifolds to be continuous, and all knots and links to be tame. While in the other chapters I usually work in the smooth category, i.e. I assume the same objects to be smooth.

Since I consider 3-manifolds only, it does not make a real difference, since every 3-manifold carries a unique smooth structure and two homeomorphic 3-manifolds are also diffeomorphic.

I assume the reader to be familiar with the basic notions of low-dimensional geometric topology on the level of the standard text books [70] and [73].

1.1 The knot complement problem

In this thesis links are mainly considered up to (coarse) equivalence.

Definition 1.1.1 (Coarse equivalence)
Let L_1 and L_2 be two links in an oriented closed 3-manifold M. Then L_1 is **(coarse) equivalent** to L_2 if there exists a homeomorphism f of M

$$f\colon M \longrightarrow M$$

that maps L_1 to L_2. Then one writes $L_1 \sim L_2$.

Remark 1.1.2 (Coarse equivalence vs. isotopy)
The (coarse) equivalence is a weaker condition than the equivalence of knots up to isotopy. For example, there is a reflection of S^3 that maps the left-handed trefoil to the right-handed trefoil, so these two knots are (coarse) equivalent, but one can show that the left handed trefoil is not isotopic to the right handed trefoil (i.e. there is no such homeomorphism isotopic to the identity).

One can also consider the **oriented (coarse) equivalence**, that means equivalence where only orientation preserving homeomorphisms of M are allowed. In S^3 oriented equivalence is equivalent to isotopy (because in S^3 every orientation preserving homeomorphism is isotopic to the identity), but in general manifolds this is a weaker condition than isotopy.

A first observation is that if two links L_1 and L_2 are equivalent, then the complements are homeomorphic. The following question, first proposed in 1908 by Heinrich Tietze [80, Section 15], is well known as the **link complement problem** (or, for one-component links, the **knot complement problem**):

Problem 1.1.3 (Link complement problem)
Are two links in the same manifold with homeomorphic complements equivalent?

Link complements are non-compact, but often it is much easier to work with compact manifolds. Therefore, pick some regular (closed) neighborhood νL of a link L in M and call the complement $M \setminus \mathring{\nu L}$ of the interior $\mathring{\nu L}$ of this neighborhood the **exterior** of L.

The corresponding problem whether the equivalence class of a link is determined by the homeomorphism type of its exterior is called the **link exterior problem**. By work of Edwards [22, Theorem 3] these two problems are equivalent. So it does not make a difference if one considers link complements or link exteriors. (However, this is an important non-trivial point; for example in contact topology this is not clear, as I will discuss later. See for example Section 6.4.)

Example 1.1.4 (The Whitehead links)
The first example of a link, that is not determined by its complement, was given in 1937 by Whitehead [85]. He considered the following two links L_1 and L_2 in S^3, now called **Whitehead links**, see Figure 1.1.

If one deletes one component out of L_1 then the remaining knot is an unknot. But if one deletes the unknot U out of L_2 then the remaining knot is a trefoil. So L_1 cannot be equivalent to L_2. But the exteriors of these links are homeomorphic, as one can see as follows: Consider the exteriors $S^3 \setminus \mathring{\nu U}$ of the unknot U. Then cut open this 3-manifold along the (Seifert) spanning disk of the unknot U, make a full

2π-twist and re-glue the two disks. In this setting, this is often called a **Rolfsen twist** and describes a homeomorphism of the link exteriors.

By twisting several times along U one can even get infinitely many non-equivalent links all with homeomorphic exteriors.

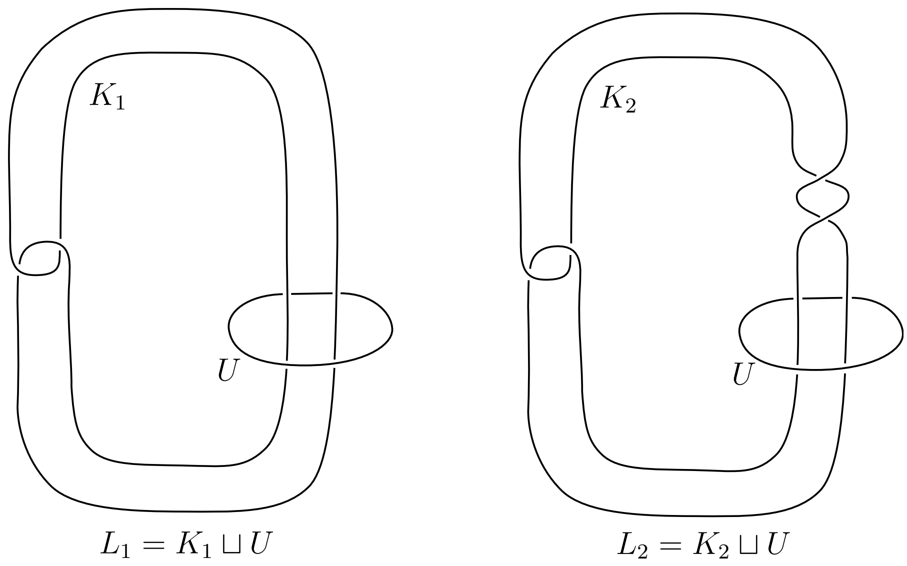


Figure 1.1: Two non-equivalent links with homeomorphic complements

The next natural question would be if this holds on the level of knots. This is the so-called knot complement theorem by Gordon–Luecke [43].

Theorem 1.1.5 (Knot complement theorem by Gordon–Luecke)
Any two knots in S^3 with homeomorphic complements are equivalent.

This highly non-trivial result does not seem to be useful at first glance, but in fact it has a very big influence on knot theory. For example in a MathSciNet review [85] Wilbur Whitten states: "The strongest and best result ever obtained in classical knot theory states that (tame) knots in S^3 are determined by their complements." The reason is that this theorem says that the knot exterior is a complete invariant of the knot. A complete invariant of a knot is an invariant that can decide if two knots are the same or not and that is also in practice computable. This invariant is computable because knot exteriors are Haken manifolds, and for Haken manifolds there exist algorithms to decide if two Haken manifolds are homeomorphic or not (in terms of Thurston's geometrization theorem for Haken manifolds, see [79] and [21, Section 4], or in terms of Haken's normal surface theory, see [44], [59] and [21, Section 4]). As far as I know, the knot exterior is the only known complete invariant of a knot.

A starting point for proving this theorem was to translate it into a problem concerning Dehn surgery.

1.2 Dehn surgery

In this section I recall the definition of Dehn surgery, which is a very effective construction method for 3-manifolds (for more information see [70, Chapter VI] or [73, Chapter 9]). First it was used by Max Dehn [12] in 1910 to construct homology spheres (see Remark 1.2.3). Roughly speaking one cuts out the neighborhood of a knot and glues a solid torus back in a different way to obtain a new 3-manifold. More precisely:

Definition 1.2.1 (Dehn surgery)
Let K be a knot in a closed oriented 3-manifold M. Take a non-trivial simple closed curve r on $\partial(\nu K)$ and a homeomorphism

$$\varphi\colon \partial(S^1 \times D^2) \longrightarrow \partial(\nu K)$$

such that

$$\{\mathrm{pt}\} \times \partial D^2 =: \mu_0 \longmapsto r.$$

Then define

$$\begin{array}{rcccc} M_K(r) & := & S^1 \times D^2 & + & M \setminus \nu\mathring{K} \\ & & \partial(S^1 \times D^2) \ni p & \sim & \varphi(p) \in \partial(\nu K). \end{array} \Big/_{\sim},$$

One says that $M_K(r)$ is obtained from M by **Dehn surgery** along K with **slope** r.

One can easily show that $M_K(r)$ is again a 3-manifold independent of the choice of φ (see [73, Chapter 9.F]). A fundamental result due to Lickorish–Wallace says that every (closed orientable connected) 3-manifold can be obtained from S^3 by Dehn surgery along a **link** (see [70, Corollary 12.4.]). So to specify a 3-manifold one only has to describe a link (in S^3) together with the slopes r_i of the link components. To do this effectively one observes that there are two special kinds of curves on $\partial(\nu K)$:

- The **meridian** μ: A simple closed curve on $\partial(\nu K)$ that is non-trivial on $\partial(\nu K)$ but trivial in νK.

- The **longitudes** λ: Simple closed curves on $\partial(\nu K)$ that are non-trivial on $\partial(\nu K)$ and intersect μ transversely exactly once.

The curves shall be oriented in such a way that the pair (μ, λ) represents the positive orientation of $\partial(\nu K)$ in M. One can show that up to isotopy the meridian μ is uniquely determined. But for the longitudes there are different choices. For a given longitude λ one gets all infinitely many other longitudes by $\tilde{\lambda} = \lambda + q\mu$, for $q \in \mathbb{Z}$. Given such a longitude λ one can write r uniquely as

$$r = p\mu + q\lambda, \text{ for } p, q \text{ coprime}.$$

For knots in S^3 (or more general nullhomologous knots in general manifolds) the usual choice for a longitude is the so-called **Seifert longitude** λ_S, obtained by pushing the knot into a **Seifert surface** of the knot, i.e. a compact orientable surface with boundary the knot. (It is easy to show that a knot is nullhomologous if and only if it admits a Seifert surface and that the Seifert longitude does not depend on the choice of the Seifert surface, see for example the text before Proposition 4.5.5 in [41].) It is easy to show that (with given Seifert longitude λ_S) for a slope $r = p\mu + q\lambda_S$ the surgered manifold $M_K(r)$ is already determined by the rational number $p/q \in \mathbb{Q} \cup \{\infty\}$ called the **(topological) surgery coefficient** (see for example [73, Section 9.G]). Depending on the context, I will denote by r the slope or the corresponding surgery coefficient.

Example 1.2.2 (Surgeries along the unknot)
I want to describe surgeries along the unknot U in S^3. In this case there is a special description of the Seifert longitude on $\partial(\nu U)$. To see this, note that the exterior $S^3 \setminus \nu\mathring{U}$ is again homeomorphic to a solid torus $S^1 \times D^2$. This corresponds to the trivial genus-1 Heegaard splitting of S^3 (see for example [70, Example 8.5.]). Write

$$\begin{aligned} T_1 &:= \quad \nu U &&\cong S^1 \times D^2, \\ T_2 &:= S^3 \setminus \nu\mathring{U} &&\cong S^1 \times D^2. \end{aligned}$$

Then, the Seifert longitude λ_1 of T_1 is given by the meridian μ_2 of T_2 (see for example [70, Figure 8.7.]). So for surgeries along the unknot in S^3 one can write the slope r uniquely as $r = p\mu_1 + q\lambda_1$.

Now, I want to show that $S^3_U(\mu_1 + q\lambda_1)$ is homeomorphic to S^3. For that, one first considers the so-called **trivial Dehn surgery** $S^3_U(\mu_1)$, where one cuts out a neighborhood of the knot and glues it back in the same way as before. So the manifold is not changing at all, and in this case it is again S^3. Then the idea is to do again a Rolfsen twist along the unknot to obtain a homeomorphism from $S^3 \cong S^3_U(\mu_1)$ to $S^3_K(\mu_1 + q\lambda_1)$. Therefore, consider the diagram that will be specified in the following.

$$\begin{tikzcd}
S^3 \cong S^3_U(\mu_1) \arrow[dddddd, dashed, "\cong"'] & := & S^1 \times D^2 \arrow[dddddd, bend right, "\mathrm{Id}"'] & + & T_2 \arrow[dddddd, bend left, "h"] & /_\sim \\
& & \mu_0 \arrow[rr, mapsto, "\varphi_1"'] & & \mu_1 & \\
& & \lambda_0 \arrow[rr, mapsto] & & \lambda_1 & \\
& & & \circlearrowright & & \\
& & \lambda_0 \arrow[rr, mapsto, "\varphi_2"'] & & \lambda_1 & \\
& & \mu_0 \arrow[rr, mapsto] & & \mu_1 + q\lambda_1 & \\
S^3_U(\mu_1 + q\lambda_1) & := & S^1 \times D^2 & + & T_2 & /_\sim
\end{tikzcd}$$

First, one chooses the gluing maps φ_i such that they map the meridians μ_0 as determined by the slope. Hence, the manifolds are fixed, but the maps φ_i are not. There are many possibilities to what a longitude λ_0 can map, but the homeomorphism type of the resulting manifold is not affected by this choice. In this example one can choose the maps φ_i such that they map λ_0 to λ_1.

To construct a homeomorphism between the two resulting manifolds, one uses on the $(S^1 \times D^2)$-factor the identity map. If one finds a homeomorphism h of the T_2-factor such that the diagram commutes, then these two maps fit together to a homeomorphism of the whole manifolds. For the map h one can choose a q-fold Dehn twist of the solid torus T_2, i.e.

$$\begin{array}{rrcl} h\colon & T_2 & \longrightarrow & T_2 \\ & \mu_1 = \lambda_2 & \longmapsto & \mu_1 + q\lambda_1 = \lambda_2 + q\mu_2 \\ & \lambda_1 = \mu_2 & \longmapsto & \lambda_1 = \mu_2. \end{array}$$

So the diagram gives rise to a homeomorphism between the two manifolds.

Remark 1.2.3 (The homology of the surgered manifolds)
For a general surgery along a knot K in S^3 one computes the homology as

$$H_1\Big(S^3_K(p\mu_1 + q\lambda_1); \mathbb{Z}\Big) = \mathbb{Z}_p.$$

So the surgeries from the above example are the only surgeries along the unknot that lead again to S^3. Such Dehn surgeries which do not change the manifold are called **cosmetic Dehn surgeries**. In fact, the cosmetic Dehn surgeries along the unknot from the foregoing example are the only ones in S^3, as the following deep theorem shows.

Theorem 1.2.4 (Surgery theorem by Gordon–Luecke)
Let K be a knot in S^3. If $S^3_K(r)$ is homeomorphic to S^3 for $r \neq \mu$, then K is equivalent to the unknot U.

Proof. see [43] □

The knot complement theorem 1.1.5 now follows easily from the surgery theorem 1.2.4. The connection is as follows. Assume first that the meridian μ of the knot K is marked on the boundary of the knot exterior. Then the knot can be recovered easily because there is a unique way to glue in a solid torus by requiring that the meridian of the solid torus should map to the meridian of the knot K. (First use the Alexander trick in dimension 2 to fill in a unique disk bounding the meridian. Then the boundary of the resulting object is a 2-sphere. By again using the Alexander trick, this time in dimension 3, there is a unique way to fill this 2-sphere with a 3-ball. This is exactly the proof why for a Dehn surgery it is enough to know the image of the meridian.) Then, one gets back the knot K as the spine of the newly glued-in solid torus. In the language of Dehn surgery this was nothing but a trivial Dehn surgery along the knot K to get back S^3.

If the meridian is not given, then the question is of course how many different curves on the boundary of the knot exterior give back S^3 by doing Dehn surgery along K with this slope. The surgery theorem says exactly that this is only possible for the unknot. To be more precise:

Proof of Theorem 1.1.5.
Choose a homeomorphism

$$h\colon S^3 \setminus \nu\mathring{K}_1 \longrightarrow S^3 \setminus \nu\mathring{K}_2$$

and write

$$S^3 \cong S^3_{K_1}(\mu_1) = S^1 \times D^2 \, + \, S^3 \setminus \nu\mathring{K}_1 \big/_{\sim}$$

where the gluing map is $\varphi_1\colon \mu_0 \mapsto \mu_1$. Then consider the surgery along K_2 with respect to the composition of maps

$$\begin{array}{ccccc} \partial\big(S^1 \times D^2\big) & \xrightarrow{\varphi_1} & \partial\big(S^3 \setminus \nu\mathring{K}_1\big) & \xrightarrow{h} & \partial\big(S^3 \setminus \nu\mathring{K}_2\big) \\ \mu_0 & \longmapsto & \mu_1 & \longmapsto & h(\mu_1) =: r_2. \end{array}$$

To determine the homeomorphism type of this new manifold $S^3_{K_2}(r_2)$ look at the following diagram:

$$\begin{tikzcd}
S^3 \cong S^3_{K_1}(\mu_1) := S^1 \times D^2 \arrow[ddd, dashed, "f"'] & & + & S^3 \setminus \nu\mathring{K}_1 \big/_{\sim} \\
& \mu_0 \arrow[rr, mapsto, "\varphi_1"'] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[drr, phantom, "\circlearrowright"] & & \mu_1 \arrow[d, bend left, "h"] \\
& \mu_0 \arrow[rr, mapsto, "h\circ\varphi_1"] & & r_2 = h(\mu_1) \\
S^3_{K_2}(r_2) := S^1 \times D^2 & & + & S^3 \setminus \nu\mathring{K}_2 \big/_{\sim}
\end{tikzcd}$$

With similar arguments as in Example 1.2.2 this induces a homeomorphism f from $S^3 \cong S^3_{K_1}(\mu_1)$ to $S^3_{K_2}(r_2)$ and with the surgery theorem 1.2.4 it follows that r_2 is equal to μ_2 or K_2 is equivalent to the unknot U.

If $r_2 = \mu_2$, then the surgery $S^3_{K_2}(r_2)$ is the trivial surgery, so the spines

$$S^1 \times \{0\} \subset S^1 \times D^2 \subset S^3$$

of the new solid tori are equal to the knots K_i. Therefore f sends K_1 to K_2.

In the other case ($K_2 \sim U$), one does the same argument again but with K_1 and K_2 reversed. It follows then that $K_1 \sim U \sim K_2$. $\square$

Remark 1.2.5 (Oriented knot complement theorem)
Exactly the same argument works also with orientations. The surgery theorem 1.2.4 also holds for oriented homeomorphism from $S^3_K(r)$ to S^3 and oriented equivalence from K to U. Then the same steps as before shows that two knots with orientation preserving homeomorphic complements are orientation preserving equivalent. For S^3 this is the same as isotopic knots (see also Remark 1.1.2).

1.3 The knot complement problem in general manifolds and cosmetic Dehn surgeries

Instead of looking only at knots in S^3 one can also look at knots in general 3-manifolds and study the knot complement problem for these knots. Of course, one can transfer this problem in an arbitrary manifold to a problem concerning Dehn surgery like before in the proof of Theorem 1.1.5 and gets by exactly the same methods the following lemma.

Lemma 1.3.1 (Criterion for the knot complement problem)
Let K be a knot in a 3-manifold M such that there is no non-trivial Dehn surgery along K resulting again in M. Then the equivalence type of K is determined by the diffeomorphism type of its complement.

But to check the hypothesis of this lemma turns out to be very difficult and in general, both the conclusion and the hypothesis of the lemma are not true. The following counterexample illustrates this (see also [74]).

Example 1.3.2 (Two non-equivalent knots with the same complements)
Consider two different Dehn surgeries along the unknot U in S^3 with surgery coefficients $r_1 = -5/2$ and $r_2 = -5/3$, leading to the lens spaces $L(5,2)$ and $L(5,3)$. By the classification of lens spaces [41, Exercise 5.3.8.(b)] these two lens spaces are orientation preserving homeomorphic and the homeomorphism is given by interchanging the two solid tori (see also [39, Section 2]).

From this Dehn surgery example it is easy to find two non-equivalent knots with the same exteriors. For this write $(i = 1, 2)$

$$L(5,2) \;\cong\; S^3_U(r_i) \;:=\; \underset{\mu_0 \longmapsto r_i}{S^1 \times D^2 \quad + \quad S^3 \setminus \nu\mathring{U}} \Big/ \sim$$

and consider the knots

$$K_i := S^1 \times \{0\} \subset S^1 \times D^2 \subset L(5,2).$$

The knots K_1 and K_2 given as the spines of the newly glued-in solid tori represent the spines of the genus-1 Heegaard splitting of $L(5,2)$. As tubular neighborhood of K_i one chooses the whole newly glued-in solid tori $S^1 \times D^2$, therefore the exterior is $S^3 \setminus \nu\mathring{U} \cong S^1 \times D^2$ in both cases.

It remains only to show that these two knots are not equivalent. Therefore, assume that there is an orientation preserving homeomorphism

$$f\colon L(5,2) \longrightarrow L(5,2)$$

mapping K_1 to K_2. By restricting f to the complementary solid tori

$$L(5,2) \setminus \nu\mathring{K}_i = S^3 \setminus \nu\mathring{U} = T_2$$

one gets a homeomorphism of T_2 which maps the slope $r_1 = -5\lambda_2 + 2\mu_2$ to the slope $r_2 = -5\lambda_2 + 3\mu_2$. But such a map cannot exist because all orientation preserving homeomorphisms of solid tori are isotopic to Dehn twists along meridians. So K_1 is not orientation preserving equivalent to K_2 in $L(5,2)$. (Similarly one shows that there do not exist orientation reversing homeomorphisms doing this.)

With the same methods as above it is easy to show (see [74]) that if K_1 and K_2 are the cores of the two solid tori in the standard Heegaard splitting of $L(p,q)$, then they have homeomorphic complements, but there is an orientation preserving (or reversing) homeomorphism of $L(p,q)$ mapping K_1 to K_2 if and only if $q^2 \equiv 1 \pmod{p}$ (or $q^2 \equiv -1 \pmod{p}$).

The key point in the foregoing example is that there is a so-called **exotic cosmetic surgery**, that means two surgeries along the same knot resulting in the same manifold but with different slopes such that there is no homeomorphism of the knot exterior mapping one slope to the other (see [7]). One can show that every knot in a given 3-manifold is determined by its complement if and only if this manifold cannot be obtained by exotic cosmetic surgery from another manifold.

Theorem 1.3.3 (Exotic cosmetic surgeries)
The following two claims are equivalent.

1. *Any two knots K_1 and K_2 in a closed 3-manifold M with homeomorphic complements are equivalent.*

2. *For any knot K' in a closed 3-manifold M' such that $M'_{K'}(r_1)$ and $M'_{K'}(r_2)$ are both homeomorphic to M (for $r_1 \neq r_2$), there exists a homeomorphism of the knot exterior*

$$\begin{array}{cccc} h\colon & M' \setminus \nu\mathring{K}' & \longrightarrow & M' \setminus \nu\mathring{K}' \\ & r_1 & \longmapsto & r_2 \end{array}$$

mapping one slope to the other.

Remark 1.3.4 (Oriented exotic cosmetic surgeries)
(1) The statement holds true in the oriented and unoriented case.
(2) Observe that the term 'cosmetic surgery' is used a bit more general than before because the manifolds M and M' are not assumed to be homeomorphic in general.

Proof of Theorem 1.3.3.
The proof of (1) $\Rightarrow$ (2) works exactly as in the foregoing example. The implication (2) $\Rightarrow$ (1) is similar to the proof of Theorem 1.1.5. □

Consequently, the study of the knot complement problem is equivalent to the study of exotic cosmetic surgeries. In the topological setting the following is known. Gabai showed in [33] that knots in $S^1 \times S^2$ (or more generally in a connected sum of arbitrary T^2- or S^2-bundles over S^1) are determined by their complements.

But in general manifolds this will not hold. Building up on work by Mathieu [58], Rong classified in [74] all knots in 3-manifolds with Seifert fibered complements that are not determined by their complements. These knots are given by the spines of the solid tori in the standard Heegaard splitting of some special lens spaces $L(p, q)$ as described at the beginning of this section or as exceptional fibers of index 2 in special Seifert fibered manifolds. But all these homeomorphisms sending one of the exceptional fibers to another one have to be orientation reversing.

Later Matignon [60] proved that all knots whose exteriors do not carry a complete hyperbolic metric in atoroidal irreducible Seifert fibered 3-manifolds are determined by their oriented complements (except the cores of the standard Heegaard splittings in lens spaces).

For knots whose exterior carry a complete hyperbolic metric there is only one counterexample until now. In [7] Bleiler, Hodgson and Weeks construct two non-equivalent hyperbolic knots in $L(49, 18)$ with orientation reversing homeomorphic complements. They also give very good reasons for the conjecture that all knots in hyperbolic 3-manifolds are determined by their oriented complements. Recently Ichihara and Jong [49] found examples of knots in hyperbolic manifolds with orientation reversing homeomorphic complements.

All together this leads to the still open oriented knot complement conjecture in general manifolds:

Conjecture 1.3.5 (Oriented knot complement conjecture)
If K_1 and K_2 are knots in a closed oriented 3-manifold M with orientation preserving homeomorphic complements (not homeomorphic to $S^1 \times D^2$), then the knots are orientation preserving equivalent.

By Theorem 1.3.3 and Remark 1.3.4 this is equivalent to the cosmetic surgery conjecture formulated in [7]:

Conjecture 1.3.6 (Oriented cosmetic surgery conjecture)
Exotic cosmetic surgeries (not resulting in a lens space) are never orientation preserving (or ***truly****) cosmetic.*

Many of the mentioned results rely on the classification of all Dehn surgeries in a solid torus resulting again in a solid torus by Berge [5] and Gabai [34]. But in the last years also Heegaard Floer homology turned out to be a very useful tool to study such problems. For example, Wang [84] showed that there are no exotic Dehn surgeries along Seifert genus-1 knots in S^3. The same holds for non-trivial algebraic knots in S^3 by [71]. Finally, in [35] and [72] it is shown that knots in L-space homology spheres are determined by their complements. In particular, knots in the Poincaré sphere are determined up to orientation preserving equivalence by the oriented homeomorphism types of their complements.

2
Contact topology

In this chapter I want to present a short introduction to some aspects of 3-dimensional contact topology. The outline follows at some points [52] and [53]. The standard reference for contact topology, Legendrian and transverse knots and the main source for most of the material presented here is [38]. Other very useful references for Legendrian and transverse knots are [69, Chapter 12] and [27].

In the first section I will explain what a contact structure is. Section 2.2 introduces Legendrian and transverse knots in contact manifolds and states the main results of this thesis, namely the extensions of the knot exterior theorem to Legendrian and transverse knots. To distinguish these knots normally one uses the so-called classical invariants, i.e. the Thurston–Bennequin invariant and the rotation number for Legendrian knots and the self-linking number for transverse knots. The definitions and some standard properties about them are given in Section 2.3. In Section 2.4 it is explained how these knots (and their classical invariants) are used to distinguish contact structures into so-called tight and overtwisted ones which is a crucial concept in contact topology.

Except the main theorems about the exteriors of Legendrian and transverse links (which will be proven later) the only new results are the facts about front projections of Legendrian knots together with transverse knots in a single front projection in Section 2.2. Everything else is well known.

From now on I will work in the smooth category, i.e. all maps, manifolds, etc. are assumed to be smooth.

2.1 Contact manifolds

The central object in contact topology is a contact structure, but before giving the precise definition I want to present the standard model of a contact structure in the 3-dimensional space $\mathbb{R}^3$.

Example 2.1.1 (The standard contact structure on $\mathbb{R}^3$)
Consider $\mathbb{R}^3$ with Cartesian coordinates (x, y, z). The so-called **standard contact structure** on $\mathbb{R}^3$ is the 2-plane field $\xi_{st} \subset T\mathbb{R}^3$ given by the following linear span of two nowhere vanishing vector fields,

$$\xi_{st} := \langle \partial_x, \partial_y - x\partial_z \rangle,$$

where ∂_x, ∂_y, and ∂_z denote the coordinate vector fields. In Figure 2.1 small parts of these planes are pictured along the xy-plane. One observes that the contact planes along the y-axis coincide with the xy-plane. If one moves along the x-axis in any direction the contact planes twist to the left and as $|x|$ gets big become arbitrary close to the xz-plane, but never tangent to it. Along all other lines parallel to the x-axis the situation is the same. (Therefore, it is enough to picture the contact planes along the xy-plane.)

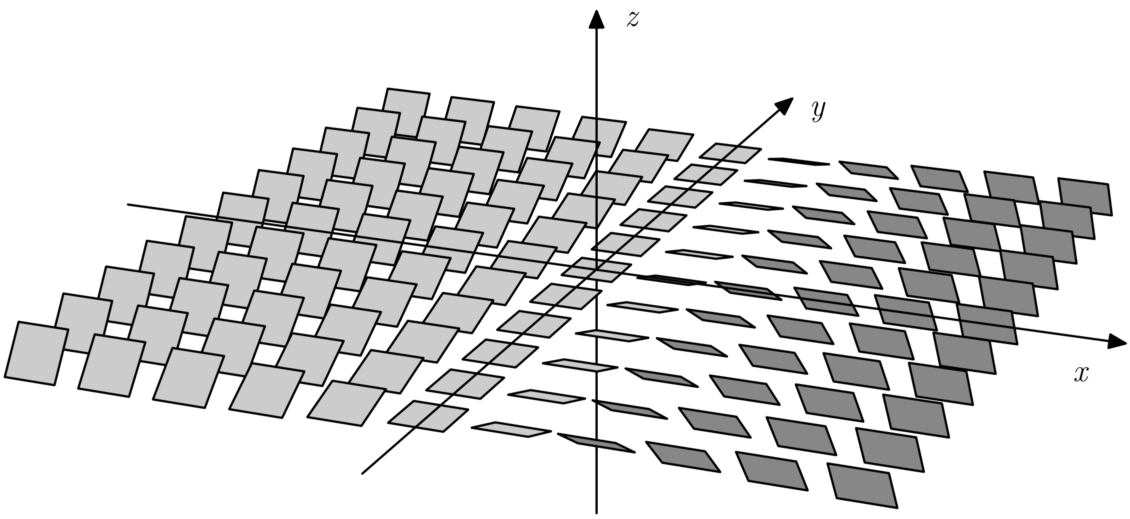


Figure 2.1: The standard contact structure ξ_{st} on $\mathbb{R}^3$. This figure is (except for some small changes in colors and axes) retrieved from Wikipedia (2016, August 31) created by user Msr657 available online at `https://en.wikipedia.org/wiki/File:Standard_contact_structure.svg`.

A special class of 2-plane fields are the **integrable** ones, i.e. 2-plane fields that arise by considering the tangent spaces of the leaves of a codimension one foliation of the manifold.

The main property of a contact structure is that it does not come from a codimension one foliation, not even locally. For the standard contact structure it can be seen in Figure 2.1 that there exists no surface (no matter if closed or open) in $\mathbb{R}^3$ that is everywhere tangent to the contact planes.

Moreover, if a surface is tangent to the contact structure at a single point then it cannot be tangent up to second order, meaning that the angle between the contact plane and the tangent space of the surface vanishes only up to first order in the point (see [38, Section 1.6]).

A way to prove that ξ_{st} admits this properties is to use the Frobenius integrability condition. For that, one first shows that every 2-plane field locally can be written as the kernel of a 1-form (see [38, Lemma 1.1.1]). For a **coorientable** 2-plane field, i.e. a 2-plane field that has a well defined normal vector field, this is even possible globally. For example the standard contact structure ξ_{st} can be written as

$$\xi_{st} = \ker(x\,dy + dz).$$

Now a 2-plane field $\xi = \ker\alpha$ is integrable if and only if α satisfies the **Frobenius integrability condition** (see [38, Section 1.1])

$$\alpha \wedge\, d\alpha = 0.$$

But for the standard contact structure ξ_{st} on $\mathbb{R}^3$ given by the kernel of the 1-form $\alpha_{st} := x\,dy + dz$, one computes that

$$\alpha_{st} \wedge\, d\alpha_{st} = dx \wedge dy \wedge dz$$

is the standard volume form on $\mathbb{R}^3$ and therefore vanishes at no point.

In this example one sees already that it is comfortable to work with 2-plane fields presented as kernels of 1-forms. Normally the definition is also given in terms of these forms.

Definition 2.1.2 (Contact manifold)
A 2-plane field $\xi = \ker\alpha$ given by the kernel of a 1-form α on a 3-manifold M is called **contact structure** if $\alpha \wedge d\alpha$ is a volume form of M. Then α is called **contact form** and the pair (M, ξ) is called **contact manifold**.

Remark 2.1.3 (Contact forms and orientations)
(1) The contact form to a given contact structure ξ is not unique. The contact form can be changed by multiplication with any function $\lambda\colon M \to \mathbb{R} \setminus \{0\}$, but the sign of the volume form does not change as

$$(\lambda\alpha) \wedge\, d(\lambda\alpha) = \lambda^2 \alpha \wedge\, d\alpha.$$

So a contact structure induces an orientation on the manifold M. Given an oriented manifold one can speak of positive or negative contact structures, depending on whether the orientation given by the contact structure coincides with the given orientation or not.
(2) In this thesis I only want to consider (as usual) coorientable contact structures, i.e. contact structures globally given as the kernel of a 1-form.
(3) A given contact form defines also an orientation and a coorientation of the contact structure. I implicitly assume that a contact structure given as the kernel of some 1-form is oriented and cooriented by this contact form.

Example 2.1.1 is fundamental because every contact structure looks locally like the standard contact structure ξ_{st} on $\mathbb{R}^3$. This is called the Theorem of Darboux (see [38, Theorem 2.5.1]).

Theorem 2.1.4 (Theorem of Darboux)
Let p be a point in an arbitrary contact manifold (M, ξ). Then there exists an open 3-ball B^3 around p and coordinates (x, y, z) on B^3 such that the contact structure ξ on B^3 is given by $\ker(x\,dy + dz)$.

The next natural step is to define maps that preserve contact structures.

Definition 2.1.5 (Contactomorphisms and isotopies)
Let (M_0, ξ_0) and (M_1, ξ_1) be contact manifolds.
(1) A diffeomorphism $f\colon M_0 \to M_1$ is called **contactomorphism** if it preserves the contact structures. For given contact forms $\xi_0 = \ker \alpha_0$ and $\xi_1 = \ker \alpha_1$ this means that there exists a function $\lambda\colon M_0 \to \mathbb{R} \setminus \{0\}$ such that $f^*\alpha_1 = \lambda\alpha_0$. Then these two contact manifolds are called **contactomorphic**.
(2) Two contact structures ξ_0 and ξ_1 on the same manifold M are called **isotopic** if there exists a smooth family of contact structures ξ_t $(t \in [0, 1])$ connecting these two contact structures.

For closed manifolds this is equivalent to the existence of a contactomorphism $f\colon (M, \xi_1) \to (M, \xi_2)$ that is isotopic (through contactomorphisms) to the identity by Gray's theorem (see [38, Theorem 2.2.2]).

A main goal in contact topology is to classify contact structures up to isotopy or contactomorphisms.

Example 2.1.6 (The standard contact structure on S^3)
Consider the 3-sphere S^3 as a subset of $\mathbb{R}^4$ with Cartesian coordinates (x, y, z, t). Then the contact form

$$\alpha_{st} := x\,dy - y\,dx + z\,dt - t\,dz$$

defines the **standard contact structure** ξ_{st} on S^3. Since $(S^3 \setminus \{p\}, \xi_{st})$ is contactomorphic to $(\mathbb{R}^3, \xi_{st})$ for every point $p \in S^3$ (see [38, Proposition 2.1.8]), this name makes sense.

Example 2.1.7 (Contact structures on $S^1 \times \mathbb{R}^2$)
Consider $S^1 \times \mathbb{R}^2$ with angular coordinate θ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and Cartesian coordinates (x, y) on $\mathbb{R}^2$. For $n \in \mathbb{Z} \setminus \{0\}$ one can define a contact structure ξ_n as the kernel of the 1-form

$$\alpha_n := \cos(n\theta)\,dx - \sin(n\theta)\,dy.$$

All these contact manifolds $(S^1 \times \mathbb{R}^2, \xi_n)$ are contactomorphic, as one can see as follows. Consider the diffeomorphisms of $S^1 \times \mathbb{R}^2$ defined by

$$f_n(\theta, x, y) := \Big(\theta, x\cos(n\theta) + \frac{y}{n}\sin(n\theta), -x\sin(n\theta) + \frac{y}{n}\cos(n\theta)\Big).$$

Then f_n is a contactomorphism of $(S^1 \times \mathbb{R}^2, \ker(dx + y\,d\theta))$ to $(S^1 \times \mathbb{R}^2, \xi_n)$. In particular all the ξ_n are contactomorphic.

2.2 Legendrian and transverse knots

As explained in Example 2.1.1 there exists no surface tangent to a contact structure. But for knots this is possible. Such knots always tangent to the contact planes are called Legendrian. In this section I will introduce these knots as well as the so-called transverse knots which are always transverse to the contact planes. These two special classes of knots fitting nicely to the contact structure are not only useful for better understanding contact structures but they also carry a lot of information about the contact structures and appear naturally in many areas of low-dimensional contact topology.

Definition 2.2.1 (Legendrian and transverse knots)
A smooth knot $K \subset (M, \xi = \ker\alpha)$ in a contact manifold is called

- **Legendrian** if $TK \subset \xi$ (or equivalently $\alpha(TK) = 0$);
- **transverse** if TK is transverse to ξ (or equivalently $\alpha(TK) \neq 0$).

Remark 2.2.2 (Positively transverse knots)
If the contact structure is cooriented (i.e. if a special contact form is given) then a transverse knot admits a canonical orientation, given by the requirement that the orientation of the transverse knot should coincide with the coorientation of the contact structure. Therefore, for oriented transverse knots it makes sense to speak of positively and negatively transverse knots. Usually, in this thesis a transverse knot is assumed to be a positive transverse knot. On the other hand, Legendrian knots do not admit a canonical orientation.

Example 2.2.3 (A Legendrian unknot)
Consider the unknot K in $(\mathbb{R}^3, \xi_{st})$ from Figure 2.2 given by

$$[0, 2\pi] \ni t \longmapsto \Big(x(t) = 3\sin(t)\cos(t), y(t) = \cos(t), z(t) = \sin^3(t)\Big) \in \mathbb{R}^3.$$

One computes $\alpha_{st}(TK) = x(t)y'(t) + z'(t) = 0$. Therefore, this unknot represents a Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$.

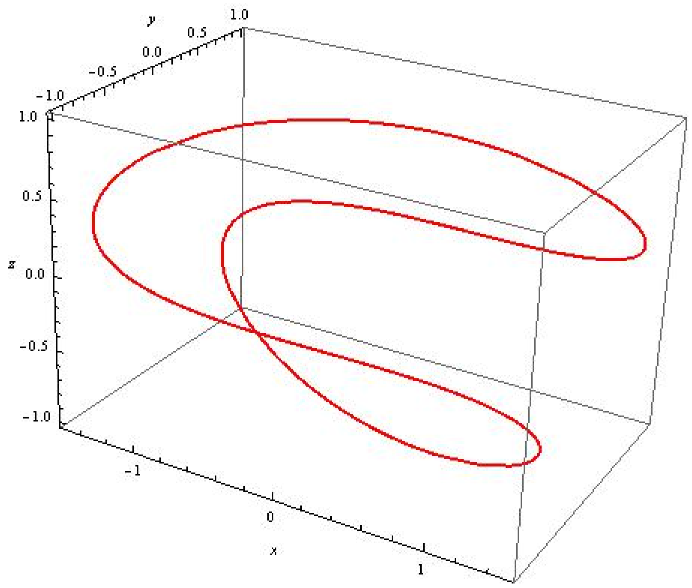


Figure 2.2: A Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$

Example 2.2.4 (Standard models)
(1) Consider the contact manifold $(S^1 \times \mathbb{R}^2, \xi_n)$ from Example 2.1.7. The knot given by $S^1 \ni \theta \mapsto (\theta, 0, 0) \in S^1 \times \mathbb{R}^2$ is a Legendrian knot because its tangent vector ∂_θ lies in the kernel of $\alpha_n = \cos(n\theta)\, dx - \sin(n\theta)\, dy$.
(2) Consider the contact manifold

$$\Big(S^1 \times \mathbb{R}^2, \ker(d\theta + r^2\, d\varphi)\Big),$$

where θ is an angular coordinate on S^1 and (r, φ) are polar coordinates on $\mathbb{R}^2$. Then the knot $S^1 \ni \theta \mapsto (\theta, 0, 0) \in S^1 \times \mathbb{R}^2$ is a transverse knot because its tangent vector never lies in the kernel of the contact form.

In fact every Legendrian or transverse knot locally looks like the knots in these two examples (for proofs see [38, Corollary 2.5.9 and Example 2.5.16]).

Theorem 2.2.5 (Neighborhood theorems)
(1) Let $K \subset (M, \xi)$ be a Legendrian knot in a contact manifold. Then there exists a ***standard tubular neighborhood*** *νK of K in M such that $(\nu K, \xi)$ is contactomorphic to $(S^1 \times D_r, \xi_n)$ for an arbitrary $r > 0$ and an arbitrary $n \in \mathbb{Z} \setminus \{0\}$ where D_r denotes the disk with radius r in $\mathbb{R}^2$. Moreover, this contactomorphism sends K to $S^1 \times \{0\} \subset S^1 \times D_r$.*
(2) Let T be a transverse knot in (M, ξ). Then there exists a ***standard tubular neighborhood*** *νT of T in M and an $\varepsilon > 0$ such that $(\nu T, \xi)$ is contactomorphic to*

$$\Big(S^1 \times D^2_\varepsilon, \ker(d\theta + r^2\, d\varphi)\Big),$$

and the contactomorphism maps T to $S^1 \times \{0\}$.

Remark 2.2.6 (Size of the standard neighborhoods)
(1) Since the map $(\theta, x, y) \mapsto (\theta, rx, ry)$ is, for every $r > 0$, a contactomorphism for the contact structures ξ_n, the size of a tubular neighborhood of a Legendrian knot is arbitrary.
(2) But in contrast to Legendrian knots, the size ε of such a neighborhood νT for transverse knots T is not arbitrary. The maximal value of this size is an invariant of the transverse knot T, as studied in [64], [11, Section 1.6] and [4].

Definition 2.2.7 (Complements and exteriors)
Let K be a Legendrian or transverse knot in (M, ξ). Then one calls $(M \setminus K, \xi)$ the **complement** of K and $(M \setminus \nu \mathring{K}, \xi)$ an **exterior** of K. If K is a Legendrian knot, then the exterior is independent of the chosen standard neighborhood. But if K is transverse then it depends on the size ε of this neighborhood.

In this thesis I want to consider Legendrian and transverse knots up to coarse equivalence.

Definition 2.2.8 (Coarse equivalence)
Let K_1 and K_2 be two Legendrian or transverse knots in (M, ξ). Then K_1 is **(coarse) equivalent** to K_2 if there exists a contactomorphism f of (M, ξ)

$$\begin{aligned} f\colon (M, \xi) &\longrightarrow (M, \xi) \\ K_1 &\longmapsto K_2 \end{aligned}$$

that maps K_1 to K_2. Then one writes $K_1 \sim K_2$.

Remark 2.2.9 (Coarse equivalence versus isotopy)
Coarse equivalence is in general a weaker condition than equivalence given by Legendrian or transverse isotopy (for example in overtwisted contact structures on S^3). But one can show that in (S^3, ξ_{st}) these two concepts are the same. (The contactomorphism group of (S^3, ξ_{st}) is trivial, but on the other hand the contactomorphism group of an overtwisted contact manifold is never trivial, see [26, Section 4.3]).

If two Legendrian or transverse knots are coarse equivalent, their complements and exteriors are contactomorphic. The Legendrian or transverse knot complement or exterior problem asks if the reverse also holds. The main results of this thesis are the following two theorems.

Theorem 2.2.10 (Legendrian knot exterior theorem)
Let K_1 and K_2 be two Legendrian knots in (S^3, ξ_{st}) with contactomorphic exteriors. Then K_1 is equivalent to K_2.

The Legendrian knot complement problem was also mentioned by Etnyre [27]. But he only discusses the relation between coarse equivalence and isotopy of Legendrian knots (compare Remark 2.2.9).

Note also that for Legendrian links Theorem 2.2.10 is in general wrong. In Section 5.2 I will give some examples of Legendrian links in (S^3, ξ_{st}) that are **not** determined by the contactomorphism type of their exteriors. Also for Legendrian knots in general contact manifolds the Legendrian knot exterior theorem is not true. Examples of non-equivalent Legendrian knots in lens spaces with contactomorphic exteriors are presented in Section 5.3. In Section 5.1 I will give the proof of this result and also explain why this does not hold true for links in (S^3, ξ_{st}).

Theorem 2.2.11 (Transverse knot exterior theorem)
Let T_1 and T_2 be two transverse knots in (S^3, ξ_{st}) with contactomorphic sufficiently big exteriors (meaning that the size of the tubular neighborhoods is sufficiently small). Then T_1 and T_2 are equivalent.

The proof of the transverse knot exterior theorem is given in Section 6.1. Counterexamples to the transverse link complement problem in (S^3, ξ_{st}) and to the transverse knot complement problem in general contact manifolds are discussed in Sections 6.2 and 6.3. For transverse knots it is not clear if the knot exterior theorem even holds for small exteriors. Also the question remains open whether this holds in both the Legendrian and the transverse case also for the complements (see Section 6.4).

Problem 2.2.12 (Contactomorphisms of closed and open knot complements)
If $(M \setminus \nu \mathring{K}_1, \xi)$ is contactomorphic to $(M \setminus \nu \mathring{K}_2, \xi)$, is $(M \setminus K_1, \xi)$ also contactomorphic to $(M \setminus K_2, \xi)$?

Remark 2.2.13 (Unoriented Legendrian links)
(1) The exterior of a knot cannot determine the orientation of the knot, so here the Legendrian knots K_1 and K_2 are normally understood to be unoriented knots. But since transverse knots carry a canonical orientation, it is sufficient to restrict in the transverse case to positive transverse knots and consequently Theorem 2.2.11 holds also for oriented transverse knots. In the Legendrian case one can fix an oriented longitude of the knot in its exterior and then the same result holds also for oriented Legendrian knots.
(2) The significance of the topological knot exterior theorem is, as explained earlier, the fact that there are algorithms to determine whether two knot exteriors are homeomorphic. One of these algorithms due to Matveev [59] (following the fundamental ideas of Haken [44]) works very roughly like this: One cuts open the knot exteriors along special so-called normal surfaces in such a way that the result is a collection of

3-balls together with a way to glue them together again. By cutting in a clever way, it is enough to consider only finitely many possibilities for cutting. Instead of comparing the knot exteriors it is then enough to compare only the resulting collections of 3-balls.

By doing the same to exteriors of Legendrian or transverse knots one ends up with collections of contact 3-balls together with a way to glue them back to the contact knot exterior. It should be possible to compare these results also in the contact case. This would result in an algorithm to determine whether two Legendrian or two transverse knots in (S^3, ξ_{st}) are isotopic or not.

Next, I want to present a way to describe and study Legendrian and transverse knots in diagrams. On the way I will present many examples of such knots (for more details see [38, Chapter 3]).

Given a Legendrian or a transverse knot K in (S^3, ξ_{st}), by Example 2.1.6 one can view this knot also as a knot in $(\mathbb{R}^3, \xi_{st})$. These knots are usually presented in the **front projection**

$$(x, y, z) \longmapsto (y, z).$$

For a generic Legendrian knot $K(t) = (x(t), y(t), z(t)) \subset (\mathbb{R}^3, \xi_{st})$ one can assume $y'(t)$ to vanish only in isolated points and then one can recover the Legendrian knot completely from its front projection via

$$x(t) = -\frac{z'(t)}{y'(t)} = -\frac{dz}{dy}$$

and its smooth extension (in the isolated points where $y'(t) = 0$). From this equation one also sees that in the front projection crossings as in Figure 2.3 are forbidden. Moreover, there are no vertical tangencies.

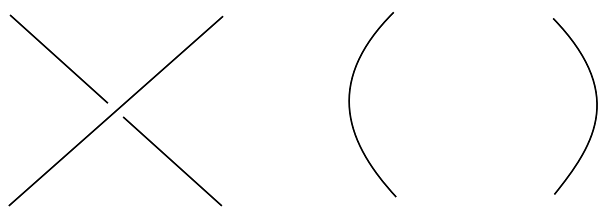


Figure 2.3: Impossible front projections of Legendrian knots

Instead of vertical tangencies a Legendrian knot has so-called **cusps** like in Figure 2.4. Typical front projections of Legendrian knots are pictured in Figure 2.4. Conversely, every such front projection containing only semi-cubical cusps (see [38, Lemma 3.2.3]) and crossings as described above represents a unique Legendrian knot.

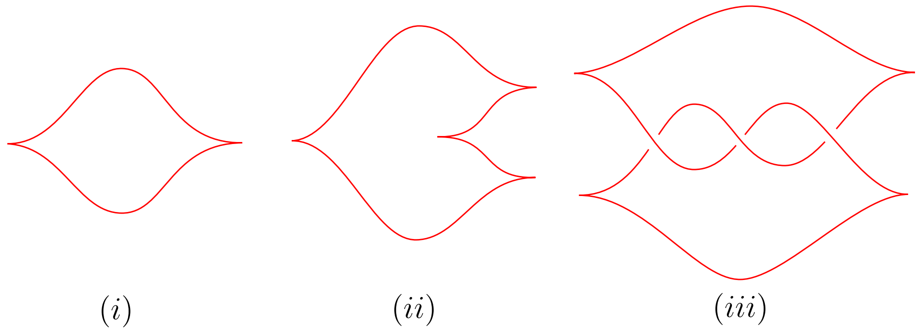


Figure 2.4: Typical Legendrian front projections

For transverse knots in the front projection the above construction works similar. The condition for a knot K to be positively transverse (negatively works similar) is given by $z'(t) + x(t)y'(t) > 0$. So the situations shown in Figure 2.5 are not possible in the front projection of positive transverse knots.

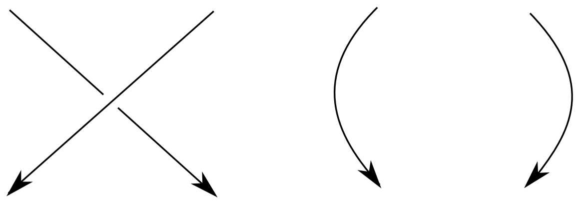


Figure 2.5: Impossible front projections of positive transverse knots

All other situations are possible, but in this case one cannot recover the transverse knot from its front projection (because there is only an inequality for the x-value). However, it is easy to show that every such diagram represents a unique positively transverse knot up to transverse isotopy. Typical examples of positively transverse knots are shown in Figure 2.6.

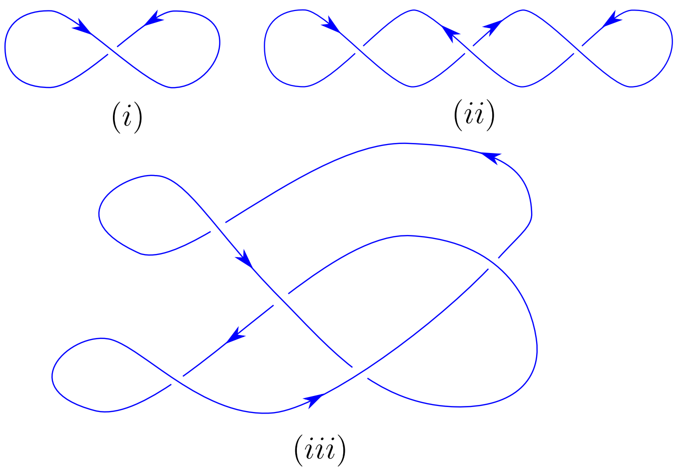


Figure 2.6: Typical front projections of positively transverse knots

From now on, Legendrian and transverse knots are normally depicted in their front projection diagrams.

By changing the front projections of an arbitrary topological knot like in Figure 2.7 one can realize every topological knot type as a Legendrian knot (and very similar also as a transverse knot). This can even be done arbitrarily close to the original knot as the next theorem shows.

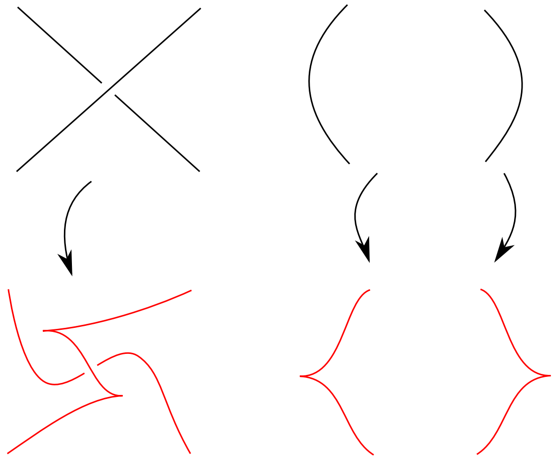


Figure 2.7: Transforming a topological knot into a Legendrian knot

Theorem 2.2.14 (Approximation theorem)
Let $K \subset (M, \xi)$ be a knot in a contact manifold. Then K can be C^0-close approximated by a Legendrian as well as by a transverse knot topologically isotopic to K.

Sketch of proof.
By the theorem of Darboux 2.1.4 it is sufficient to consider Legendrian knot segments in $(\mathbb{R}^3, \xi_{st})$.

By approximating a given topological knot K as in Figure 2.8 by a Legendrian knot L (with the negative slope of L arbitrarily close to the x-coordinate of K) one can stay arbitrarily close to the original knot.

The next step is to show that a given oriented Legendrian knot L can be approximated arbitrarily close by a transverse knot T. Therefore, look at a standard neighborhood (see Theorem 2.2.5) of the Legendrian knot L given by

$$\Big(S^1 \times D_r^2, \ker \cos(\theta)\, dx - \sin(\theta)\, dy\Big).$$

The two knots $T_\pm$ given by

$$\theta \longmapsto T_\pm(\theta) := \Big(\theta, x = \pm\varepsilon \sin(\theta), y = \pm\varepsilon \cos(\theta)\Big)$$

are positive/negative transverse knots since $\alpha_n(T'_\pm) = \pm\varepsilon$. Different choices of ε give isotopic transverse knots which become for small values of $\varepsilon > 0$ arbitrarily close to the original Legendrian knot L. $\square$

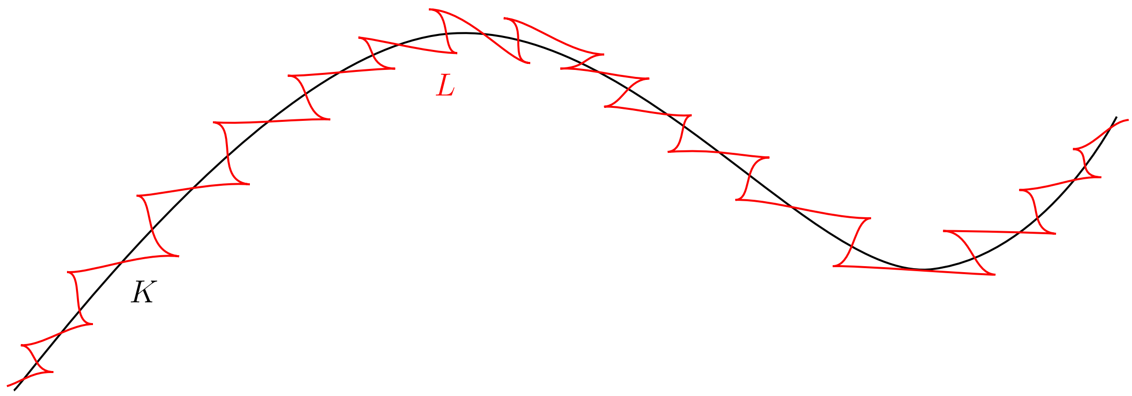


Figure 2.8: Approximating a topological knot by a Legendrian knot

Definition 2.2.15 (Transverse push-off)
The positively/negatively transverse knots $T_\pm$ constructed in the sketch of the proof of Theorem 2.2.14 are called the positively/negatively **transverse push-off** of the oriented Legendrian knot L.

Remark 2.2.16 (Transverse push-offs in front projections)
In a front projection of a Legendrian knot L one can construct the positively (and similar the negatively) transverse push-off T as in Figure 2.9 (compare [69, Section 12.5]).

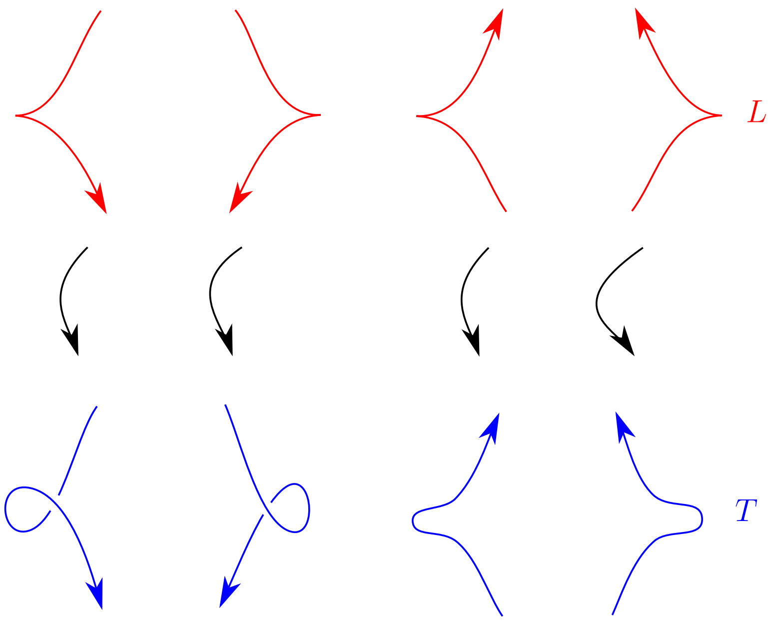


Figure 2.9: Constructing the front projection of the positively transverse push-off

By working Figure 2.9 backwards it is not hard to see that every transverse knot can be obtained as the push-off of a Legendrian knot, see [69, Section 12.5], but this Legendrian knot is not unique.

Next, I want to study links in contact manifolds consisting of Legendrian and transverse knots at the same time. Although this is very easy and similar to the other cases, the following results are new and only appeared very briefly in my paper in joint work with Sebastian Durst [18]. In Section 6.1 the following will be used to prove the transverse knot exterior theorem.

One wants to present these links in front projections. Therefore, one has to find out how Legendrian and transverse knots behave together in a front-projection diagram. A parametrized knot $(x(t), y(t), z(t))$ is Legendrian if and only if $z'(t) + x(t)y'(t) = 0$. And similarly such a knot is (positively) transverse if and only if $z'(t) + x(t)y'(t) > 0$. In Figure 2.10 one can see the four possible intersections of a Legendrian knot L and a transverse knot T in the front-projection. From the conditions above it follows immediately that the intersection of type (2) and (4) are uniquely determined, but in type (1) and (3) in general both are possible. For negatively transverse knots it is similar, see Figure 2.10.

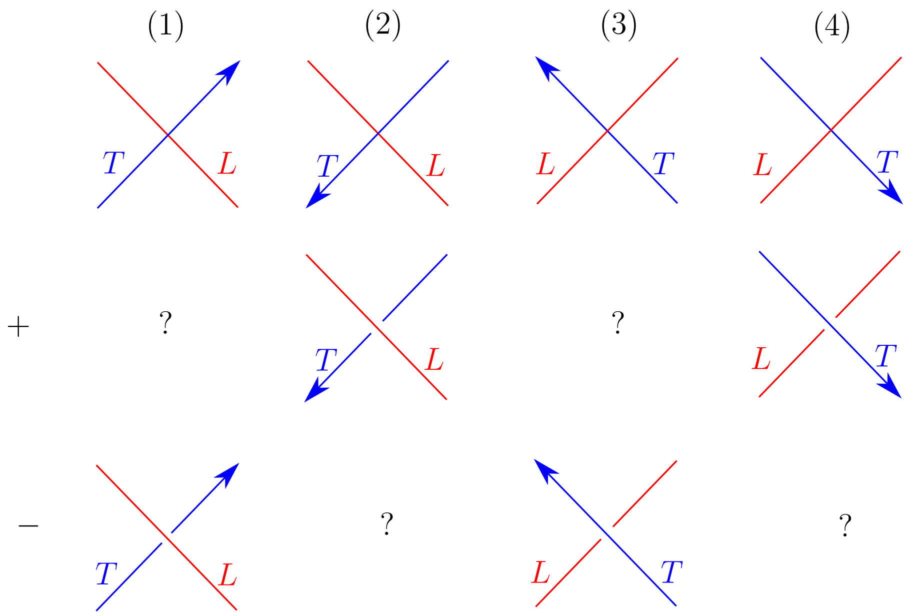


Figure 2.10: Crossings between Legendrian and transverse knots. The transverse knots in the second row are positive, the ones in the third row negative.

So in a front projection of a link consisting of positively transverse and Legendrian knots the configurations in Figure 2.11 can never occur (and neither, of course, the old forbidden configurations of Legendrian and transverse knots). On the other hand, every front projection without such a configuration represents uniquely such a link.

Similar to the case for Legendrian or transverse knots (see for example [69, Appendix B]) one can also prove a Reidemeister-type theorem for links consisting of Legendrian and transverse knots.

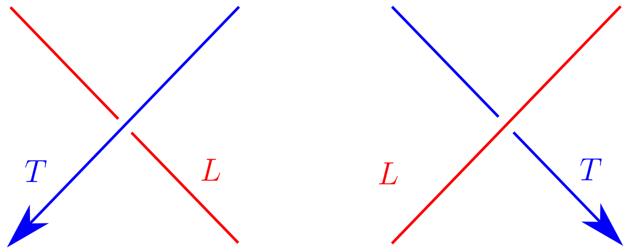


Figure 2.11: Impossible crossings of positive transverse with Legendrian knots

2.3 The classical invariants

A natural question is how to distinguish two knots from each other which are both Legendrian or transverse. Of course, if they do not represent the same topological knot type then they cannot be equivalent as Legendrian or transverse knots, either. But what for example about the Legendrian unknots in Figure 2.4?

To distinguish such knots one usually uses the so-called classical invariants which I will define in this section.

Definition 2.3.1 (Contact longitude and Thurston–Bennequin invariant)
Let K be a Legendrian knot in (M, ξ). Then one obtains the so-called **contact longitude** λ_C by pushing K into a direction transverse to the contact structure. So if K is nullhomologous and Legendrian then there exist two special longitudes, the Seifert longitude λ_S and the contact longitude λ_C, which differ only by a multiple of the meridian. Then the so-called **Thurston–Bennequin invariant** $\mathrm{tb}(K) \in \mathbb{Z}$ is defined by the equation

$$\lambda_C = \mathrm{tb}(K)\mu + \lambda_S \in H_1(\partial\nu K, \mathbb{Z}).$$

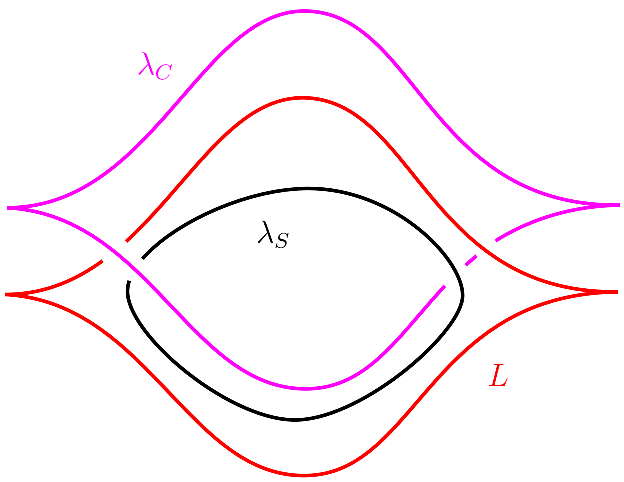


Figure 2.12: The contact and the Seifert longitude of a Legendrian unknot

A Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ is always nullhomologous, so it is natural to ask how to compute the Thurston–Bennequin invariant in front projections.

Example 2.3.2 (The Thurston–Bennequin invariant of a Legendrian unknot)
Let $L \subset (\mathbb{R}^3, \xi_{st})$ be the Legendrian unknot in Figure 2.4(i). The contact longitude λ_C is obtained from L by pushing it into a direction transverse to the contact structure. For ξ_{st} one can use the ∂_z-direction and obtain the contact longitude λ_C as a Legendrian knot again (see Figure 2.12).

This new Legendrian knot is also called a **Legendrian push-off** and is Legendrian isotopic to the old one. In Figure 2.13 one can compare on $\partial(\nu L)$ the contact longitude λ_C with the Seifert longitude λ_S to obtain

$$\lambda_C = -\mu + \lambda_S.$$

Therefore, it follows that $\text{tb}(L) = -1$.

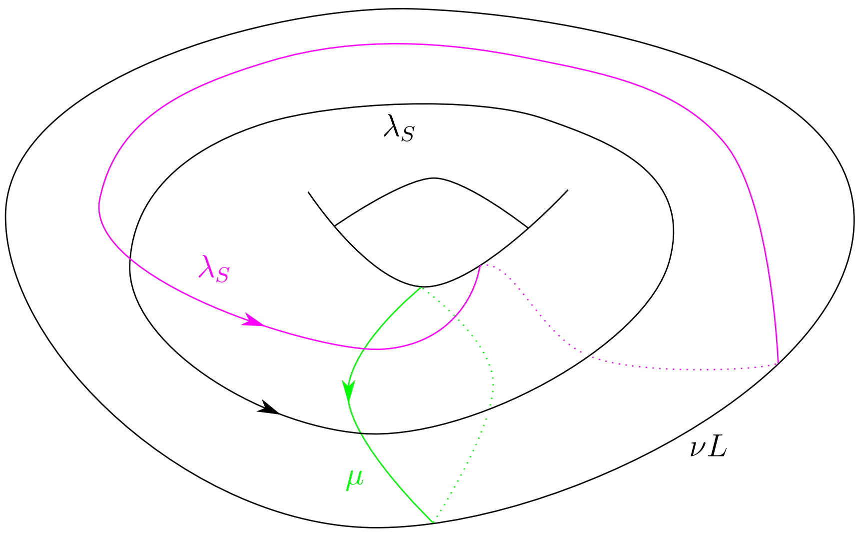


Figure 2.13: The contact and the Seifert longitude on a tubular neighborhood

By using the same method as in this example one gets the following formula for computing the Thurston–Bennequin invariant in the front projection.

Lemma 2.3.3 (Computation of tb in front projections)
Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ represented in the front projection. Then the Thurston–Bennequin invariant is given by

$$\text{tb}(K) = -\frac{1}{2}c + w$$

where c denotes the number of cusps in the front projection and w is the ***writhe****, i.e. the signed sum over all self-crossings in the front projection.*

With this lemma it is easy to distinguish many Legendrian knots of the same topological knot type. For example the Legendrian unknots in Figure 2.4 have tb equal to -1 and -2, so they are different. On the other hand, there exist different Legendrian knots with the same tb numbers. For example the Legendrian unknots in Figure 2.14 both have $\mathrm{tb} = -3$, but they will turn out to be different. To show this one can use the other classical invariant for Legendrian knots, the so-called rotation number.

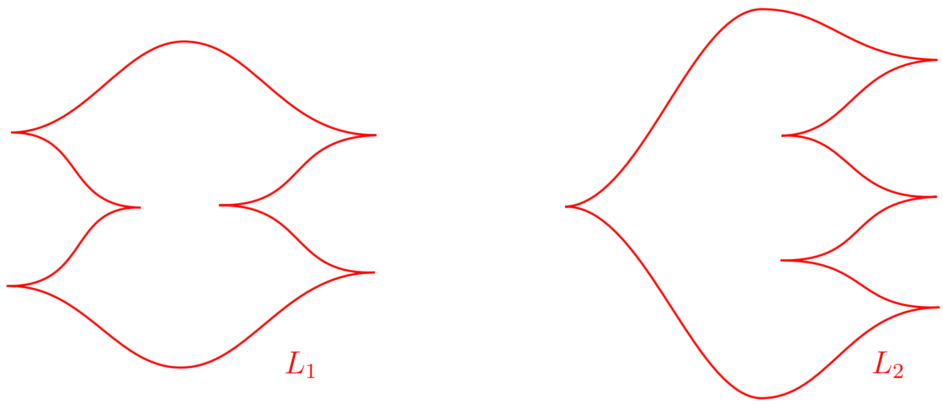


Figure 2.14: Different Legendrian unknots with same tb invariants

Definition 2.3.4 (Rotation number)
Let K be a nullhomologous oriented Legendrian knot in (M, ξ) and Σ a Seifert surface for K. The **rotation number** of K with respect to the Seifert surface Σ is defined by

$$\mathrm{rot}(K, \Sigma) := \langle \mathrm{e}(\xi, K), [\Sigma] \rangle = \mathrm{PD}(\mathrm{e}(\xi, K)) \bullet [\Sigma]$$

where $\mathrm{e}(\xi, K)$ is the relative Euler class of the contact structure ξ relative to the trivialization given by a positive tangent vector field along the knot K and $[\Sigma]$ the relative homology class represented by the surface Σ.

This definition of the rotation number is useful for calculations (see also [69]). For an alternative equivalent definition see [38, Definition 3.5.12]. The rotation number depends only on the homology class of the chosen Seifert surface, not on the particular choice of the surface itself. Note also that the rotation number is independent of the class of the Seifert surface if the Euler class $\mathrm{e}(\xi)$ of the contact structure vanishes (see Proposition 3.5.15 in [38]). Moreover, the rotation number changes its sign if the orientation of the knot changes. Therefore, the absolute value $|\mathrm{rot}|$ is an invariant for unoriented Legendrian knots.

Since in $(\mathbb{R}^3, \xi_{st})$ there exists a global trivialization of the contact structure, one can give (using the above-mentioned equivalent definition of the rotation number) a formula to compute rot in front projections (see [38, Proposition 3.5.19]).

Lemma 2.3.5 (Computation of rot in front projections)
Let K be a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ represented in the front projection. Then the rotation number is given by

$$\mathrm{rot}(K) = \frac{1}{2}(c_- - c_+)$$

where $c_\pm$ denotes the total number of cusps oriented upwards or downwards in the front projection.

With this lemma the rotation numbers of the Legendrian unknots in Figure 2.14 compute as $\mathrm{rot}(L_1) = 0$ and $\mathrm{rot}(L_2) = \pm 2$ (depending on the orientation), so they represent different Legendrian knots.

For transverse knots there exists only one classical invariant, the so-called self-linking number.

Definition 2.3.6 (Self-linking number)
Let K be a nullhomologous transverse knot in a contact manifold (M, ξ) and let Σ be a Seifert surface. The **self-linking number** $\mathrm{sl}(K, \Sigma)$ of K is defined as the linking number of K and K' where K' is obtained by pushing K in the direction of a non-vanishing section of $\xi|_\Sigma$.

The self-linking number of a knot is independent of its orientation and only depends on the homology class of the chosen Seifert surface (cf. Section 3.5.2 in [38]). Again, if $\mathrm{e}(\xi) = 0$ then the self-linking number is independent of the Seifert surface.

There is also a simple formula for computing the self-linking number of a transverse knot in its front projection.

Lemma 2.3.7 (Computing sl in front projections)
Let K be a transverse knot in $(\mathbb{R}^3, \xi_{st})$ represented in the front projection. Then its self-linking number is given by

$$\mathrm{sl}(K) = w$$

where w denotes again the writhe of the front projection.

Proof.
From Example 2.1.1 it follows that ∂_x is a global section of ξ_{st}. That means that the parallel knot K' from the definition can be obtained by pushing K into the ∂_x-direction. Therefore, the linking number $\mathrm{lk}(K, K')$ is given by the writhe of this front projection. □

It follows that the self-linking numbers of the transverse unknots in Figure 2.6 are -1 and -3, so these transverse knots are different.

Given an oriented Legendrian knot L in $(\mathbb{R}^3, \xi_{st})$ one can compute (using Figure 2.9 and the formulas for computing the classical invariants in front projections) the self-linking number of its positive/negative transverse push-offs $T_\pm$ easily as

$$\mathrm{sl}(T_\pm) = \mathrm{tb}(L) \mp \mathrm{rot}(L).$$

This also holds in general contact manifolds (see [38, Proposition 3.5.36]).

Lemma 2.3.8 (Self-linking number of transverse push-offs)
Let L be an oriented nullhomologous Legendrian knot in a contact manifold (M, ξ) with Seifert surface Σ. Denote by $T_\pm$ its positive/negative push-off. Then Σ can also be seen as a Seifert surface for $T_\pm$ and

$$\mathrm{sl}(T_\pm, \Sigma) = \mathrm{tb}(L) \mp \mathrm{rot}(L, [\Sigma]).$$

A natural question arising now is which values can be realized as classical invariants of Legendrian or transverse knots.

First one observes that one can change the classical invariants by local moves, so-called **stabilizations**, depicted in Figure 2.15 in its front projections.

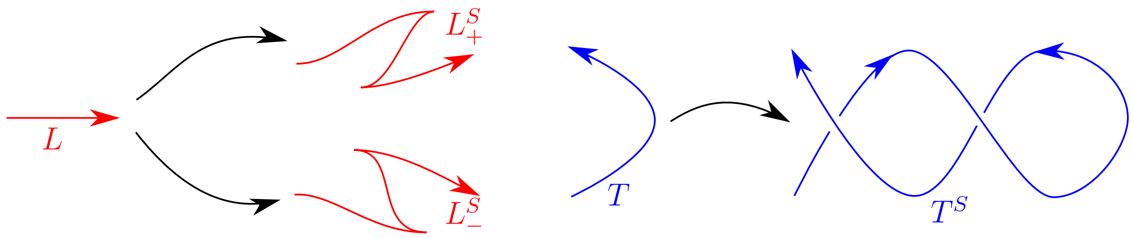


Figure 2.15: Stabilizations of a Legendrian knot L and a transverse knot T

The classical invariants change under stabilizations as

$$\begin{aligned}\mathrm{tb}(L_\pm^S) &= \mathrm{tb}(L) - 1,\\ \mathrm{rot}(L_\pm^S) &= \mathrm{rot}(L) \pm 1,\\ \mathrm{sl}(T^S) &= \mathrm{sl}(T) - 2.\end{aligned}$$

So in particular it is easy to give a Legendrian or transverse realization of a given topological knot type with arbitrarily small classical invariants. But on the other hand, it is not always possible to make them larger. The first result in this direction is the following theorem due to Bennequin.

Theorem 2.3.9 (Bennequin inequality)
Let K be a topological knot in $\mathbb{R}^3$. Then for every Legendrian realization L of K and every transverse realization T of K in $(\mathbb{R}^3, \xi_{st})$ the following inequalities hold

$$\begin{aligned}\mathrm{sl}(T) &\leq 2g(K) - 1,\\ \mathrm{tb}(L) + |\,\mathrm{rot}(L)| &\leq 2g(K) - 1,\end{aligned}$$

*where $g(K)$ denotes the **genus** of the knot K, i.e. the smallest genus of a Seifert surface of the knot K.*

This theorem can be seen as the starting point of contact topology. It connects contact geometry (in terms of the classical invariants of Legendrian and transverse knots) to topology (in terms of the genus of a knot). In the next section it is explained how one can use this result and a generalization by Eliashberg to distinguish contact structures.

In particular, this theorem says that the Legendrian unknot from Figure 2.4(i) and the transverse unknot from Figure 2.6(i) represent the maximal values of tb, $\mathrm{tb} + |\operatorname{rot}|$ and of sl. But for a general topological knot type this bound is far from optimal (see for example [63] for other bounds). Observe also that the so-called Bennequin bound $2g(K) - 1$ is at least -1, but in general the maximal value is much smaller.

It follows also that for a given topological knot type K in S^3 the maximal value of sl of all its transverse realizations in (S^3, ξ_{st}) is an invariant of K, denoted by $\mathrm{SL}(K)$. (And similarly one defines $\mathrm{TB}(K)$ to be the maximal value of tb of all Legendrian realizations of K.)

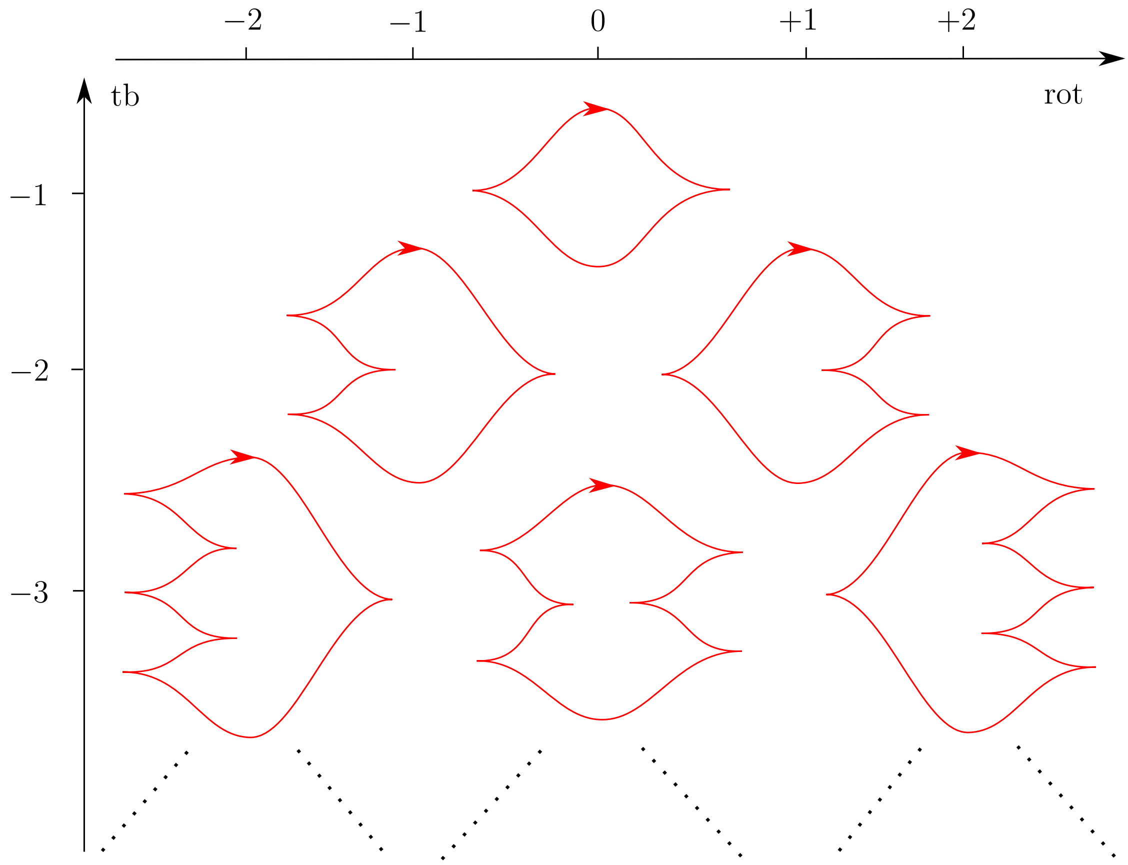


Figure 2.16: A complete list of Legendrian unknots

On the other hand, one could also ask if the classical invariants classify Legendrian or transverse knots. The primary results in this direction are:

Theorem 2.3.10 (Classification of Legendrian unknots)
Every oriented Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$ or (S^3, ξ_{st}) is isotopic to exactly one in Figure 2.16. In particular, two oriented Legendrian unknots are isotopic if and only if tb *and* rot *coincide.*

Proof. see [26] □

Theorem 2.3.11 (Classification of transverse unknots)
Every positive transverse unknot in $(\mathbb{R}^3, \xi_{st})$ or (S^3, ξ_{st}) is isotopic to exactly one in Figure 2.17. In particular, two positive transverse unknots are isotopic if and only if their self-linking numbers coincide.

Proof. see [24] □

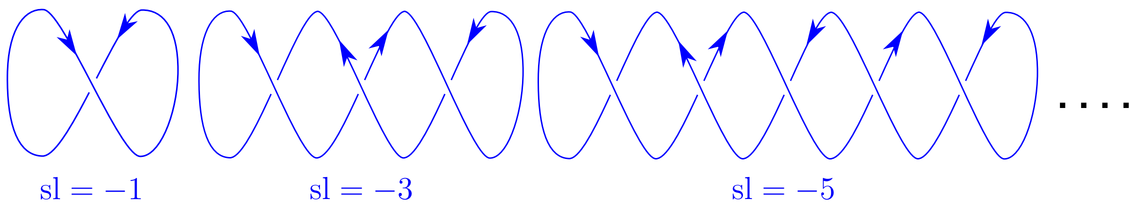


Figure 2.17: A complete list of transverse unknots

Similar results have been obtained for other knot types, for example torus knots are also classified by their classical invariants by work of Etnyre and Honda [29]. But in general this is not true; Chekanov [9] constructed examples of topologically isotopic Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ with the same classical invariants which are not Legendrian isotopic. For similar examples of transverse knots see for example [68].

2.4 Tight versus overtwisted and the classification of contact structures

In this section I will explain how Legendrian and transverse knots influence the global contact topology. The starting point is the Bennequin inequality 2.3.9. By this inequality every Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$ has $\mathrm{tb} \leq -1$. This can be used to distinguish contact structures from each other.

Example 2.4.1 (The standard overtwisted contact structure)
Consider $\mathbb{R}^3$ with cylindrical coordinates (r, φ, z). Define a 1-form by

$$\alpha_{ot} = \cos r \, dz + r \sin r \, d\varphi.$$

For $r = 0$ it is not clear that α_{ot} is well defined, since $d\varphi$ is not well defined for $r = 0$. But the 1-form $r^2\, d\varphi = x\, dy - y\, dx$ is smooth and the function $r \mapsto \frac{\sin r}{r}$ admits a smooth extension to $r = 0$. Therefore, α_{ot} is a well-defined contact form and defines the so-called **overtwisted contact structure** $\xi_{ot} := \ker \alpha_{ot}$.

In terms of non-vanishing vector fields this overtwisted contact structure, for $r \neq 0$, can be written as

$$\xi_{ot} = \langle \partial_r, \cos r \partial_\varphi - r \sin r \partial_z \rangle.$$

First observe that ξ_{ot} does not depend on z, so the contact planes look the same on every plane parallel to the xy-plane. Therefore, it is enough to understand the contact planes in the xy-plane. For $r = 0$ the contact plane coincides with the xy-plane. If one moves outward along a ray in the xy-plane, the contact planes twist to the left and will make a full turn for every integer multiple of 2π distance from the origin. These infinitely many full twists are the reason for the name. In Figure 2.18 a few contact planes are pictured in the xy-plane.

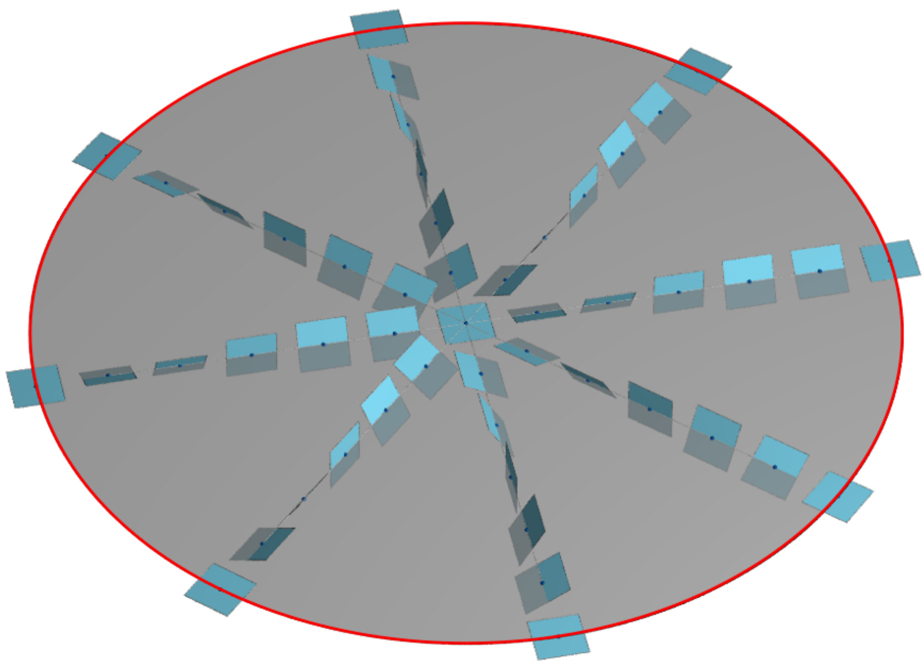


Figure 2.18: An unknot with $\mathrm{tb} = 0$ in an overtwisted contact structure. This figure is (except for a small change in color) retrieved from Wikipedia (2016, August 31) created by user Pmassot available online at `https://commons.wikimedia.org/wiki/File:Overtwisted_contact_structure.png?uselang=fr`.

The difference to the standard contact structure lies in the fact that the standard contact structure never makes a full twist, while the overtwisted one twists infinitely many times. Using the Bennequin inequality one can distinguish these two contact structures. To do so, consider the unknot given by

$$\left\{(\pi, \varphi, 0) \in \mathbb{R}^3 \,\middle|\, \varphi \in S^1\right\}.$$

In Figure 2.18 one observes that this knot (pictured in red) is Legendrian and that the contact longitude coincides with the Seifert longitude. So this knot is an unknot with $\mathrm{tb} = 0$. Therefore, the overtwisted contact structure cannot be contactomorphic to the standard contact structure by the Bennequin inequality 2.3.9.

Later the concept of studying Legendrian unknots with $\mathrm{tb} = 0$ was generalized by Eliashberg to arbitrary contact manifolds and this turned out to be the starting point of the classification of contact structures.

Definition 2.4.2 (Tight and overtwisted)
A contact manifold (M, ξ) is called **overtwisted** if there exists a tb-0-unknot in (M, ξ). (Then an embedded disk bounded by a tb-0-unknot is called **overtwisted disk**.) Otherwise, a contact manifold is called **tight**.

Example 2.4.3 ((S^3, ξ_{st}) is tight)
Due to Example 2.1.6 every Legendrian unknot in (S^3, ξ_{st}) also represents a Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$ with the same Thurston–Bennequin invariant. Consequently, (S^3, ξ_{st}) is also tight.

The importance of the definitions of tightness and overtwistedness are demonstrated in the following deep results due to Eliashberg.

Theorem 2.4.4 (Classification of overtwisted contact manifolds)
Let M be a closed orientable manifold. Then in every homotopy class of 2-plane fields tangential to M there exists (up to isotopy) exactly one overtwisted contact structure.

Proof. See [38, Chapter 4.7]. □

So two overtwisted contact structures are isotopic if and only if they are homotopic as tangential 2-plane fields. In particular, there exists an overtwisted contact structure on every 3-manifold. But observe that this theorem does only hold for closed manifolds. For example there exist infinitely many overtwisted contact structure on $\mathbb{R}^3$ as shown in [25].

On the other hand, the tight contact structures are much more related to the topology of the underlying manifold, as can be seen for example from the following result.

Theorem 2.4.5 (Bennequin–Eliashberg inequality)
Let (M, ξ) be a contact manifold. Then the following statements are equivalent.

1. (M, ξ) *is tight.*

2. $\mathrm{sl}(K, \Sigma) \leq 2g(K) - 1$ *for all nullhomologous transverse knots K and all their Seifert surfaces Σ in (M, ξ).*

3. $\mathrm{tb}(K) + |\,\mathrm{rot}(K, [\Sigma])| \leq 2g(K) - 1$ *for all nullhomologous Legendrian knots K and all their Seifert surfaces Σ in (M, ξ).*

4. $\mathrm{tb}(K) \leq 2g(K) - 1$ *for all nullhomologous Legendrian knots K in (M, ξ).*

5. *There do not exist* tb-0*-unknots in (M, ξ).*

Sketch of proof.
The only non-trivial implication is (1) ⇒ (2) (see proof of Theorem 4.6.43 in [38] for more information). (For (2) ⇒ (3) one uses Lemma 2.3.8 for both the positive and the negative transverse push-off.) The idea to prove (1) ⇒ (2) is to take a Seifert surface Σ of a transverse knot K in a contact manifold (M, ξ). Then the intersections of the contact planes with the tangent space $T\Sigma$ of Σ give a vector field X on Σ (called the **characteristic foliation**). On the one hand the Poincaré–Hopf Index Theorem relates the genus of Σ to a signed count of zeros of X and on the other hand it is possible to compute also the self-linking number as a signed count of zeros of X. By putting these together one gets an equation with the self-linking number, the genus and some signed counts of zeros of X. Using now the tightness of the contact structure one can exclude several configurations of the zeros of X (because these configurations would yield an unknot with $\mathrm{tb} = 0$). This allows one to cancel all zeros of X that contribute with a negative sign. By ignoring all remaining positive counting zeros of X one gets the desired inequality. □

The classification of tight contact manifolds is only known for a few simple manifolds. The most fundamental result is the following theorem again due to Eliashberg.

Theorem 2.4.6 (Classification of tight contact structures on $\mathbb{R}^3$ and S^3)
S^3 and $\mathbb{R}^3$ both admit only one positive tight contact structure, the standard contact structure.

Proof. See for example [38, Theorem 4.10.1]. □

Later (see the sketch of proof of Theorem 3.2.1) I will also use the classification of tight contact structures on solid tori $S^1 \times D^2$ due to Honda [46]. On these there exist infinitely many different tight contact structures. There also exist closed manifolds with either no (see for example [30]) or infinitely many tight contact structures (see for example [50]).

3
Contact Dehn surgery

In this chapter I explain how to generalize the Dehn surgery construction from Section 1.2 to contact manifolds. There are two natural settings to do this. The first setting is to do a topological Dehn surgery along a transverse knot in a contact manifold and the second one is to do the same along a Legendrian knot. (Because of the approximation Theorem 2.2.14 this is topologically no restriction.) The question is then if one can extend the old contact structure over the newly glued-in solid torus, and if so, if the resulting contact structure is unique.

In Section 3.1 I will recall the classical construction due to Martinet to extend a given contact structure on the exterior of a transverse knot to a new glued-in solid torus of an arbitrary Dehn surgery. As a corollary one obtains that every closed orientable 3-manifold carries a contact structure. However, this resulting contact structure is in general not unique.

For Legendrian knots it is also possible to always extend a given contact structure on a knot exterior to the surgered manifold. The advantage here is that for special slopes this new contact manifold is unique.

A fundamental theorem due to Ding and Geiges states that one can get every closed contact 3-manifold by contact Dehn surgery along a Legendrian link out of (S^3,ξ_{st}). The proof of this theorem together with the basic facts about contact Dehn surgery along Legendrian knots and several examples are presented in Section 3.2.

A surprising theorem due to Etnyre–Ghrist (Theorem 3.1.3) states that there are transverse knots in (S^3,ξ_{st}) such that for every surgery coefficient there exists a surgery along this knot with this coefficient resulting in a tight contact manifold. Such knots are called tight transverse knots.

3.1 Contact Dehn surgery along transverse knots

A Dehn surgery along a transverse knot that results again in a contact manifold with contact structure outside the new glued-in solid torus coinciding with the old contact structure is called contact Dehn surgery. More precisely:

Theorem 3.1.1 (Contact Dehn surgery along transverse knots)
Let T be a transverse knot in (M,ξ) and νT a standard tubular neighborhood of T. Then the result $M_T(r)$ of an arbitrary surgery along T with respect to this chosen

tubular neighborhood carries again a contact structure $\xi_T(r)$ which coincides with the old contact structure ξ on $M \setminus \mathring{\nu T}$. One says $(M_T(r), \xi_T(r))$ is obtained from (M, ξ) by ***contact Dehn surgery*** *along the transverse knot T with tubular neighborhood νT and slope r.*

Sketch of proof.
By the neighborhood theorem 2.2.5 one can assume $(\nu T, \xi) \subset (M, \xi)$ to look like

$$\left(S^1 \times D^2_\varepsilon, \ker(d\theta + r^2 d\varphi)\right) \subset \left(S^1 \times D^2_\delta, \ker(d\theta + r^2 d\varphi)\right).$$

Now take another copy V_0 of $S^1 \times D^2_\varepsilon$ and denote its meridian by μ_0 and its longitude $S^1 \times \{pt\}$ by λ_0 and similarly write (μ, λ) for the same meridian and longitude of νT. Then glue V_0 to $M \setminus \mathring{\nu T}$ with an arbitrary gluing map of the boundaries given by

$$\begin{aligned} \mu_0 &\longmapsto p\mu + q\lambda \\ \lambda_0 &\longmapsto m\mu + n\lambda \end{aligned}$$

with $pn - qm = 1$. If one denotes the coordinates on V_0 by $(\theta_0, r_0, \varphi_0)$, the gluing map can be written as

$$\begin{aligned} \theta &= n\theta_0 + q\varphi_0, \\ \varphi &= m\theta_0 + p\varphi_0. \end{aligned}$$

Now one pulls back the contact form $d\theta + r^2 d\varphi$ representing the contact structure ξ along the boundary $\partial M \setminus \mathring{\nu T}$ to the boundary ∂V_0. Then it is not difficult to construct in a local computation a contact form on V_0 coinciding with this pulled back contact form along the boundary (see for example the proof of Lemma 4.1.3 in [38]). This contact structure on V_0 fits together with the old contact structure on $M \setminus \mathring{\nu T}$ to a global contact structure on the new surgered manifold. $\square$

But observe that this contact structure is not unique. Therefore, the resulting contact structure $\xi_T(r)$ on the new manifold is also not unique. Moreover, the size of the tubular neighborhood νT is not arbitrary as described in Remark 2.2.6. Because of that the resulting contact structure depends also on the chosen tubular neighborhood νT.

Since the resulting contact structure is not unique, this construction is difficult to use in explicit constructions. But if one is only interested in the existence of contact structures, this is very useful. For example there is the following theorem due to Martinet.

Corollary 3.1.2 (Martinet's Theorem)
On every closed oriented 3-manifold there exists a contact structure.

Proof.
By the theorem of Lickorish–Wallace [70, Corollary 12.4.] every closed oriented 3-manifold can be obtained by Dehn surgery along a link in S^3. Now equip S^3 with the standard contact structure and make the surgery link with the Approximation Theorem 2.2.14 to a transverse link. By doing contact Dehn surgery along this transverse link one obtains a contact structure on the resulting 3-manifold. □

But in general the resulting contact structures will be overtwisted. It is much harder to construct tight contact structures in this way. However, there is the following result due to Etnyre and Ghrist.

Theorem 3.1.3 (Existence of tight transverse knots)
There exists a transverse unknot T in (S^3, ξ_{st}) such that for every surgery coefficient $r \in \mathbb{Q} \cup \{\infty\}$ there exists a contact Dehn surgery along T yielding a tight contact manifold. In particular there are cosmetic contact surgeries from (S^3, ξ_{st}) to itself for every surgery coefficient of the form $1/n$.

Such a transverse knot is called **tight** knot. But the surgeries here are done with respect to a sufficiently big tubular neighborhood of the knot T to ensure tightness of the resulting manifold as explained in [28, Remark 2.1]. For small tubular neighborhoods this theorem is not true, as we will see in Chapter 6.

In Section 6.1 I will present another application of contact Dehn surgery along transverse knots, namely the proof of the transverse knot exterior theorem 2.2.11.

Other applications are given for example in [11] by constructing tight contact structures and in [36] and [37] by constructing symplectic manifolds.

3.2 Contact Dehn surgery along Legendrian knots

First, I want to recall the definition of contact Dehn surgery along Legendrian knots following [13], [14], [17], [52] and [53].

Recall first from Definition 2.3.1 that every Legendrian knot admits a specified longitude, the so-called contact longitude λ_C.

Now one wants to do Dehn surgery along Legendrian links with respect to this contact longitude. For an expression of a slope r with the contact longitude λ_C as $r = p\mu + q\lambda_C$ the corresponding rational number $p/q \in \mathbb{Q} \cup \{\infty\}$ is called the **(contact) surgery coefficient**.

When doing Dehn surgery along K one can show that the contact structure of the old manifold extends to a contact structure of the resulting manifold.

Theorem 3.2.1 (Contact Dehn surgery along Legendrian knots)
Let K be a Legendrian knot in a contact 3-manifold (M, ξ).
(1) Then $M_K(r)$ carries a contact structure $\xi_K(r)$ which coincides with the old contact structure ξ on $M \setminus \nu\mathring{K}$.
(2) For $r \neq \pm\lambda_C$ one can choose $\xi_K(r)$ to be tight on the new glued-in solid torus.
(3) For $r = \mu + q\lambda_C$ this tight contact structure on the new solid torus is unique.
*One says $(M_K(r), \xi_K(r))$ is obtained from (M, ξ) by **contact Dehn surgery** along the Legendrian knot K with slope r.*

Sketch of proof.
The neighborhood νK is chosen such that its boundary is a convex surface (see [38, Section 4.8]). The germ of a contact structure along a convex surface is determined by some simple data on that surface. Therefore, it is easy to glue contact structures along convex surfaces. Tight contact structures on solid tori with prescribed boundary conditions have been classified by Honda [46]. In particular, such a contact structure always exists if $r \neq \pm\lambda_C$ and is unique if $r = \mu + q\lambda_C$. But in general there are more possibilities to choose this contact structure, so for other slopes the result of contact Dehn surgery is in general not unique. For details see [13] or [52].

For $r = \pm\lambda_C$ it is also easy to construct a contact structure on the new glued-in solid torus that fits together to a global contact structure on the new surgered manifold (see [14, pages 586–587]), but this contact structure has to be overtwisted. Therefore, one usually does not discuss this case and only considers tight contact structures on the new glued-in solid tori. □

Notice that as a corollary one gets another proof of Martinet's theorem 3.1.2 completely similar to the proof given before.

Before I discuss some examples of contact Dehn surgeries along Legendrian knots, I want to present some basic properties. In order to do this, it is useful to introduce the following notation. Let K be a Legendrian knot in (M, ξ), then one can write also as short notation

$$K(r) := (M_K(r), \xi_K(r)),$$

where it is important to notice that the contact structure $\xi_K(r)$ is in general not unique. For K together with a Legendrian push-off of K I write $K \times K$, compare Example 2.3.2. A copy of K with n extra stabilizations I denote by K_n, if this knot K_n is again stabilized m-times this is denoted by $K_{n,m}$. For example let U be a Legendrian unknot in (S^3, ξ_{st}) with $\mathrm{tb} = -1$, then

$$U(+1) \times U_1(-1) \times U_{1,1}(-1)$$

denotes the collection of the surgery diagrams in Figure 3.1.

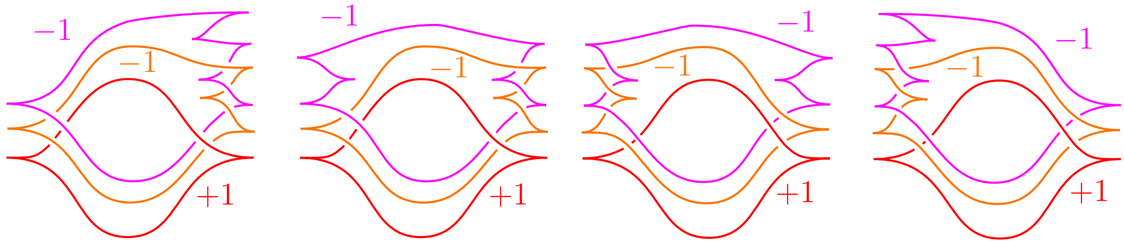


Figure 3.1: Different choices of stabilizations in surgery diagrams

The following three lemmas due to Ding and Geiges are fundamental in the study of contact Dehn surgeries along Legendrian links.

Lemma 3.2.2 (Cancellation lemma)
Contact $(1/n)$-surgery ($n \in \mathbb{Z}\backslash\{0\}$) along a Legendrian knot $K \subset (M,\xi)$ and contact $(-1/n)$-surgery along a Legendrian push-off of K cancel each other, i.e. the result is contactomorphic to (M,ξ). In the notation introduced above this reads

$$K\left(\frac{1}{n}\right) \times K\left(-\frac{1}{n}\right) = (M,\xi).$$

Lemma 3.2.3 (Replacement lemma)
Contact $(\pm 1/n)$-surgery ($n \in \mathbb{N}$) along a Legendrian knot $K \subset (M,\xi)$ yields the same contact manifold as contact (± 1)-surgeries along n Legendrian push-offs of K, i.e.

$$K\left(\pm\frac{1}{n}\right) = K(\pm 1) \times \cdots \times K(\pm 1).$$

Lemma 3.2.4 (Transformation lemma)
(1) Let K be a Legendrian knot in (M,ξ) with given contact surgery coefficient $r \in \mathbb{Q} \setminus \{0\}$. Then

$$K(r) = K\left(\frac{1}{k}\right) \times K\left(\frac{1}{\frac{1}{r} - k}\right)$$

holds for all integers $k \in \mathbb{Z}$.
(2) If the contact surgery coefficient r is negative, one can write r as a continued fraction

$$r = r_1 + 1 - \cfrac{1}{r_2 - \cfrac{1}{\cdots - \cfrac{1}{r_n}}}$$

with integers $r_1, \ldots, r_n \leq -2$. Then

$$K(r) = K_{|2+r_1|}(-1) \times \cdots \times K_{|2+r_1|,\ldots,|2+r_n|}(-1).$$

Remark 3.2.5 (Choices of stabilizations in the transformation lemma)
Observe that by choosing k such that $\frac{1}{r} - k$ is negative one can transform with the transformation lemma every contact r-surgery (for $r \neq 0$) into a sequence of contact $(1/n)$-surgeries along a different Legendrian link. Moreover in [14] and [17] it is shown that all the different choices of stabilizations in this Legendrian link correspond exactly to the different tight contact structures one can choose on the new glued-in solid torus in the contact r-surgery (compare the sketch of the proof of Theorem 3.2.1).

For a proof of the cancellation and the replacement lemma see [13] or [52] and compare also [17, Section 1]. The proof of the transformation lemma is given in [14] (see also [17, Section 1]). The ideas of the proofs of all three lemmas are the same. One chooses a standard neighborhood of the Legendrian knot K and does the corresponding contact surgeries completely inside this local model. By analyzing the boundary convex surfaces of the new glued-in solid tori and by using the classification of tight contact structures on solid tori with prescribed boundary convex surfaces by Honda [46] it is easy to compare the resulting contact manifolds.

Observe that the first part of the transformation lemma in [14] and [17, Section 1] is only formulated for natural numbers $k \in \mathbb{N}$ and positive contact surgery coefficients $r < 0$, but the proof given there works exactly the same for all integers $k \in \mathbb{Z}$ and all surgery coefficients $r \in \mathbb{Q} \setminus \{0\}$.

Next it is interesting to understand how the properties of the contact structures behave under contact Dehn surgery. Since contact (-1)-surgery corresponds to a symplectic handle attachment (see [14, Section 3]), contact (-1)-surgery preserves Stein fillability, strong symplectic fillability [13, Proposition 10] and weak symplectic fillability [38, Lemma 6.5.2] of a given contact manifold. And because of the replacement lemma 3.2.3 and the transformation lemma 3.2.4 the same holds true for every negative contact surgery coefficient.

Recently, Wand [83] proved with completely 3-dimensional methods that even tightness is preserved by negative contact surgeries.

Theorem 3.2.6 (Negative contact surgery preserves tightness)
Let K be a Legendrian knot in a tight contact manifold (M, ξ). Then the result of contact (-1)-surgery $(M_K(-1), \xi_K(-1))$ is also tight (and hence the same is true for every contact surgery coefficient $r < 0$).

Example 3.2.7 (Contact Dehn surgeries along Legendrian unknots)
(1) Consider the contact (-1)-surgery along the Legendrian unknot in (S^3, ξ_{st}) with $\operatorname{tb} = -1$ as shown in Figure 3.2. The resulting contact manifold has to be a lens space (since it is a surgery along the unknot) with tight contact structure (due to

Theorem 3.2.6). To determine the exact lens space one can recompute the contact surgery coefficient into the topological surgery coefficient.

In general it works like this: Let $r_C = p_C/q_C$ be the contact surgery coefficient (measured with respect to the contact longitude λ_C) and $r_S = p_S/q_S$ be the topological surgery coefficient (measured with respect to the Seifert longitude λ_S). This means μ_0 is glued to the curve

$$p_C\mu + q_C\lambda_C = p_S\mu + q_S\lambda_S.$$

Since the contact longitude λ_C and the Seifert longitude λ_S are related by

$$\lambda_C = \mathrm{tb}\,\mu + \lambda_S,$$

one gets

$$r_S = \frac{p_S}{q_S} = \frac{p_C + \mathrm{tb}\,q_C}{q_C} = r_C + \mathrm{tb}\,.$$

Therefore, this contact (-1)-surgery corresponds to a topological (-2)-surgery yielding the lens space $L(2,1) = \mathbb{R}P^3$. Since this lens space is known to admit only one tight contact structure [46] (denoted by ξ_{st}), the resulting contact manifold is $(\mathbb{R}P^3, \xi_{st})$.

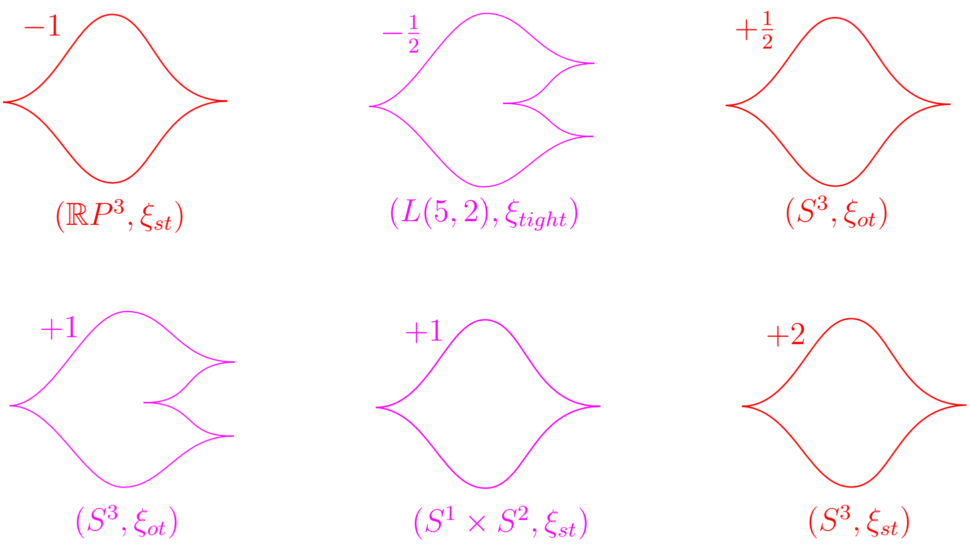


Figure 3.2: Some examples of contact Dehn surgeries along unknots

(2) With the same methods one can show that contact $(-1/2)$-surgery along a Legendrian unknot with $\mathrm{tb} = -2$ as in Figure 3.2 yields a tight contact structure on $L(5,2)$. But this time the classification of tight contact structures on lens spaces [46] says that on this lens space there are two different tight contact structures distinguished by their Euler classes. In Section 4.8 I will explain how to compute this Euler class

to determine which of the two tight contact structures on $L(5,2)$ is given by this contact surgery.
(3) A contact $(+1/2)$-surgery along a Legendrian unknot with $\mathrm{tb} = -1$ corresponds topologically to a $(-1/2)$-surgery, so yields by Example 1.2.2 a contact 3-sphere. I want to show that the resulting contact structure has to be overtwisted. Therefore, look at a Legendrian unknot with $\mathrm{tb} = -1$ in the complement of the surgery unknot as in Figure 3.3 on the left.

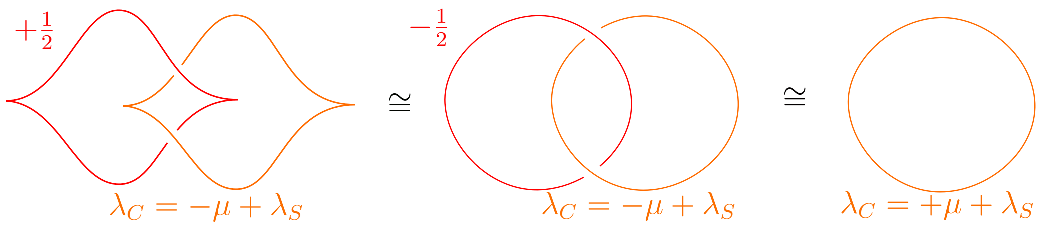


Figure 3.3: A contact Dehn surgery yielding (S^3, ξ_{ot})

I want to compute the new Thurston–Bennequin invariant of this knot in the new contact S^3. The contact longitude and the Seifert longitude before the surgery are related by $\lambda_C = -\mu + \lambda_S$. The contact longitude depends only on the contact structure in the neighborhood of the knot, and, since this contact structure in the neighborhood of the knot does not change under the surgery, the contact longitude stays the same. However, the Seifert longitude changes under the surgery, but only depends on the topological information. Therefore, it is enough to look at the topological picture in the middle of Figure 3.3. By doing a 2-fold Rolfsen twist along the surgery knot one obtains again S^3 and sees that the knot in the complement is again an unknot. But the longitudes changes as $\lambda_C = +\mu + \lambda_S$. Hence, the resulting contact 3-sphere contains a Legendrian unknot with $\mathrm{tb} = 1$, which is the boundary of an overtwisted disk after stabilizing once.
(4) With the same argument as above one shows that contact $(+1)$-surgery along a Legendrian unknot with $\mathrm{tb} = -2$ yields also an overtwisted 3-sphere. This idea of computing the Thurston–Bennequin invariant of Legendrian knots in the complements of surgery links is explained in more detail in Section 4.2 (where also a general formula is given). This is one of the main ingredients to prove the knot exterior theorems for Legendrian and transverse knots.

Observe also that knowing that the contact structure is overtwisted does not determine the contact structure, since overtwisted contact structures on closed manifolds are in one-to-one correspondence with tangential 2-plane fields (Theorem 2.4.4). In Section 4.8 I will present how to compute the so-called d_3-invariant to distinguish such tangential 2-plane fields which is well-known to be a complete invariant for overtwisted contact structures on S^3.

(5) But there are also positive contact surgeries yielding tight contact manifolds. An important example is given by contact $(+1)$-surgery along the Legendrian unknot with $\mathrm{tb} = -1$. Topologically this corresponds to a 0-surgery along the unknot resulting in $S^1 \times S^2$. One can show that the resulting contact structure is the unique tight ξ_{st} on $S^1 \times S^2$ (see [17, Lemma 4.2.]).

(6) Consider the contact surgery along the Legendrian unknot U with $\mathrm{tb} = -1$ and contact surgery coefficient $+2$ (see Figure 3.4). The resulting manifold is topologically again S^3.

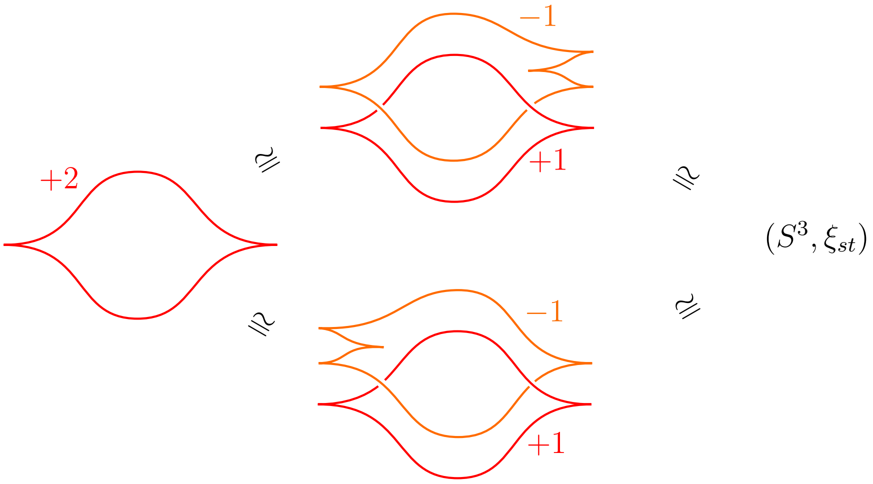


Figure 3.4: A unique contact $(+2)$-Dehn surgery resulting again in (S^3, ξ_{st})

Next I want to show that the resulting contact structure is unique and leads again to ξ_{st} (if one requires as usual the contact structure on the new glued-in solid torus to be tight). In general a contact Dehn surgery with surgery coefficient not of the form $1/q$ is not unique. But actually in this example it is. Too see this, one first uses the transformation lemma 3.2.4 to change the contact surgery diagram into contact surgeries along a link with only ± 1 surgery coefficients (see Figure 3.4) and gets

$$U(+2) = U(+1) \times U(-2) = U(+1) \times U_1(-1).$$

Observe that different choices of stabilizations yield different contact structures in general (and correspond exactly to the different contact structures on the glued-in solid torus), but in this case the resulting contact structures are contactomorphic. The contactomorphism of the resulting manifold is induced by the contactomorphism $(x, y, z) \mapsto (-x, -y, z)$ of the old (S^3, ξ_{st}) that maps one link to the other (see also [16, Section 9]).

Since by the foregoing example the contact $(+1)$-surgery yields a tight manifold and by Theorem 3.2.6 contact (-1)-surgery preserves tightness, the claim follows.

With this background now it is not difficult to prove the fundamental theorem due to Ding and Geiges that it is not only possible to construct a contact structure on every closed contact 3-manifold, but it is also possible to get every given contact structure on every given closed contact 3-manifold by contact Dehn surgery along a Legendrian link in (S^3, ξ_{st}).

Theorem 3.2.8 (Theorem of Ding–Geiges)
Every closed contact 3-manifold (M, ξ) can be obtained from (S^3, ξ_{st}) by contact (± 1)-surgery along a Legendrian link.

Proof.
Let (M, ξ) be some arbitrary closed contact 3-manifold. By the theorem of Lickorish–Wallace there exists a link in M such that one can get S^3 by Dehn surgery along this link. By the approximation theorem 2.2.14 one can approximate this link by a Legendrian link with the same topological link type. By stabilizations one can avoid contact 0-surgeries and by the transformation lemma one can obtain a contact 3-sphere out of (M, ξ) by contact $(1/n)$-surgery along a Legendrian link.

Because of the cancellation lemma and because there exists only one tight contact structure on S^3, it is now enough to give contact $(1/n)$-surgery diagrams along Legendrian links in (S^3, ξ_{st}) to every overtwisted contact 3-sphere. This is done in Remark 4.8.4 with help of the d_3-invariant. □

4

Computing the classical invariants

As explained in Chapter 2 a main method to study contact manifolds is via the classical invariants of their Legendrian and transverse knots. In this chapter I want to explain how to compute the classical invariants of knots in contact surgery diagrams. A given knot in the exterior of a surgery link can be seen as a knot in the new resulting manifold. Here I want to present formulas for computing the classical invariants of such knots in the new surgered manifold.

The main problem is that the classical invariants are only defined for nullhomologous (or more general rationally nullhomologous) knots, and a surgery in general destroys this property. But as long as the resulting manifold is a homology sphere (or a rational homology sphere) every knot has to be nullhomologous (or rationally nullhomologous). In this case formulas for computing the new Thurston–Bennequin invariant and the new rotation number out of the old ones and out of the algebraic surgery data are given in [55, Lemma 6.6], [39, Lemma 2] and [11, Lemma 6.4].

Here, I first want to give an easy (and easy to check) condition (out of the algebraic surgery data) when such a knot is nullhomologous in the new manifold. In this case I show how to compute out of these data the new classical invariants.

Finally, in Section 4.7 I explain that every knot in a given manifold can be presented, as a knot in the exterior of a surgery link.

There is another useful homotopical invariant of tangential 2-plane fields (with Euler class torsion), the so-called d_3-invariant. Many contact structures can be distinguished by this invariant. In [17] a formula is given how to compute the d_3-invariant out of a contact (± 1)-surgery diagram. This turns out to be closely related to the rotation numbers. In Section 4.8 we generalize this result to arbitrary contact $(1/n)$-surgeries, which often simplifies combined with the replacement lemma 3.2.3 computations a lot.

The formulas to compute the Thurston–Bennequin invariant are based on Section 8 in my paper [53]. The formulas for the rotation, self-linking number and the d_3-invariant were obtained in joint work with Sebastian Durst. They appeared in very similar form in our paper [18]. The part on the rational invariants (Section 4.5) is partially based on joint work with Sebastian Durst and Mirko Klukas [19].

If no coefficient group is specified, in the whole chapter homology groups are understood to be over the integers.

4.1 Computing the homology class of a knot

Let $L = L_1 \sqcup \cdots \sqcup L_n \subset S^3$ be an oriented link (where the choice of the orientations is needed for the computations but does not affect the final result). Let M be the 3-manifold obtained from S^3 by Dehn surgery along L with topological surgery coefficients $r_i = p_i/q_i$, for $i = 1, \dots, n$. Denote by $L_0 \subset S^3 \setminus \mathring{\nu L}$ an oriented knot in S^3 or M, depending on the context. For simplicity write the linking numbers as $l_{ij} := \operatorname{lk}(L_i, L_j)$, for $i = 0, \dots, n$. Also set

$$Q := \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1n} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{n1} & & & p_n \end{pmatrix} \quad \text{and} \quad \mathbf{l} := \begin{pmatrix} l_{01} \\ \vdots \\ l_{0n} \end{pmatrix}.$$

The matrix Q is a generalization of the linking matrix, since for integer surgeries ($q_i = 1$) the matrix Q is the linking matrix.

The knot L_0 is called **nullhomologous** in M if $[L_0] = 0 \in H_1(M; \mathbb{Z})$. As already mentioned this is equivalent to the existence of a Seifert surface for the knot.

With the following lemma one can decide from the algebraic surgery data if such a knot is nullhomologous in the surgered manifold.

Lemma 4.1.1 (Nullhomologous knots)
L_0 is nullhomologous in M if and only if there exists an integer solution $\mathbf{a} \in \mathbb{Z}^n$ of the equation $\mathbf{l} = Q\mathbf{a}$.

Proof.
It is easy to compute the homology of M (see for example [41, Proposition 5.3.11]) as

$$H_1(M; \mathbb{Z}) = \mathbb{Z}_{\mu_1} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i \mu_i + q_i \sum_{\substack{j=1 \\ j \neq i}}^{n} l_{ij} \mu_j = 0 \,|\, i = 1, \dots, n \rangle,$$

where the generators of the $\mathbb{Z}$-factors are given by right-handed meridians μ_i corresponding to the components L_i. Next we express L_0 as a linear combination of the μ_i. One can show that the coefficients are the linking numbers l_{i0}, i.e.

$$[L_0] = \sum_{i=1}^{n} l_{i0} \mu_i.$$

So L_0 is nullhomologous if and only if one can express $[L_0] = \sum l_{i0} \mu_i$ as a linear combination of the relations, i.e. if there exists integers a_i, $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} l_{i0} \mu_i = \sum_{i=1}^{n} a_i \Big(p_i \mu_i + q_i \sum_{\substack{j=1 \\ j \neq i}}^{n} l_{ij} \mu_j \Big) = \sum_{i=1}^{n} \Big(a_i p_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j l_{ij} a_j \Big) \mu_i.$$

By comparing the coefficients and by writing the corresponding equations in vector form one sees that this is true if and only if there exists a vector $\mathbf{a} \in \mathbb{Z}^n$ such that $\mathbf{l} = Q\mathbf{a}$. $\square$

4.2 Computing the Thurston–Bennequin invariant

Now assume L_0 is a Legendrian link in $(S^3 \setminus \nu\mathring{L}, \xi_{st}) \subset (S^3, \xi_{st})$, and let ξ be a contact structure on M that coincides with ξ_{st} outside a tubular neighborhood of L, for example if L is also a Legendrian link in (S^3, ξ_{st}) and (M, ξ) is the result of a contact Dehn surgery along L. But it is important to notice that the setting here is more general, one can also use contact surgery along transverse knots or surgery along a knot that is not adapted to the contact structure.

As explained in Chapter 3 the contact structure ξ is not unique, but it will turn out that the value of the Thurston–Bennequin invariant of the knot is the same for all choices of contact structures.

Here all surgery coefficients are understood to be topological surgery coefficients, i.e. with respect to the Seifert longitude λ_s. (So if one has a Legendrian surgery diagram one first has to change the contact surgery coefficients to topological surgery coefficients with the formula $r_{i,top} = r_{i,cont} + \operatorname{tb}(L_i)$.)

Lemma 4.2.1 (Computing the Thurston–Bennequin invariant)
If L_0 is nullhomologous in M, then one can compute the new Thurston–Bennequin number tb_{new} of L_0 in (M, ξ) from the old one tb_{old} of L_0 in (S^3, ξ_{st}) as

$$\operatorname{tb}_{new} = \operatorname{tb}_{old} - \sum_{i=1}^{n} a_i q_i l_{i0}$$

where $\mathbf{a} \in \mathbb{Z}^n$ is a solution vector from the formula of Lemma 4.1.1.

Proof.
Let λ_s be the Seifert longitude of L_0 in S^3 and let λ_c be the contact longitude of L_0 in (S^3, ξ_{st}). Then tb_{old} is given by

$$\lambda_c = \operatorname{tb}_{old} \mu_0 + \lambda_s \in H_1(\partial\nu L_0).$$

Since the contact longitude is defined by the contact structure along L_0 and the contact structure does not change near L_0 by doing the surgery along L, the contact longitude λ_c also represents the contact longitude in (M, ξ). But in general the Seifert longitude λ_s changes. Since the knot L_0 is nullhomologous in M, there is a unique $f \in \mathbb{Z}$ such that $f\mu_0 + \lambda_s = 0 \in H_1(M \setminus \nu\mathring{L}_0; \mathbb{Z})$ (this is the new Seifert longitude). Then tb_{new} is given by

$$\lambda_c = \operatorname{tb}_{new} \mu_0 + (f\mu_0 + \lambda_s) \in H_1(\partial\nu L_0).$$

Putting this together leads to

$$\mathrm{tb}_{new} = \mathrm{tb}_{old} - f.$$

So it is only left to compute f. Similar to the proof of the previous lemma one computes the homology of $M \setminus \nu \mathring{L}_0$ as

$$H_1(M \setminus \nu \mathring{L}_0; \mathbb{Z}) = \mathbb{Z}_{\mu_0} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i\mu_i + q_i \sum_{\substack{j=0 \\ j\neq i}}^{n} l_{ij}\mu_j = 0 \,|\, i = 1, \ldots, n\rangle$$

and expresses λ_s as

$$\lambda_s = \sum_{i=1}^{n} l_{i0}\mu_i.$$

So $f\mu_0 + \lambda_s$ is zero in $H_1(M \setminus \nu \mathring{L}_0; \mathbb{Z})$ if and only if there exist integers b_i, $i = 1, \ldots, n$, such that

$$\begin{aligned} f\mu_0 + \sum_{i=1}^{n} l_{i0}\mu_i &= \sum_{i=1}^{n} b_i\Big(p_i\mu_i + q_i \sum_{\substack{j=0 \\ j\neq i}}^{n} l_{ij}\mu_j\Big) \\ &= \Big(\sum_{j=1}^{n} b_j q_j l_{j0}\Big)\mu_0 + \sum_{i=1}^{n}\Big(b_i p_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} q_j l_{ij} b_j\Big)\mu_i. \end{aligned}$$

By comparing the coefficients this is equivalent to the existence of a vector $\mathbf{b} \in \mathbb{Z}^n$ such that

$$\mathbf{l} = Q\mathbf{b} \text{ and}$$
$$f = \sum_{j=1}^{n} b_j q_j l_{j0}.$$

If one chooses for $\mathbf{b}$ a solution $\mathbf{a}$ from Lemma 4.1.1 the formula follows. $\square$

Example 4.2.2 (Computing tb-numbers with these formulas)
(1) Consider the surgery diagram from Figure 4.1(i). The old Thurston–Bennequin invariant of L_0 is -1. Since the surgery along the unknot L_1 with topological coefficient $1/n$ leads again to S^3 the knot L_0 is again nullhomologous in the new manifold. This can be checked also with the formula from above. For $\mathbf{a}$ one gets the equation $1 = l_{10} = Qa_1 = p_1a_1 = a_1$. Therefore $\mathrm{tb}_{new} = -1 - l_{10}q_1a_1 = -1 - n$. Observe again that this result does not depend on the explicit contact structure chosen for the surgery.
(2) Consider the surgery diagram from Figure 4.1(ii). Again the surgery leads to S^3, so the resulting knot is again nullhomologous. The new Thurston–Bennequin invariant can be computed as follows. First one looks for a solution $\mathbf{a}$ of the equation

$2 = l_{10} = Qa_1 = p_1a_1 = a_1$ and then $\mathrm{tb}_{new} = -1 - l_{10}q_1a_1 = -1 - 4n$.

(3) There are also examples where the solution a of $\mathbf{l} = Q\mathbf{a}$ is not unique, but the result of tb_{new}, of course, is not affected by this. For example consider the surgery diagram from Figure 4.6(i). First, one has to check if the knot L_0 is again nullhomologous in the surgered manifold. In order to do this, one has to check if there exists a solution $\mathbf{a} \in \mathbb{Z}$ of

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{l} = Q\mathbf{a} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Two obvious solutions are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the new knot is nullhomologous and the new Thurston–Bennequin invariant can be computed out of the old one as follows:

$$\mathrm{tb}_{new} = \mathrm{tb}_{old} - \langle \mathbf{a}, \begin{pmatrix} q_1 l_{10} \\ q_2 l_{20} \end{pmatrix} \rangle = \mathrm{tb}_{old} - \langle \mathbf{a}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle = \mathrm{tb}_{old} - 2$$

Observe that this result does not depend on the choice of $\mathbf{a}$.

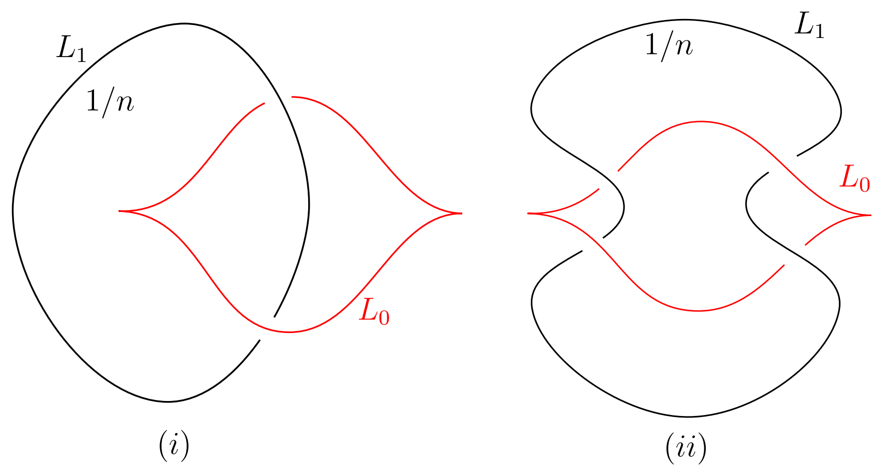


Figure 4.1: Computing tb-numbers in surgery diagrams

Remark 4.2.3 (Different choices of solution vectors yield the same result)
The particular choice of a solution $\mathbf{a}$ does not influence the result since two different solutions differ only by a vector $\mathbf{c}$ in the kernel of Q and the vector $(q_i l_{i0})_{i=1,\ldots,n}$ lies in the image of Q. Thus, the scalar product of $(q_i l_{i0})_{i=1,\ldots,n}$ and $\mathbf{c}$ vanishes.

4.3 Computing the rotation number

In the previous section we saw that the Thurston–Bennequin invariant does not depend on the special choice of the contact structure on the new glued-in solid torus, but it will turn out (see Example 4.3.3) that the rotation number does. Therefore, a formula for the rotation number makes sense only if the contact structure on the new glued-in solid tori is unique, for example for contact $(1/q)$-surgeries along Legendrian knots.

So assume now that the surgery link L and the knot L_0 in its exterior are Legendrian and that the contact surgery coefficients are all of the form $1/q_i$ for $q_i \in \mathbb{Z}$.

Lemma 4.3.1 (Computing the rotation number)
The knot L_0 is nullhomologous in M if and only if there is an integral vector $\mathbf{a}$ *solving* $\mathbf{l} = Q\mathbf{a}$ *as above, in which case its new rotation number* $\operatorname{rot}_{new,\widehat{\Sigma}}$ *in* (M,ξ) *with respect to a special Seifert class* $\widehat{\Sigma}$ *is equal to*

$$\operatorname{rot}_{new,\widehat{\Sigma}} = \operatorname{rot}_{old} - \sum_{i=1}^{n} a_i q_i \operatorname{rot}_i$$

where rot_i *denotes the rotation number of L_i in (S^3, ξ_{st}).*

Recall again that if the Euler class of the contact structure vanishes (see Section 4.8 how to compute this), then the rotation number does not depend on the homology class of the particular Seifert surface. The proof proceeds in two steps. First, following [11], we construct the class of a Seifert surface for L_0 in M. We then use the description of the rotation number in terms of relative Euler classes to compute rot.

Proof.
Assume that L_0 is nullhomologous in M and fix Seifert surfaces $\Sigma_0, \dots, \Sigma_n$ for $L_0, \dots, L_n$ in S^3 such that intersections of surfaces and link components are transverse. Our aim is to use these surfaces to construct the class of a Seifert surface for L_0 in the surgered manifold M. By abuse of notation, we will identify Σ_i with its class in $H_1(S^3 \setminus \nu\mathring{L}_i, \partial\nu L_i)$ and we will denote the class in $H_1(S^3 \setminus (\nu\mathring{L}_0 \sqcup \nu\mathring{L}), \partial L_0 \sqcup \partial\nu L)$ induced by restriction also as Σ_i.

The idea is to construct a class of the form

$$\Sigma = \Sigma_0 + \sum_{i=1}^{n} k_i \Sigma_i$$

such that its image under the boundary homomorphism ∂ in the long exact sequence of the pair $(S^3 \setminus (\nu\mathring{L}_0 \sqcup \nu\mathring{L}), \partial L_0 \sqcup \partial\nu L)$ is a linear combination of the surgery slopes

r_i and a longitude of L_0, i.e. we want

$$\partial\Sigma = f\mu_0 + \lambda_0 + \sum_{i=1}^{n} m_i r_i = f\mu_0 + \lambda_0 + \sum_{i=1}^{n} m_i(p_i\mu_i + q_i\lambda_i).$$

The surgery longitudes r_i bound discs in the surgered manifold M, so Σ can be extended to give rise to a class in $H_1(M \setminus \nu L_0, \partial\nu L_0)$ which we denote by $\widehat{\Sigma}$. Geometrically, the boundary homomorphism sends Σ to its intersection with the boundary of the link complement. So we have:

$$\partial\colon\ \Sigma_j \longmapsto \lambda_j - \sum_{i\neq j} l_{ij}\mu_i,$$

and thus

$$\begin{aligned}\partial\colon\ \Sigma \longmapsto &\sum_{i=0}^{n} k_i\lambda_i - \sum_{j=0}^{n}\sum_{i\neq j} k_j l_{ij}\mu_i \\ &= -\sum_{j=1}^{n} k_j l_{0j}\mu_0 + \lambda_0 + \sum_{i=1}^{n} k_i\lambda_i - \sum_{i=1}^{n} l_{0i}\mu_i - \sum_{i=1}^{n}\sum_{j\neq i} k_j l_{ij}\mu_i,\end{aligned}$$

where we set $k_0 = 1$. Note that the minus sign stems from the induced boundary orientation of Σ (see Figure 4.2).

By comparing the coefficients and using that $\mathbf{l} = Q\mathbf{a}$ since L_0 is nullhomologous in M, we obtain $k_i = -a_i q_i$, $m_i = -a_i$ and $f = \sum_{j=1}^{n} a_j q_j l_{0j}$.

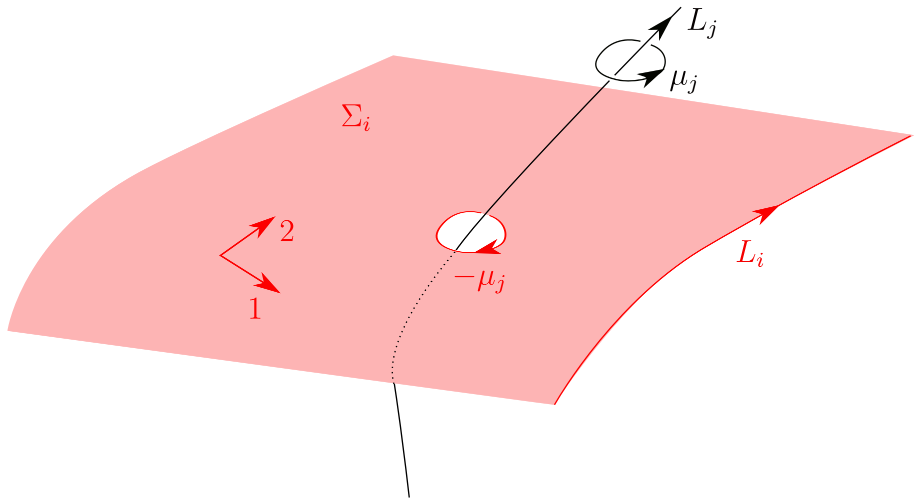


Figure 4.2: Orientation of the meridian induced by the intersection

Observe that we can also directly obtain an embedded surface representing the capped-off class $\widehat{\Sigma}$ by resolving self-intersections in Σ. In particular, f is the negative change of the Thurston–Bennequin number of L_0 in the surgery, i.e. we get again the formula from Lemma 4.2.1

$$\mathrm{tb}_{new} = \mathrm{tb}_{old} - \sum_{j=1}^{n} a_j q_j l_{j0}.$$

Moreover, observe that one can give an upper bound on the new genus g_{new} of the knot L_0 in the new surgered manifold since resolving the singularities is done by removing annuli and gluing them back in a different way. Therefore, the Euler characteristic of the surface with resolved singularities computes as

$$\chi(\Sigma) = \chi(\Sigma_0) - \sum_{i=1}^{n} a_i q_i \chi(\Sigma_i),$$

which leads to a bound on the genus as required in the Bennequin inequality (by choosing Seifert surfaces Σ_i of minimal genus)

$$2g_{new} - 1 \leq -\chi(\Sigma) = 2g_{old} - 1 - \sum_{i=1}^{n} a_i q_i (2g_i - 1).$$

Sometimes this bound can be used to prove overtwistedness of surgered contact manifolds (compare [11]).

Now take L and L_0 to be Legendrian in (S^3, ξ_{st}) and the surgeries to be contact $(1/q)$-surgeries. We claim that the rotation number of L_0 in the surgered contact manifold (M, ξ) with respect to $\widehat{\Sigma}$ is equal to

$$\mathrm{rot}_{new,\widehat{\Sigma}} = \mathrm{rot}_{new} - \sum_{i=1}^{n} a_i q_i \, \mathrm{rot}_i.$$

In complete analogy to [55], [39] and [11] we have the following lemma.

Lemma 4.3.2 (Behavior of the relative Euler classes under surgery)
The homomorphism $H_1(S^3 \setminus (L_0 \sqcup L)) \to H_1(M \setminus L_0)$ *induced by inclusion sends* $\mathrm{PD}(e(\xi_{st}, L_0 \sqcup L))$ *to* $\mathrm{PD}(e(\xi, L_0))$.

The proof is completely analogous to the proof of Lemma 3 in [39], where one uses the Legendrian rulings of the surgery tori induced by $(1/q)$-surgery instead of (± 1)-surgery.

We thus have

$$\begin{aligned}
\mathrm{rot}_{new,\widehat{\Sigma}} &= \mathrm{PD}\left(\mathrm{e}(\xi, L_0)\right) \bullet \widehat{\Sigma} \\
&= \mathrm{PD}\left(\mathrm{e}(\xi_{st}, L_0 \sqcup L)\right) \bullet \Sigma \\
&= \left(\sum_{i=0}^{n} \mathrm{rot}_i \, \mu_i\right) \bullet \Sigma \\
&= \left(\sum_{i=0}^{n} \mathrm{rot}_i \, \mu_i\right) \bullet \left(\Sigma_0 + \sum_{j=1}^{n} (-a_j q_j) \Sigma_j\right) \\
&= \mathrm{rot}_{old} - \sum_{i=1}^{n} a_i q_i \, \mathrm{rot}_i,
\end{aligned}$$

which proves the theorem. □

If the contact surgeries are not unique the rotation number is – in contrast to the Thurston–Bennequin invariant – not independent of the chosen contact structures on the solid tori, as the following example illustrates.

Example 4.3.3 (Rotation depends on the choice of the contact structure)
Consider the diagram depicted in Figure 4.3, where L is a Legendrian trefoil with contact surgery coefficient $-3/4$ and L_0 a Legendrian unknot in its complement.

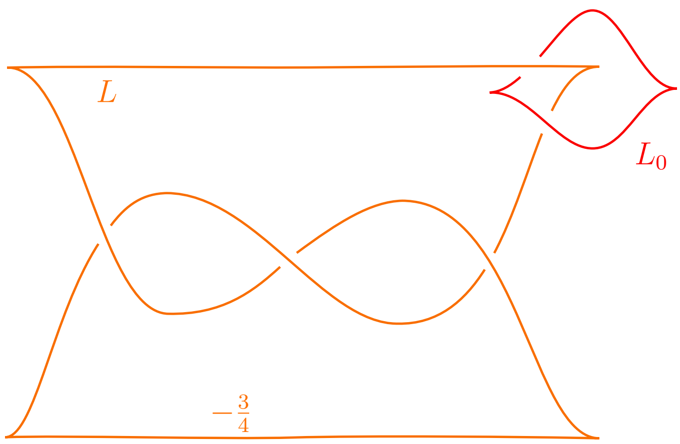


Figure 4.3: Non-unique contact surgery yielding a homology sphere

We have $\mathrm{tb}(L) = 1$, so the topological surgery coefficient is $1/4$. Thus, the surgered manifold M is a homology sphere and the rotation number of L_0 is independent of the choice of the Seifert surface. The contact surgery coefficient $-3/4$ has a continued fraction expansion $1 - 2 - \frac{1}{-4}$, so by the transformation lemma 3.2.4 there are three distinct tight contact structures on the solid torus compatible with the surgery resulting in the contact manifolds which are shown in Figure 4.4.

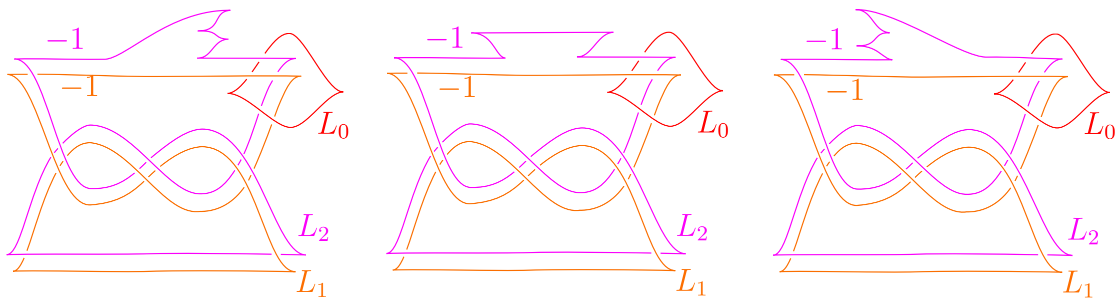


Figure 4.4: Three unique contact surgeries corresponding to Figure 4.3

Topologically these are the same, i.e. for all three diagrams we have

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } \mathbf{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and hence $\mathbf{a} = (3, 1)$. Furthermore, we have $\mathrm{rot}_{old} = 0$, $\mathrm{rot}_1 = 0$ and $\mathrm{rot}_2 \in \{-2, 0, 2\}$. This yields

$$\mathrm{rot}_{new} = \mathrm{rot}_{old} - 3\,\mathrm{rot}_1 - \mathrm{rot}_2 \in \{-2, 0, 2\},$$

depending on the chosen contact structure.

Example 4.3.4 (Computing the classical invariants after a single surgery)
We consider the case of L being a one component link with contact surgery coefficient $\pm\frac{1}{n}$ and classical invariants tb and rot, so the topological surgery coefficient is $\frac{n\,\mathrm{tb}\pm 1}{n}$. We then have $Q = p = n\,\mathrm{tb}\pm 1$ and L_0 is nullhomologous in the surgered manifold if and only if the linking number of L_0 and L is divisible by $n\,\mathrm{tb}\pm 1$, in which case $\mathbf{a}$ is the quotient $\frac{\mathrm{lk}}{n\,\mathrm{tb}\pm 1}$. Then the rotation number of L_0 in the surgered manifold is

$$\mathrm{rot}_{new,\widehat{\Sigma}} = \mathrm{rot}_{old} - \frac{n\,\mathrm{lk}}{n\,\mathrm{tb}\pm 1}\,\mathrm{rot},$$

and its Thurston–Bennequin invariant is

$$\mathrm{tb}_{new} = \mathrm{tb}_{old} - \frac{n\,\mathrm{lk}^2}{n\,\mathrm{tb}\pm 1}.$$

Observe that if $n\,\mathrm{tb}\pm 1$ is non-zero, the knot L_0 is rationally nullhomologous. Then the computed numbers represent the rational invariants (cf. Section 4.5).

4.4 Computing the self-linking number

The computation of self-linking numbers of transverse knots is similar to rotation numbers of Legendrian knots. Therefore, we consider a similar setting.

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an oriented Legendrian link in (S^3, ξ_{st}) and T_0 an oriented transverse knot in its complement. Let (M, ξ) be the contact manifold obtained from S^3 by contact $(1/q)$-surgery along L and write p_i/q_i for the topological surgery coefficients.

Lemma 4.4.1 (Computing the self-linking number)
The knot T_0 is nullhomologous in M if and only if there is an integer vector $\mathbf{a}$ solving $\mathbf{l} = Q\mathbf{a}$ as above, in which case its new self-linking number $\mathrm{sl}_{new,\widehat{\Sigma}}$ in (M, ξ) with respect to a special Seifert class $\widehat{\Sigma}$ is equal to

$$\mathrm{sl}_{new,\widehat{\Sigma}} = \mathrm{sl}_{old} - \sum_{i=1}^{n} a_i q_i (l_{i0} \mp \mathrm{rot}_i)$$

where the sign is $-$ when T_0 is positively transverse and $+$ when L_0 is negatively transverse.

Proof.
As explained in Remark 2.2.16, any transverse knot is a transverse push-off of a (non-unique) Legendrian knot. Now for $L_0^\pm$ the positive or negative push-off of the Legendrian knot L_0 and Σ a Seifert surface we have

$$\mathrm{sl}(L_0^\pm, [\Sigma]) = \mathrm{tb}(L_0) \mp \mathrm{rot}(L_0, [\Sigma])$$

in any contact manifold (see Lemma 2.3.8). Hence,

$$\begin{aligned}\mathrm{sl}_{new}(L_0^\pm, \widehat{\Sigma}) &= \mathrm{tb}_{new}(L_0) \mp \mathrm{rot}_{new}(L_0, \widehat{\Sigma})\\ &= \Big(\mathrm{tb}_{old}(L_0) - \sum_{j=1}^n a_j q_j l_{j0}\Big) \mp \Big(\mathrm{rot}_{old}(L_0) - \sum_{i=1}^n a_i q_i \,\mathrm{rot}_i\Big)\\ &= \mathrm{sl}_{old}(L_0^\pm) - \sum_{i=1}^n a_i q_i \Big(l_{i0} \mp \mathrm{rot}_i\Big).\end{aligned}$$

□

Remark 4.4.2 (Orientations of Legendrian knots)
An oriented transverse knot T is either positively or negatively transverse. If we pick a Legendrian knot L such that T is a transverse push-off, we orient L accordingly. Then the class of an oriented Seifert surface of T is also the class of an oriented Seifert surface of L and vice versa. With these orientations, T is a positive (negative) push-off of L if T is positively (negatively) transverse. In particular, the topological data used in the formula in Corollary 4.4.1 coincides for the two knots.

Example 4.4.3 (Computing the self-linking number with this formula)
(1) The left diagram in Figure 4.5 shows a positive transverse knot T_0 in an overtwisted 3-sphere. We have $\mathbf{l} = -1$, $Q = p = -1$, and thus $\mathbf{a} = 1$. The rotation number of L is 1, so we have

$$\mathrm{sl}_{new} = \mathrm{sl}_{old} - a_1 q_1 (l_{10} - \mathrm{rot}_1) = -1 - (-1 - 1) = 1.$$

Therefore, T_0 violates the Bennequin inequality in the surgered 3-sphere.

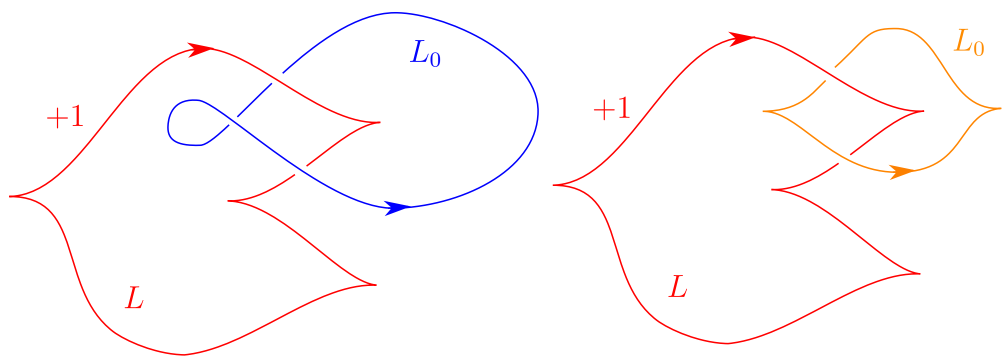


Figure 4.5: Computing the self-linking number

Alternatively, we can consider a Legendrian unknot such that L_0 is its positive push-off, as shown on the right in Figure 4.5, and one can compute again (compare Example 3.2.7(4)) its Thurston–Bennequin invariant in the surgered 3-sphere as $-1+1=1$, and its rotation number is $0-1=-1$.
(2) We can also consider T_0 as a negative transverse knot by reversing its orientation. Then $\mathbf{l}=1$, $Q=p=-1$ and $\mathbf{a}=-1$, so

$$\mathrm{sl}_{new} = \mathrm{sl}_{old} - a_1 q_1 \Big(l_{10} - \mathrm{rot}_1 \Big) = -1 - (1+1) = 1,$$

as expected, since the self-linking number is independent of the choices of orientation. We can again consider the corresponding Legendrian knot, which then has vanishing Thurston–Bennequin invariant and rotation number 1. As L_0 is now its negative push-off, we also get

$$\mathrm{sl}_{new} = \mathrm{tb}_{new} + \mathrm{rot}_{new} = 1.$$

4.5 Computing the rational invariants

The results in the previous sections can easily be generalized to rationally nullhomologous knots. A knot L_0 in M is called **rationally nullhomologous** if there exists a natural number $d \in \mathbb{N}$ such that $d[L_0] = 0 \in H_1(M;\mathbb{Z})$, i.e. $[L_0]$ vanishes in $H_1(M;\mathbb{Q})$.

One of the most fundamental examples of rationally nullhomologous knots are given by the spines of the standard genus-1 Heegaard tori of lens spaces.

For rationally nullhomologous Legendrian knots in contact 3-manifolds one can generalize the classical invariants to the so-called **rational classical invariants** $\mathrm{tb}_\mathbb{Q}$, $\mathrm{rot}_\mathbb{Q}$ and $\mathrm{sl}_\mathbb{Q}$ (see for example [3], [2] and [39]). We first review these definitions and then extend the results from the previous sections to this setting.

Definition 4.5.1 (Rational Seifert longitude and surface)
A **(rational) Seifert longitude** of an oriented rationally nullhomologous knot K of order d is a class $r \in H_1(\partial\nu K)$ such that

$$\mu \bullet r = d \qquad\qquad \text{and} \qquad\qquad r = 0 \text{ in } H_1(M \setminus \nu K).$$

A **rational Seifert surface** for an oriented rationally nullhomologous knot K is a surface with boundary in the exterior of K whose boundary represents a (rational) Seifert longitude of K.

It is obvious that every rationally nullhomologous knot has a Seifert longitude. However, its uniqueness was not clear but appeared recently in my paper in joint work with Sebastian Durst and Mirko Klukas [19].

Lemma 4.5.2 (Uniqueness of the Seifert longitude)
The Seifert longitude of a rationally nullhomologous knot is unique.

Proof.
Let r_1 and r_2 be Seifert longitudes. Let (μ, λ) be an oriented basis of $H_1(\partial\nu K)$ where μ is represented by a meridian of K. Then we can write

$$r_i = p_i\mu + q_i\lambda.$$

As r_i is a Seifert longitude we have $q_i = d$ with d the order of K. The classes r_1 and r_2 are equal if considered in $H_1(M \setminus \nu K)$, therefore we have $p_1\mu = p_2\mu$ in $H_1(M \setminus \nu K)$. But a meridian of K intersects a rational Seifert surface non-trivially, so μ cannot be a torsion element. Hence, $p_1 = p_2$, i.e. the longitudes coincide. $\square$

Existence and uniqueness of the Seifert longitude enable us to define the rational Thurston–Bennequin invariant $\mathrm{tb}_{\mathbb{Q}}$, which coincides with the usual definition in the nullhomologous case and is well defined in arbitrary contact 3-manifolds.

Definition 4.5.3 (The rational Thurston–Bennequin invariant)
The **rational Thurston–Bennequin invariant** of a rationally nullhomologous Legendrian knot K is defined as

$$\mathrm{tb}_{\mathbb{Q}}(K) = \frac{1}{d}\left(\lambda_c \bullet r\right)$$

where λ_c denotes the contact longitude and r the Seifert longitude, and the intersection product $\bullet$ is taken in $H_1(\partial\nu K)$.

In the same situation as in Lemma 4.2.1 one gets the following generalization.

Lemma 4.5.4 (Computing the rational Thurston–Bennequin invariant)
(1) L_0 is rationally nullhomologous in M if and only if there exists a natural number d and a vector $\mathbf{a} \in \mathbb{Z}^n$ such that $d\mathbf{l} = Q\mathbf{a}$.
(2) If L_0 is rationally nullhomologous in M, then one can compute the new rational Thurston–Bennequin number $\mathrm{tb}_{\mathbb{Q},new}$ of L_0 in (M, ξ) from the old one tb_{old} of L_0 in (S^3, ξ_{st}) as follows

$$\mathrm{tb}_{\mathbb{Q},new} = \mathrm{tb}_{old} - \frac{1}{d}\sum_{i=1}^{n} a_i q_i l_{i0}.$$

Proof.
The proofs are similar to the ones in the foregoing subsections. For the first part one has to change the condition $[L_0] = 0$ to $d[L_0] = 0$ and do the same computations again.

The second part works similar. If L_0 is only rationally nullhomologous in M then the Thurston–Bennequin invariants are related as follows (see also [39, Proof of Lemma 2])

$$d\lambda_s + d\,\mathrm{tb}_{old}\,\mu_0 = d\lambda_c = d\,\mathrm{tb}_{\mathbb{Q},new}\,\mu_0 + (f\mu_0 + d\lambda_s).$$

Then the same computations as in the proof of Lemma 4.2.1 yield the result. □

Example 4.5.5 (Computing tb-numbers of rationally Legendrian unknots)
This formula for computing the rational Thurston–Bennequin invariant is very useful for calculating $\mathrm{tb}_{\mathbb{Q}}$ in lens spaces. For example, consider the surgery diagram from Figure 4.6(ii). The $(-p/q)$-surgery along L_1 leads to the lens space $L(p,q)$. To check if the knot L_0 is nullhomologous one has to solve the equation $1 = \mathbf{l} = Q\mathbf{a} = -pa_1$. For $p \neq 1$ this equation has no integral solution and therefore L_0 is not nullhomologous in the surgered manifold. But for $p \neq 0$ the equation $d = d\mathbf{l} = Q\mathbf{a} = -pa_1$ has a solution, for example $d = p$ and $a_1 = -1$. So L_0 is rationally nullhomologous in $L(p,q)$. The rational Thurston–Bennequin invariant is computed as follows

$$\mathrm{tb}_{\mathbb{Q},new} = \mathrm{tb}_{old} - \frac{1}{d} a_1 l_{01} q_1 = \mathrm{tb}_{old} + \frac{q}{p}.$$

Observe again that the result is independent of the chosen solution $\mathbf{a}$ and independent of the chosen contact structure on the new glued-in solid torus.

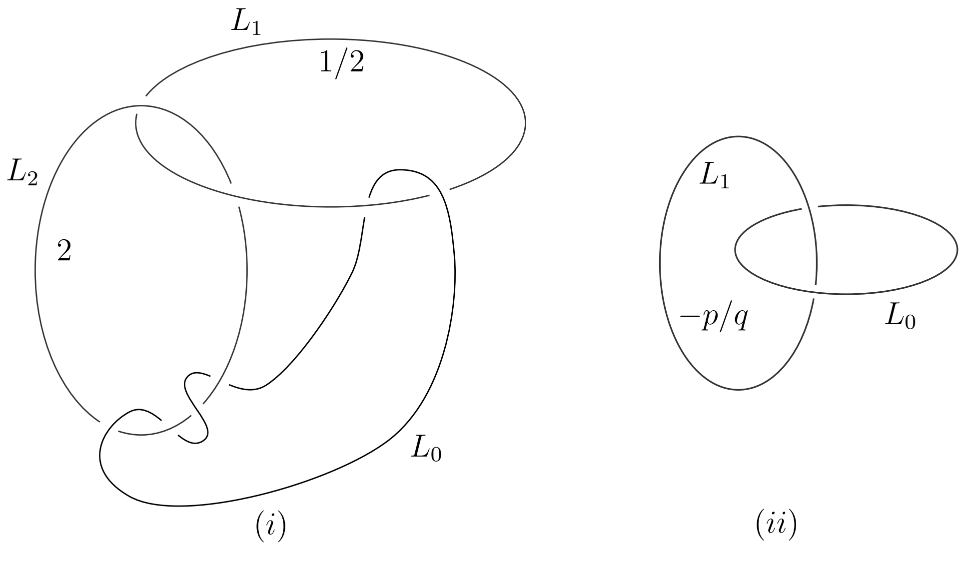


Figure 4.6: Computing rational tb-numbers in surgery diagrams

The definition of the rational rotation number also generalizes naturally to Legendrian knots which are only rationally nullhomologous.

Definition 4.5.6 (The rational rotation number)
The **rational rotation number** of an oriented rationally nullhomologous Legendrian knot K of order d with respect to a rational Seifert surface Σ is equal to

$$\operatorname{rot}_{\mathbb{Q}}(K,\Sigma)=\frac{1}{d}\langle \mathrm{e}(\xi,K),[\Sigma]\rangle=\frac{1}{d}\operatorname{PD}(\mathrm{e}(\xi,K))\bullet[\Sigma]$$

where $\mathrm{e}(\xi,K)$ is the relative Euler class of the contact structure ξ relative to the trivialization given by a positive tangent vector field along the knot K and $[\Sigma]$ the relative homology class represented by the surface Σ, and the intersection is taken in $H_1(\partial\nu K)$.

Again in the same situation as in Lemma 4.3.1 one can generalize this result.

Lemma 4.5.7 (Computing the rational rotation number)
The knot L_0 is rationally nullhomologous of order d in M if and only if there is an integer vector $\mathbf{a}$ solving $d\mathbf{l}=Q\mathbf{a}$ as above with d the minimal natural number for which a solution exists, in which case its rational rotation number $\operatorname{rot}_{\mathbb{Q},new,\widehat{\Sigma}}$ in (M,ξ) with respect to a special (rational) Seifert class $\widehat{\Sigma}$ is equal to

$$\operatorname{rot}_{\mathbb{Q},new,\widehat{\Sigma}}=\operatorname{rot}_{old}-\frac{1}{d}\sum_{i=1}^{k}a_iq_i\operatorname{rot}_i.$$

Proof.
Again following [11], we want to construct a rational Seifert class of the form

$$\Sigma=d\Sigma_0+\sum_{i=1}^{k}k_i\Sigma_i$$

such that its image under the boundary homomorphism ∂ in the long exact sequence of the pair $(S^3\setminus(\nu\mathring{L}_0\sqcup\nu\mathring{L}),\partial\nu L_0\sqcup\partial\nu L)$ is a linear combination of the surgery slopes r_i and a rational longitude of L_0. Exactly the same computations as in the proof of Lemma 4.3.1 lead to the result. □

The definition of the self-linking number of a transverse knot also generalizes to the setting of rationally nullhomologous knots by choosing a rational Seifert surface and doing the same construction as in the nullhomologous case. Furthermore, the rational invariants of a Legendrian knot and the rational self-linking of a transverse push-off are, as in the nullhomologous case, related by

$$\operatorname{sl}_{\mathbb{Q}}(L_0^{\pm},[\Sigma])=\operatorname{tb}_{\mathbb{Q}}(L_0)\mp\operatorname{rot}_{\mathbb{Q}}(L_0,[\Sigma])$$

(see Lemma 1.2 in [2]). Hence, we have the following corollary.

Lemma 4.5.8 (Computing the rational self-linking number)
In the situation of Lemma 4.4.1 the knot T_0 is rationally nullhomologous of order d in M if and only if there is an integral vector $\mathbf{a}$ solving $d\mathbf{l} = Q\mathbf{a}$ as above, in which case its rational self-linking number $\mathrm{sl}_{\mathbb{Q},new,\widehat{\Sigma}}$ in (M,ξ) with respect to a special (rational) Seifert class $\widehat{\Sigma}$ is equal to

$$\mathrm{sl}_{\mathbb{Q},new,\widehat{\Sigma}} = \mathrm{sl}_{old} - \frac{1}{d}\sum_{i=1}^{k} a_i q_i (l_i \mp \mathrm{rot}_i).$$

Remark 4.5.9 (Solutions over $\mathbb{Q}$)
Observe that the formulas for rationally nullhomologous knots coincide with the ones for nullhomologous knots presented in previous sections if one allows rational coefficients.

4.6 Extensions to general manifolds

One can also study the same problem for a surgery in a general contact manifold (not in (S^3,ξ_{st})). This is motivated by and similar to [11, Lemma 6.4].

Consider now an oriented nullhomologous link $L = L_1 \sqcup \cdots \sqcup L_n \subset N^3$ in some contact 3-manifold (N,ξ_N). Denote by (M,ξ) the result of some contact surgery along L and by $L_0 \subset (N \setminus \mathring{\nu} L, \xi_N)$ an oriented nullhomologous Legendrian knot in (N,ξ_N) or (M,ξ), depending on the context. Then one gets exactly the same formula for tb as before.

The same is true if $L = L_1 \sqcup \cdots \sqcup L_n \subset N^3$ is an oriented nullhomologous Legendrian link in some contact 3-manifold (N,ξ_N). Denote by (M,ξ) the result of contact $(1/q)$-surgery along L and by $L_0 \subset (N \setminus \mathring{\nu} L, \xi_N)$ an oriented nullhomologous Legendrian or transverse knot in (N,ξ_N) or (M,ξ), depending on the context. Then one gets exactly the same formula for rot and sl as before.

The only parts changing in the proofs are that the homologies are different, for example

$$H_1(M;\mathbb{Z}) = H_1(N;\mathbb{Z}) \oplus \mathbb{Z}_{\mu_1} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i\mu_i + q_i \sum_{j\neq i} l_{ij}\mu_j = 0 \,|\, i = 1,\ldots,n\rangle.$$

For more details see [11, Proof of Lemma 6.4].

If one has a surgery diagram with 1-handles included, then one can use the methods above as well. The first possibility is to change all 1-handles into topological 0-surgeries along unknots (observe that this is also possible for contact surgery along Legendrian knots, see [15]), and the second possibility is to think of the surgery diagram as a surgery diagram in $(\#_n S^1 \times S^2, \xi_{st})$ (represented by n 1-handles) and then use the extensions above.

4.7 Presenting knots in surgery diagrams

In the previous sections I explained how to study knots in a surgered manifold presented as knots in the exterior of the surgery link. In this section I want to explain that this is no restriction, i.e. that every knot in a given manifold can be presented like this.

The purely topological case is easy. Let L be a link in S^3 such that a Dehn surgery along L yields M. By pushing K out of the new glued-in solid tori in M one can assume that K lies completely in the exterior of L in S^3.

This works similar for Legendrian knots in surgery diagrams.

Lemma 4.7.1 (Presenting Legendrian knots in contact surgery diagrams)
Let K be an arbitrary Legendrian knot in a general contact manifold (M, ξ). Then there exists a surgery diagram L in (S^3, ξ_{st}) such that K can be presented as a knot in the exterior $(S^3 \setminus \nu L, \xi_{st})$.

Proof.
By the Ding–Geiges theorem 3.2.8 there exists a contact (± 1)-surgery diagram of (M, ξ). Then by the classification result of Honda [46] for tight contact structures on $S^1 \times D^2$ all new glued-in solid tori are standard neighborhoods of Legendrian knots L_i in (M, ξ) and by the cancellation lemma 3.2.2 contact (∓ 1)-surgeries along the Legendrian knots L_i in (M, ξ) reproduces (S^3, ξ_{st}).

Since the standard neighborhoods of the Legendrian knots L_i, used to construct (S^3, ξ_{st}), can be chosen arbitrary small it is enough to show that an arbitrary Legendrian knot K in an arbitrary contact manifold (M, ξ) can made disjoint (by a Legendrian isotopy) of an arbitrary Legendrian link $L = L_1 \sqcup \cdots \sqcup L_n$.

As in the proof of the approximation theorem 2.2.14 it is by the theorem of Darboux 2.1.4 sufficient to show the same statement for Legendrian knot segments in $(\mathbb{R}^3, \xi_{st})$.

For this consider the front projection of the Legendrian knot segment of K and the Legendrian link segments L_i. By the transversality theorem K can be C^∞-close approximated by a curve that is transverse to all L_i and represents a Legendrian knot segment in $(\mathbb{R}^3, \xi_{st})$, which is in $(\mathbb{R}^3, \xi_{st})$ disjoint from the L_i. $\square$

This proof shows a bit more. Namely, that one can represent any Legendrian knot K in a contact manifold (M, ξ) in every contact (± 1)-surgery diagram of (M, ξ). The same argument works for a contact surgery diagram with only $(\pm 1/n)$-contact surgeries. But in contrast to the topological case in surgery diagrams with general surgery coefficients this is in general not possible, as one can see in the following example.

Example 4.7.2 (A contact surgery diagram with non-unique contact structure) Consider contact Dehn surgery along a Legendrian unknot with $\mathrm{tb} = -2$ and surgery coefficient $+3$ (see Figure 4.7 on the left). Then by the transformation lemma 3.2.4 there are two different tight extensions over the new glued-in solid torus, represented in Figure 4.7 on the right.

By changing these two surgery diagrams into open book decompositions it is easy to show that one of them represents the tight contact structure on S^3 and the other an overtwisted contact structure on S^3.

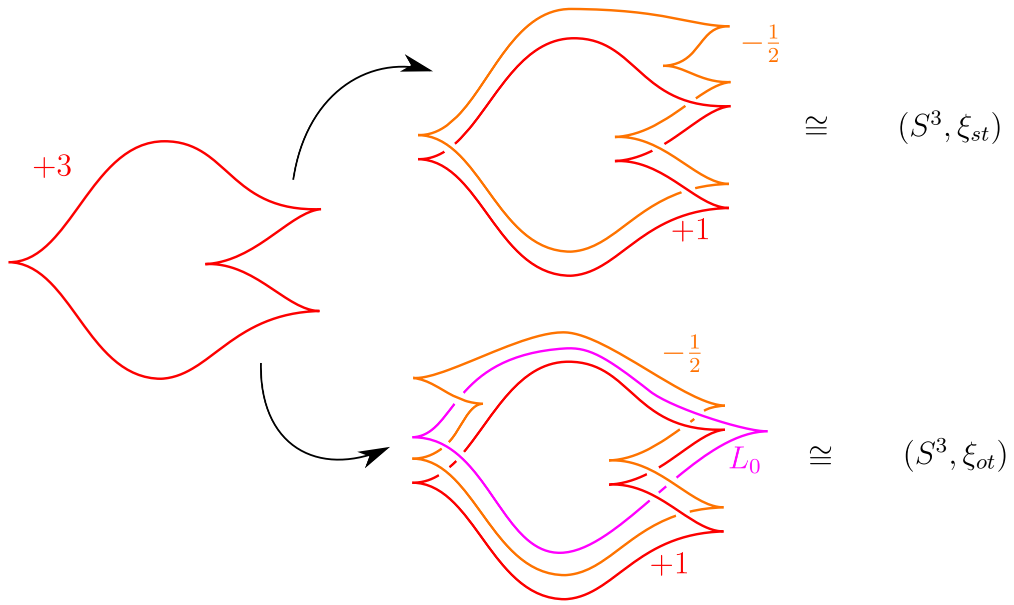


Figure 4.7: A contact surgery with non-unique contact structure

Since there exists an extension over the solid torus such that the whole surgered manifold is tight it follows from Lemma 4.2.1 that every Legendrian knot in the new surgered manifold represented in a contact surgery diagram fulfills the Bennequin inequality $\mathrm{tb} \leq 2g - 1$.

On the other hand in every overtwisted contact manifold there exist Legendrian knots violating the Bennequin inequality. Since the Thurston–Bennequin invariant of a Legendrian knot represented in a surgery diagram depends only on the topological surgery diagram (as explained in Section 4.2), such a knot cannot be realized in the surgery diagram of Figure 4.7 on the left. (However, in Figure 4.7 on the right this is possible by the argument before.)

For example the purple Legendrian knot L_0 in the contact surgery diagram of (S^3, ξ_{ot}) in Figure 4.7 represents a tb-0-unknot in the new manifold (as can be seen by the methods developed before), but one cannot represent this knot in the other two surgery diagrams in Figure 4.7.

By describing a transverse knot as the transverse push-off of a Legendrian knot the same results also hold for transverse knots presented in the exterior of a Legendrian contact surgery diagram. But in contrast to this it is in general **not** possible to present a Legendrian knot in the exterior of a transverse surgery link. This is explained in Remark 6.1.2.

4.8 Computing the d_3-invariant

The so-called d_3-invariant is a homotopical invariant of a tangential 2-plane field on a 3-manifold which is defined if the Euler class (or first Chern class) of the 2-plane field is torsion, see [41, Definition 11.3.3]. Many contact structures can be distinguished by computing the d_3-invariants of the underlying topological 2-plane fields. In [17, Corollary 3.6] Ding, Geiges and Stipsicz, building on work of Gompf [40], present a formula to compute first the Euler class and then the d_3-invariant of a contact structure given by a (± 1)-contact surgery diagram. Both invariants are closely related to the rotation numbers of the surgery links.

By expressing an arbitrary $(1/n)$-contact surgery diagram as a (± 1)-contact surgery diagram and then using the result of Ding–Geiges–Stipsicz, we obtain a similar result for arbitrary $(1/n)$-contact surgery diagrams, which often simplifies computations a lot.

First we recall some results from [17]: For $L = L_1 \sqcup \cdots \sqcup L_k$ an oriented Legendrian link in (S^3, ξ_{st}) and (M, ξ) the contact manifold obtained from S^3 by contact (± 1)-surgeries along L, the Poincaré dual of the Euler class is given by

$$\mathrm{PD}\Big(\mathrm{e}(\xi)\Big) = \sum_{i=1}^{k} \mathrm{rot}_i \, \mu_i \in H_1(M).$$

The meridians μ_i generate the first homology $H_1(M)$, and the relations are given by $Q\mu = 0$. Observe that the generalized linking matrix Q coincides with the ordinary linking matrix since we only have integer surgeries here. Then $\mathrm{e}(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b} \in \mathbb{Q}^k$ of $Q\mathbf{b} = \mathbf{rot}$. If this is the case, the d_3-invariant computes as

$$d_3 = \frac{1}{4}\Big(\langle \mathbf{b}, \mathrm{rot}\rangle - 3\sigma(Q) - 2k\Big) - \frac{1}{2} + q$$

where $\sigma(Q)$ denotes the signature of Q (i.e. the number of positive eigenvalues minus the number of negative ones), and q is the number of Legendrian knots in L with $(+1)$-contact surgery coefficient.

With the help of these results we can now state and prove a corresponding theorem for arbitrary $(1/n)$-contact surgeries.

Lemma 4.8.1 (Computing the d_3-invariant)
Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3, ξ_{st}) and denote by (M, ξ) the contact manifold obtained from S^3 by contact $(\pm 1/n_i)$-surgeries along L ($n_i \in \mathbb{N}$).

1. *The Poincaré dual of the Euler class is given by*
$$\mathrm{PD}\left(e(\xi)\right) = \sum_{i=1}^{k} n_i \operatorname{rot}_i \mu_i \in H_1(M).$$
The first homology group $H_1(M)$ of M is generated by the meridians μ_i and the relations are given in terms of the generalized linking matrix Q by $Q\mu = 0$.

2. *The Euler class $e(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b} \in \mathbb{Q}^k$ of $Q\mathbf{b} = \mathbf{rot}$. In this case, the d_3-invariant computes as*
$$d_3 = \frac{1}{4}\left(\sum_{i=1}^{k} n_i b_i \operatorname{rot}_i + (3 - n_i)\operatorname{sign}_i\right) - \frac{3}{4}\sigma(Q) - \frac{1}{2}$$
where sign_i denotes the sign of the contact surgery coefficient of L_i.

Remark 4.8.2 (The signature of the generalized linking matrix)
In the proof we will show that all eigenvalues of Q are real. Therefore, it makes sense to speak of the signature, even if Q is not symmetric.

Proof.
The replacement lemma 3.2.3 states that a contact $(\pm 1/n)$-surgery along a Legendrian knot L is equivalent to n contact (± 1)-surgeries along Legendrian push-offs of L. Using this, we translate the contact $(\pm 1/n_i)$-surgeries along L in contact (± 1)-surgeries along a new Legendrian link L' and compute the invariants there.

Denote by L_i^j $(j = 1, \ldots, n_i)$ the Legendrian push-offs of L_i in the new Legendrian link L'. Write μ_i for the meridian of L_i $(i = 1, \ldots, k)$ and μ_i^j for the meridian of L_i^j $(i = 1, \ldots, k;\ j = 1, \ldots, n_i)$. We now have two surgery descriptions of the manifold M – one in terms of L and one in terms of L' – and hence two presentations of its first homology:

$$\begin{aligned} H_1(M) &= \langle \mu_i \mid Q\mu = 0\rangle \text{ for the surgery presentation along } L,\\ H_1(M) &= \langle \mu_i^j \mid Q'\mu' = 0\rangle \text{ for the surgery presentation along } L'. \end{aligned}$$

An isomorphism between these two presentations is given by $\mu_i^j \mapsto \mu_i$ for all i, j, and hence, as $\operatorname{rot}_i^j = \operatorname{rot}_i$,
$$\mathrm{PD}\left(\mathrm{e}(\xi)\right) = \sum_{i=1}^{k}\sum_{j=1}^{n_j} \operatorname{rot}_i^j \mu_i^j \longmapsto \sum_{i=1}^{k} n_i \operatorname{rot}_i \mu_i.$$

And part (1) of the lemma follows.

The numbers k and q compute easily as

$$k = \sum_{i=1}^{k} n_i,$$
$$q = \sum_{i=1}^{k} \frac{1}{2} (1 + \operatorname{sign}_i) n_i.$$

For reasons of readability we will assume $k = 3$ in the following. The general case works exactly the same. Write $\mathbf{1}_n$ for the vector $(1, \dots, 1)^T \in \mathbb{Q}^n$.

Let $\mathbf{b} \in \mathbb{Q}^3$ a solution of $Q\mathbf{b} = \mathbf{rot}$, i.e

$$Q\mathbf{b} = \begin{pmatrix} \pm 1 + n_1 \operatorname{tb}_1 & n_2 l_{12} & n_3 l_{13} \\ n_1 l_{12} & \pm 1 + n_2 \operatorname{tb}_2 & n_3 l_{23} \\ n_1 l_{13} & n_2 l_{23} & \pm 1 + n_3 \operatorname{tb}_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \operatorname{rot}_1 \\ \operatorname{rot}_2 \\ \operatorname{rot}_3 \end{pmatrix} = \mathbf{rot}$$

Then for $\mathbf{b}' := (b_1, \dots, b_1, b_2, \dots, b_2, b_3, \dots, b_3)^T \in \mathbb{Q}^{n_1+n_2+n_3}$ we have

$$\begin{aligned} Q'\mathbf{b}' &= \begin{pmatrix} \pm E_{n_1} + \operatorname{tb}_1 \mathbf{1}_{n_1} \mathbf{1}_{n_1}^T & l_{12} \mathbf{1}_{n_1} \mathbf{1}_{n_2}^T & l_{13} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T \\ l_{12} \mathbf{1}_{n_2} \mathbf{1}_{n_1}^T & \pm E_{n_1} + \operatorname{tb}_2 \mathbf{1}_{n_2} \mathbf{1}_{n_2}^T & l_{23} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T \\ l_{13} \mathbf{1}_{n_3} \mathbf{1}_{n_1}^T & l_{23} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T & \pm E_{n_3} + \operatorname{tb}_3 \mathbf{1}_{n_3} \mathbf{1}_{n_3}^T \end{pmatrix} \mathbf{b}' \\ &= \begin{pmatrix} \operatorname{rot}_1 \mathbf{1}_{n_1} \\ \operatorname{rot}_2 \mathbf{1}_{n_2} \\ \operatorname{rot}_3 \mathbf{1}_{n_3} \end{pmatrix} = \mathbf{rot}'. \end{aligned}$$

And therefore,

$$\langle \mathbf{b}', \mathbf{rot}' \rangle = \sum_{i=1}^{3} n_i b_i \operatorname{rot}_i .$$

It remains to compute the signature $\sigma(Q')$ from $\sigma(Q)$. Let λ be an eigenvalue of Q with eigenvector $\mathbf{v}$. Similarly to above, one computes

$$Q'\mathbf{v}' = \lambda \mathbf{v}'$$

for $\mathbf{v}' := (v_1, \dots, v_1, v_2, \dots, v_2, v_3, \dots, v_3)^T \in \mathbb{Q}^{n_1+n_2+n_3}$. Thus, every eigenvalue of Q is also an eigenvalue of Q'. In particular, all eigenvalues of Q are real. Now we only have to find the other $\sum_{i=1}^{3}(n_i - 1)$ eigenvectors of Q'. To that end, consider the vector $\mathbf{v}_1 \in \mathbf{1}_{n_1}^{\perp}$ and write $\mathbf{v}_1' := (\mathbf{v}_1, 0, \dots, 0, 0, \dots, 0)^T \in \mathbb{Q}^{n_1+n_2+n_3}$. Then, as before, one computes

$$Q'\mathbf{v}_1' = \operatorname{sign}_1 \mathbf{v}_1',$$

and therefore

$$\sigma(Q') = \sigma(Q) + \sum_{i=1}^{k} (n_i - 1) \operatorname{sign}_i . \qquad \square$$

With this formula for the d_3-invariant it is often easy to distinguish contact manifolds given as surgery diagrams.

Example 4.8.3 (Computing the d_3-invariants for contact 3-spheres)
(1) Consider the contact $(1/2)$-surgery along the Legendrian unknot with $\operatorname{tb} = -1$ in (S^3, ξ_{st}) from Figure 3.2 resulting in a contact 3-sphere (S^3, ξ_1). By computing the d_3-invariant one can see again that the resulting contact structure is overtwisted. In this case Q is given by -1, so the d_3-invariant computes as

$$d_3(S^3, \xi_1) = \frac{1}{4}(3-2) + \frac{3}{4} - \frac{1}{2} = 1 - \frac{1}{2}.$$

On the other hand the d_3-invariant of (S^3, ξ_{st}) can be computed by looking at the empty diagram or at the contact $(+2)$-surgery along the Legendrian unknot with $\operatorname{tb} = -1$ (see Example 3.2.7(6)) as

$$d_3(S^3, \xi_{st}) = -\frac{1}{2}.$$

Therefore, the contact manifold (S^3, ξ_1) is different from (S^3, ξ_{st}), and since ξ_{st} is the only tight contact structure on S^3, the contact structure ξ_1 has to be overtwisted.
(2) Consider the contact surgery in Figure 4.8 along a Legendrian unknot with $\operatorname{tb} = -3$ and $\operatorname{rot} = -2$. Topologically this surgery corresponds to a (-1)-surgery and therefore represents S^3. Using the transformation lemma 3.2.4 one gets two different surgery diagrams with only (± 1)-surgery coefficients, both representing contact structures on S^3. By computing the d_3-invariants one can distinguish these contact structures.

In both cases one gets

$$Q = \begin{pmatrix} -2 & -3 \\ -3 & -4 \end{pmatrix}.$$

The determinant of Q is equal to 1 and the trace is -10. Since the determinant is the product and the trace the sum of the eigenvalues, the signature $\sigma(Q)$ has to be -2. So the formula for the d_3-invariant simplifies to

$$d_3 = \frac{1}{4}\Big(b_1 \operatorname{rot}_1 + b_2 \operatorname{rot}_2 + (3-1) - (3-1)\Big) + \frac{6}{4} - \frac{1}{2} = \frac{1}{4}(b_1 \operatorname{rot}_1 + b_2 \operatorname{rot}_2 + 6) - \frac{1}{2}.$$

By solving the equation $Q\mathbf{b} = \mathbf{rot}$ one gets

$$\begin{aligned} b_1 &= -5 \operatorname{rot}_1 + 3 \operatorname{rot}_2 \\ b_2 &= 3 \operatorname{rot}_1 - 2 \operatorname{rot}_2 . \end{aligned}$$

For both surgery diagrams one has $\operatorname{rot}_1 = -2$. But rot_2 is in the first case -3 and in the second case -1. Therefore one gets two different values for the d_3-invariants:

$$d_3 = \begin{cases} \frac{1}{4}(-2+6) - \frac{1}{2} = 1 - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = -3 \\ \frac{1}{4}(-14+4+6) - \frac{1}{2} = -1 - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = -1 \end{cases}$$

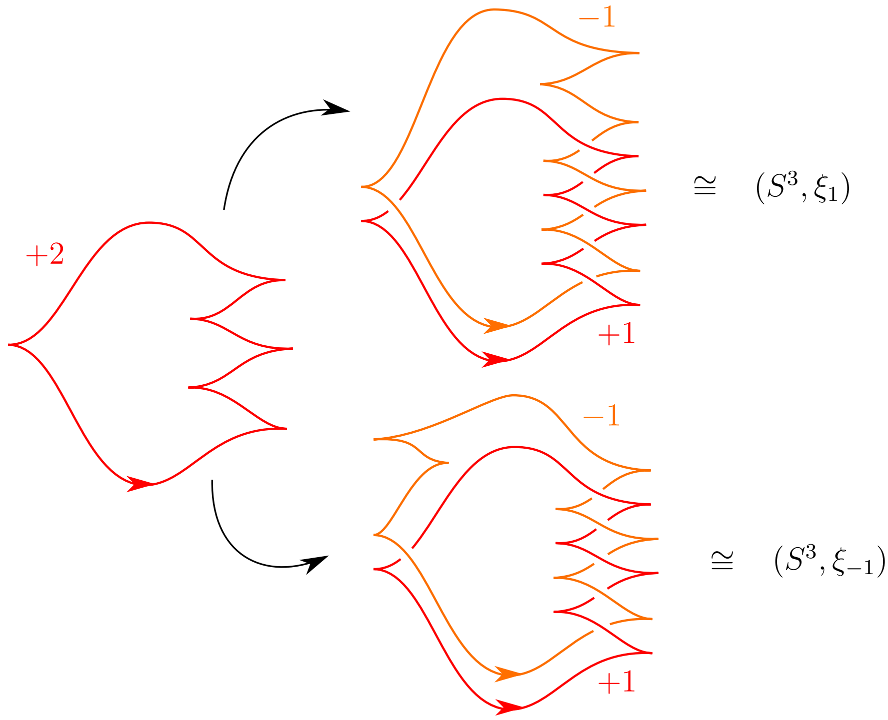


Figure 4.8: Surgery diagrams for (S^3, ξ_1) and (S^3, ξ_{-1})

Remark 4.8.4 (Surgery diagrams of all contact structures on S^3)
(1) Let K_1 and K_2 be two Legendrian knots in S^3 divided by an embedded 2-sphere. Then this 2-sphere is an embedded 2-sphere in the new surgered manifold which divides the manifold into two parts. Therefore, the result of contact r_1-surgery along K_1 and contact r_2-surgery along K_2 is contactomorphic to the contact connected sum $(S^3_{K_1}(r_1)\#S^3_{K_2}(r_2), \xi_{K_1}(r_1)\#\xi_{K_2}(r_2))$.

If $d_3(S^3_{K_i}(r_i), \xi_{K_i}(r_i)) = d_i - 1/2$, the formula for computing d_3 yields that the d_3-invariant for the contact connected sum is given by $d_3 = d_1 + d_2 - 1/2$.
(2) By taking several connected sums of the contact structures (S^3, ξ_1) from Example 4.8.3(1) and (S^3, ξ_{-1}) from Example 4.8.3(2) one gets overtwisted contact structures ξ_n on S^3 with d_3-invariant equal to $n - 1/2$ for all $n \in \mathbb{Z}$.
(3) Moreover, it is easy to show that for any tangential 2-plane field on S^3 the quantity $d_3 - 1/2$ is an integer (see for example [67, Page 199]). By the classification theorem for overtwisted contact structures of Eliashberg 2.4.4 the contact structures ξ_n represent all overtwisted contact structures on S^3. In particular, we constructed contact surgery diagrams for all overtwisted contact structures on S^3. This was the last part missing in the proof of the theorem of Ding–Geiges 3.2.8.
(4) Note also that the two contact structures from Example 4.8.3(1) and (2) both denoted by ξ_1 are really the same. It is also easy to compute that the d_3-invariant of the contact 3-sphere from Example 3.2.7(4) is $1 - 1/2$, too. Therefore, this contact

structure is also contactomorphic to ξ_1.

Example 4.8.5 (Computing the Euler class of contact structures)
In manifolds with $H_1 \neq 0$ the Euler class is often also a very good invariant to distinguish contact structures. In [46] for example it is proven that the Euler class is a complete invariant for tight contact structures on a given lens space.

Consider for example the tight contact structure ξ on the lens space $L(5,2)$ from Example 3.2.7(2). Since there $\operatorname{rot} = \pm 1$ and $Q = -5$, the Euler class computes as

$$\operatorname{PD}\left(\mathrm{e}(\xi)\right) = \operatorname{rot}\mu = \pm\mu \in H_1(M) = \langle \mu \mid -5\mu \rangle \cong \mathbb{Z}_5.$$

5

The Legendrian knot complement problem

In this chapter the Legendrian link exterior problem is discussed. In Section 5.1 I will prove the Legendrian knot exterior theorem for knots in (S^3, ξ_{st}). This is done very similarly to the topological case by first generalizing the surgery theorem of Gordon–Luecke.

Then in Section 5.2 I want to present counterexamples in the case of Legendrian links in (S^3, ξ_{st}). For doing this I will develop a move for contact surgery diagrams along Legendrian knots corresponding to a topological (-1)-Rolfsen twist.

The Legendrian knot exterior problem is also not true for Legendrian knots in general contact manifolds. Counterexamples in lens spaces are presented in Section 5.3. Also the relation to exotic cosmetic contact surgeries is discussed. But it will turn out that, in contrast to the topological case, the non-existence of these is not equivalent to the Legendrian knot exterior problem.

However, cosmetic contact surgeries are interesting on their own. In Section 5.4 I present examples of exotic cosmetic contact surgeries resulting in lens spaces, and finally I will classify all surgeries along a Legendrian knot in (S^3, ξ_{st}) resulting again in a contact S^3. In particular there exist infinitely many contact surgeries resulting again in (S^3, ξ_{st}).

Sections 5.1, 5.2 and 5.3 are based on and very similar to my paper [54].

5.1 The Legendrian knot exterior theorem

In this section we want to give a proof of the Legendrian knot exterior theorem 2.2.10, namely that two Legendrian knots in (S^3, ξ_{st}) are equivalent if and only if their exteriors are contactomorphic. To prove this we generalize the surgery theorem by Gordon–Luecke 1.2.4.

For stating this result we introduce the Legendrian knots U_n as Legendrian unknots with classical invariants $\operatorname{tb}(U_n) = -n$ and $\operatorname{rot}(U_n) = |n-1|$.

The classification of Legendrian unknots in (S^3, ξ_{st}) by Eliashberg–Fraser (Theorem 2.3.10) says that two Legendrian unknots are equivalent (as oriented knots) if and only if they have the same tb and rot where the orientation of the knot is given by the sign of rot.

Here we want to prove that two Legendrian knots are equivalent if and only if their exteriors are contactomorphic. Since a knot exterior cannot determine the orientation of the knot this result can only hold for equivalence as unoriented knots. This is the reason why we consider Legendrian knots up to equivalence as unoriented knots.

From the theorem of Eliashberg–Fraser 2.3.10 it follows that the knots U_n are unique up to equivalence (of unoriented Legendrian knots). A front projection of a Legendrian unknot of type U_n is shown in Figure 5.1.

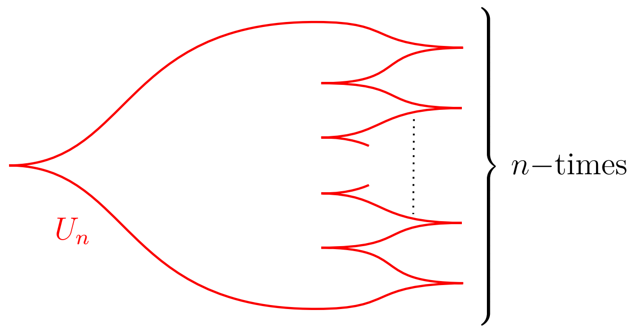


Figure 5.1: A front projection of a Legendrian unknot of type U_n

The generalization of the surgery theorem by Gordon–Luecke 1.2.4 is as follows.

Theorem 5.1.1 (Contact Dehn surgery theorem)
Let K be a Legendrian knot in (S^3, ξ_{st}). If $(S^3_K(r), \xi_K(r))$ is contactomorphic to (S^3, ξ_{st}) for some $r \neq \mu$, then K is equivalent to a Legendrian unknot U_n with $\mathrm{tb}(U_n) = -n$ *and* $\mathrm{rot}(U_n) = |n-1|$.

Remark 5.1.2 (Non-uniqueness of the contact structure)
For general slopes the contact structure $\xi_K(r)$ is not unique. So one should read Theorem 5.1.1 as follows: If there is a contact structure $\xi_K(r)$ on $S^3_K(r)$ such that $(S^3_K(r), \xi_K(r))$ is obtained from (M, ξ) by contact Dehn surgery along K with slope r and if $(S^3_K(r), \xi_K(r))$ is contactomorphic to (S^3, ξ_{st}) for $r \neq \mu$, then the conclusion holds.

Proof of the contact Dehn surgery theorem 5.1.1.
Let K be a Legendrian knot in (S^3, ξ_{st}) such that $S^3_K(r)$ is diffeomorphic to S^3 for some $r \neq \mu$. From the (topological) surgery theorem 1.2.4 it follows that K is topologically equivalent to an unknot U. (So I will write U instead of K.) In Remark 1.2.3 I explained that the topological surgery coefficient (with respect to the Seifert longitude λ_S) has to be of the form $1/n$, for $n \in \mathbb{Z}$.

Now I want to show that every resulting contact structure $\xi_U(r)$ is overtwisted if U is not coarse equivalent to an unknot of the form U_n. To do this I will show that in the resulting contact 3-spheres there exist Legendrian knots that cannot be realized in ξ_{st}.

For this consider Examples 4.2.2(1) and (2). By doing a Rolfsen twist along U one sees that the Legendrian knot L_0 from the first example remains an unknot in the new surgered manifold. For $n < 0$ one gets $\mathrm{tb}_{new} = -n - 1 > 0$ (independent of the choice of the contact structure on the new glued-in solid torus).

According to the Bennequin inequality this knot cannot lie in a tight contact structure. So for $n < 0$ it is not possible to have a non-trivial contact surgery from (S^3, ξ_{st}) to itself.

For $n > 0$ one looks at Example 4.2.2(2). In Figure 5.2 it is shown (again by doing a Rolfsen twist) that L_0 becomes a negative $(2, 2n+1)$-torus knot $T_{2,2n+1}$ in the new surgered manifold.

In [29, Section 1] it is shown that the maximal Thurston–Bennequin invariant of such a knot in (S^3, ξ_{st}) is given by $-2 - 4n$, which is smaller than $\mathrm{tb}_{new} = -1 - 4n$. For U not coarse equivalent to an Legendrian unknot of the form U_n this example can be realized as a contact surgery along a Legendrian knot, which proves the result. $\square$

The proof of the Legendrian knot exterior theorem 2.2.10 is now similar to the topological case.

Proof of the Legendrian knot exterior theorem 2.2.10.
Pick a contactomorphism

$$h\colon \big(S^3 \setminus \nu\mathring{K}_1, \xi_{st}\big) \longrightarrow \big(S^3 \setminus \nu\mathring{K}_2, \xi_{st}\big).$$

Then consider again the following diagram:

$$\begin{tikzcd}
\big(S^3,\xi_{st}\big) \cong \big(S^3_{K_1}(\mu_1),\xi_{K_1}(\mu_1)\big) \arrow[ddd, dashed, "f"'] & {:=}\big(S^1\times D^2,\xi'\big) & + & \big(S^3\setminus\nu\mathring{K}_1,\xi_{st}\big)\big/_{\sim} \\
& \mu_0 \arrow[rr, mapsto, "\varphi_1"'] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[drr, phantom, "\circlearrowright"] & & \mu_1 \arrow[d, bend left, "h"] \\
& \mu_0 \arrow[rr, mapsto, "h\circ\varphi_1"] & & r_2 := h(\mu_1) \\
\big(S^3_{K_2}(r_2),\xi_{K_2}(r_2)\big) & {:=}\big(S^1\times D^2,\xi'\big) & + & \big(S^3\setminus\nu\mathring{K}_2,\xi_{st}\big)\big/_{\sim}
\end{tikzcd}$$

Here the contact structure ξ' denotes the unique tight contact structure on $S^1 \times D^2$ with convex boundary corresponding to the slope μ_1. Because the contactomorphisms Id and h on the two factors agree on the boundary convex surfaces which determine the germ of the contact structures, these two maps glue together to a contactomorphism f of the whole contact manifolds. From the contact Dehn surgery theorem 5.1.1 it follows that r_2 is equal to μ_2, or K_2 is equivalent to U_n for some n.

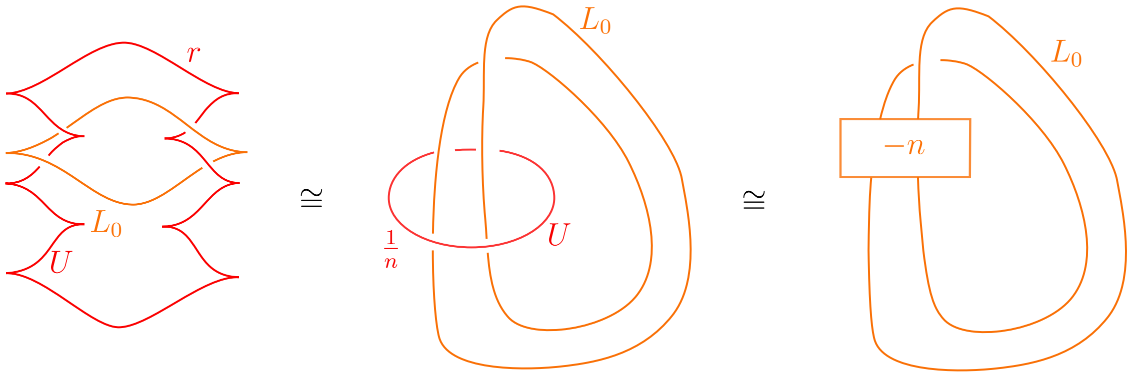


Figure 5.2: The unknot L_0 becomes a $(2, 2n+1)$-torus knot after surgery

If $r_2 = \mu_2$ then this is the trivial contact Dehn surgery, and so the contactomorphism f maps K_1 to K_2.

In the other case one makes the same argument with K_1 and K_2 reversed and concludes that K_1 is equivalent to U_m for some m. Again the classification result of Eliashberg–Fraser (Theorem 2.3.10) implies that $K_1 \sim U_m$ is equivalent to $U_n \sim K_2$ if and only if $n = m$. To show the last statement one observes that $(S^3 \setminus \nu \mathring{K}_1, \xi_{st})$ is a solid torus with tight contact structure and convex boundary. Therefore, one can compute $-n = \mathrm{tb}(U_n)$ also as half the number of intersection points of the Seifert disk of U_n with the dividing set of the convex boundary (see [51, Proposition 6.6]). Because the Seifert disks of U_n and U_m are both given by the D^2-factors of the exterior solid tori and because the exteriors are contactomorphic, the number of intersection points stays the same. □

Remark 5.1.3 (Dependence on the classification of Legendrian unknots)
Observe that these theorems (both the contact surgery theorem and the Legendrian knot exterior theorem) use the classification of Legendrian unknots by Eliashberg–Fraser. So for proving this classification result of unknots one cannot use the Legendrian knot exterior theorem. For example, assuming the Legendrian knot exterior theorem there is an easy proof that there exists a unique Legendrian unknot with $\mathrm{tb} = -1$ and $\mathrm{rot} = 0$. Since the knot exterior of such a knot is a tight solid torus with longitudinal dividing curves, the contact structure on the knot exterior is unique [46] and therefore the corresponding Legendrian knot is by the Legendrian knot exterior theorem also unique.

It would be nice to find an argument for proving the Legendrian knot exterior theorem without using the classification of Legendrian unknots.

Observe also that the original proof of Eliashberg–Fraser [26], as well as the proofs by Etnyre [27] and Etnyre–Honda [29] do not use the Legendrian knot exterior theorem.

5.2 The Legendrian link complement problem

The next natural question is if there exist (like in the topological case) Legendrian links not determined by the contactomorphism type of their exteriors. If one wants to generalize the examples of Whitehead one needs a contact analogue of a Rolfsen twist.

In Example 3.2.7 I gave a contact surgery from (S^3, ξ_{st}) to (S^3, ξ_{st}). Topologically this surgery represents a $(+1)$-Dehn surgery along an unknot. Deleting such components from contact surgery diagrams leads to a contact analogue of a (-1)-Rolfsen twist. But it is not clear how the rest of the diagram changes then.

The next lemma answers this question and will also be used in Chapter 7 to realize crossing changes of Legendrian and transverse knots.

Lemma 5.2.1 (A contact Rolfsen twist)
Figure 5.3 shows two moves for contact Dehn surgery diagrams. There, L is an arbitrary Legendrian link with surgery coefficients ± 1, that can go arbitrarily often through the unknot.

Remark 5.2.2 (Doing n-fold contact Rolfsen twists)
Of course one can do such a contact Rolfsen twist more than once, this corresponds to doing $(+2)$-contact Dehn surgeries along n disjoint copies of the Legendrian unknot. By translating the contact surgery diagram into an open book as in the following proof one can show that this is the same as doing a single contact Dehn surgery along the unknot with contact surgery coefficient $1 + \frac{1}{n}$.

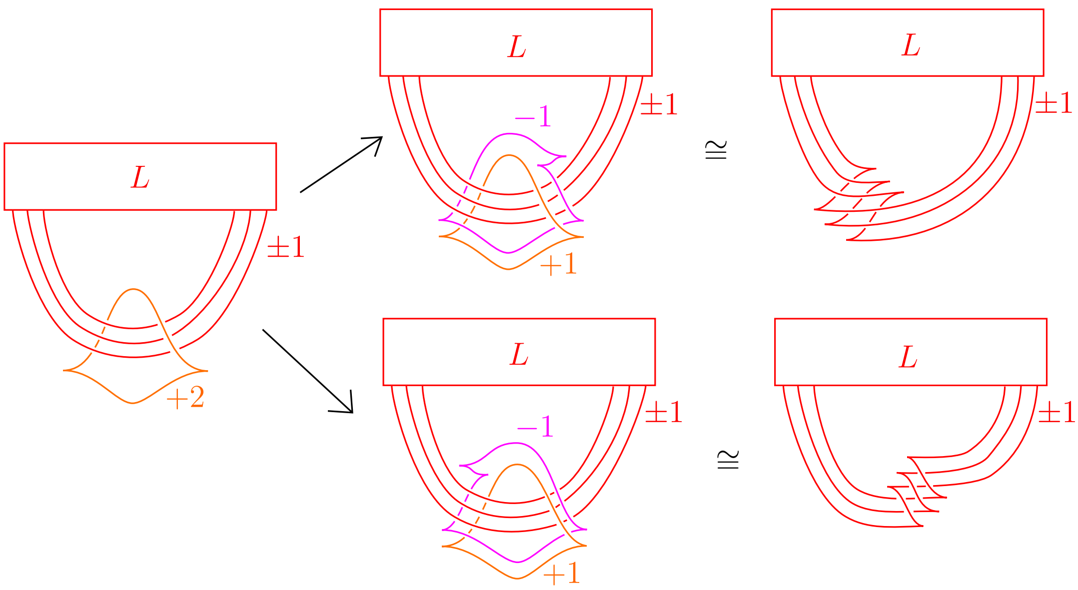


Figure 5.3: A contact Rolfsen twist

Proof of Lemma 5.2.1.
First observe that the first row in Figure 5.3 maps to the second row under the contactomorphism $(x, y, z) \mapsto (-x, -y, z)$. So it is enough to prove the second row.

For this, one observes that one can put the whole Legendrian link L together with the Legendrian unknot U on the pages of an abstract open book for (S^3, ξ_{st}) (see for example [1, Proof of Theorem 5.5]). A part of this abstract open book is shown in Figure 5.4. The page is pictured in gray and a part of the monodromy is described as a right-handed Dehn twist along the blue curve. One has to read the red curve in the open book as that many parallel copies as in the surgery diagram on the left. Observe that the abstract open book determines the contact manifold only up to contactomorphism, as the surgery diagram does.

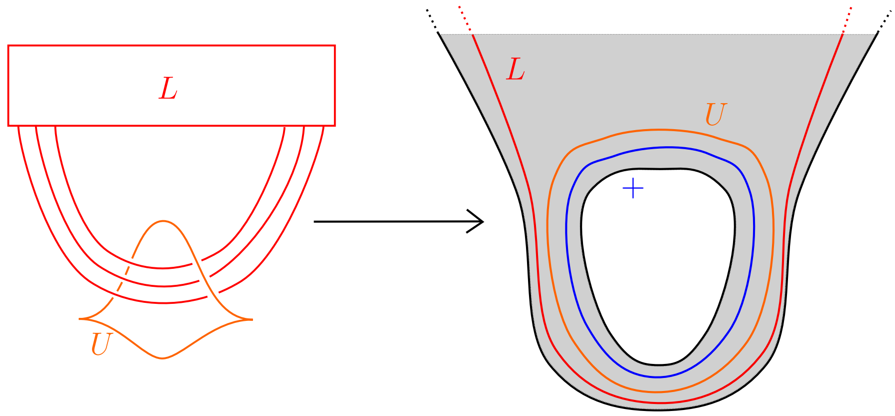


Figure 5.4: Putting L and U on an open book for (S^3, ξ_{st})

Next one constructs from this an abstract open book for the surgered manifold as explained in [66]. First, one puts the Legendrian link L together with the Legendrian unknots U_1 and U_2 from the (± 1)-surgery diagram on the page of an abstract open book for (S^3, ξ_{st}). This is shown in Figure 5.5 on the upper right side. The additional stabilization of the Legendrian unknot U_2 corresponds to a stabilization of the open book as shown in [66, Figures 1 and 2].

The monodromy of the open book for (S^3, ξ_{st}) is given by right-handed Dehn twists along the blue curves. The monodromy of the surgered manifold is obtained by composing this old monodromy with right-handed Dehn twists along the (-1)-surgery knot and left-handed Dehn twists along the $(+1)$-surgery knot (see [66, Proposition 8]).

Two of the Dehn twists cancel each other and one gets the open book in the bottom right corner of Figure 5.5. That open book represents the surgery diagram of the once stabilized Legendrian link L shown in the bottom left corner of Figure 5.5.

Observe also that the topological surgery coefficients of L change exactly as prescribed by the topological Rolfsen twists. $\square$

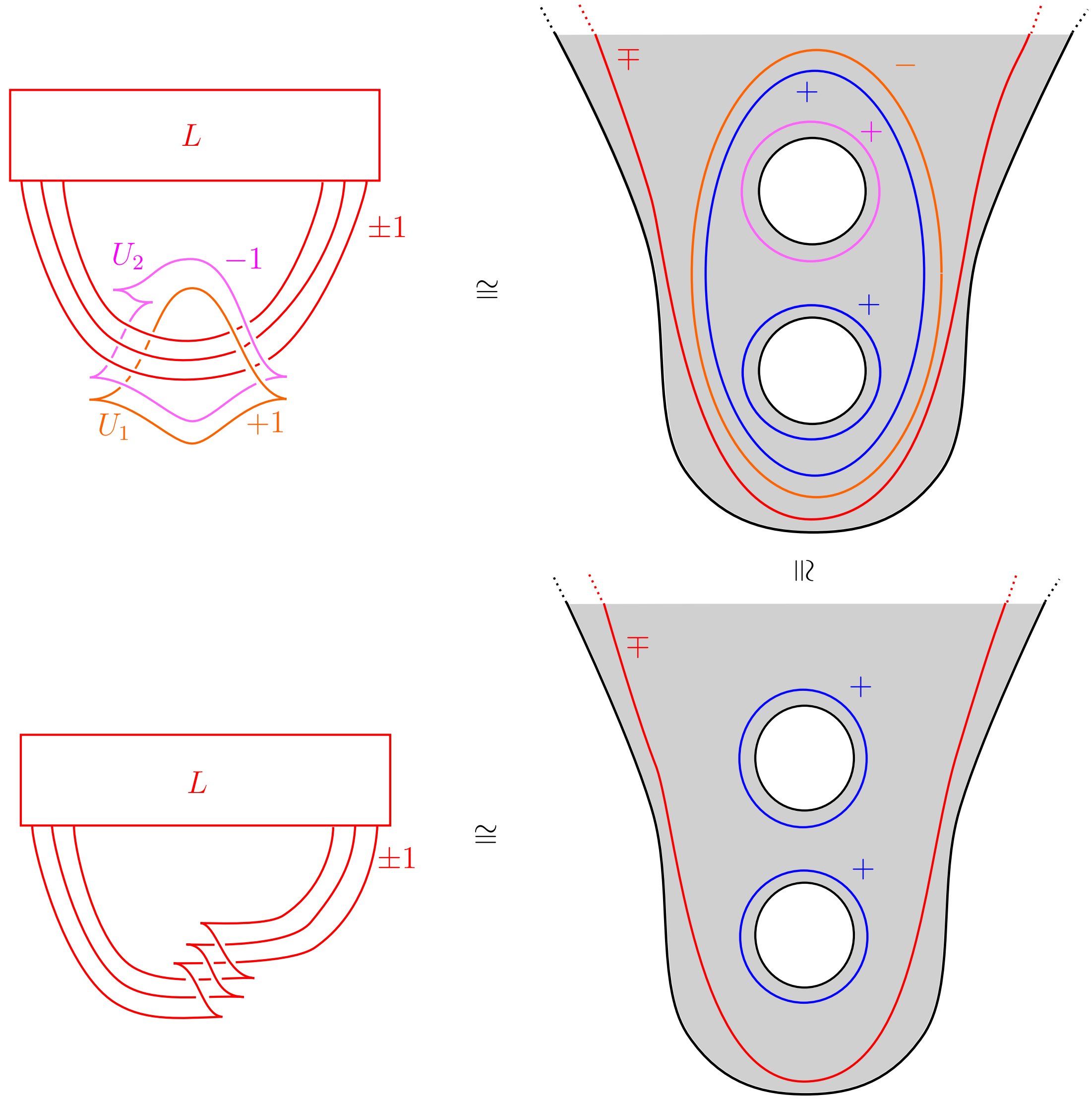


Figure 5.5: Proof of the contact Rolfsen twists via open book decompositions

Example 5.2.3 (Counterexamples to the Legendrian link complement problem) With this contact Rolfsen twist one can give counterexamples to the Legendrian link complement problem, but it is not as easy as in the topological setting. The main point there was that the new glued-in solid tori is again a tubular neighborhood of the spine of this torus. This is not the case in the contact setting.

To see this, consider a Legendrian knot K in a contact 3-manifold (M, ξ) and the result of contact surgery along this knot $(M_K(r), \xi_K(r))$. Then the slope of the new glued-in solid torus is r. But if this new glued-in solid torus were the standard neighborhood of a Legendrian knot then the slope were of the form $\lambda + n\mu$. So for general r this is not the case.

Therefore, one looks at the contact surgery diagram with one $(+1)$-surgery along U_1 and one (-1)-surgery along U_2. The new glued-in solid tori are again standard

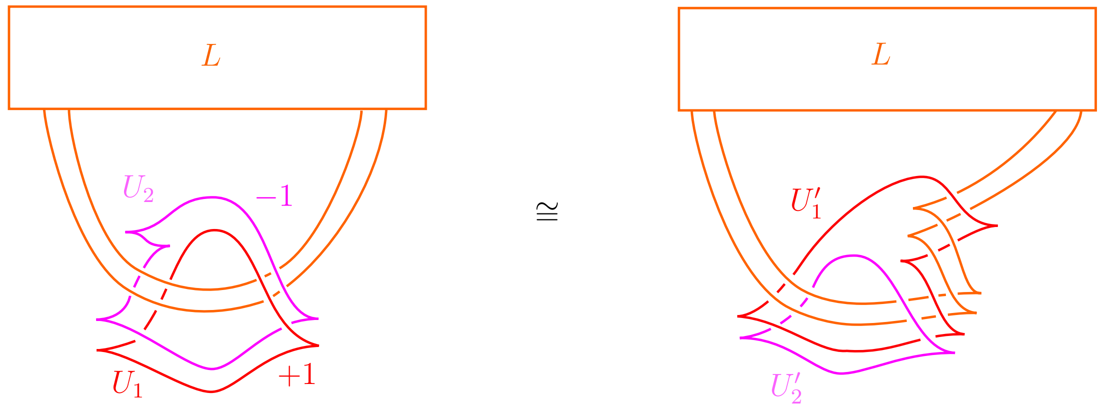


Figure 5.6: The spines of the new glued-in solid tori

neighborhoods of the Legendrian spines U_1' and U_2' in a canonical way. From the proof of the lemma it follows that these spines U_1' and U_2' lie in the resulting surgery diagram as shown in Figure 5.6.

Observe also that by doing a contact $(+1)$-surgery along U_2' and a contact (-1)-surgery along U_1' one gets the old picture back by the cancellation lemma 3.2.2.

(1) Consider the two Legendrian links $L \sqcup U_1 \sqcup U_2$ and $L' \sqcup U_1' \sqcup U_2'$ in (S^3, ξ_{st}) as depicted in Figure 5.7. These two links are not equivalent because their triples of tb-numbers are different: $\mathrm{tb}(L \sqcup U_1 \sqcup U_2) = (-1, -1, -2)$; $\mathrm{tb}(L' \sqcup U_1' \sqcup U_2') = (-2, -2, -1)$.

But their exteriors are contactomorphic, as one can see as follows. One does a $(+1)$-surgery along U_1 and a (-1)-surgery along U_2. The resulting manifold is again (S^3, ξ_{st}) in which the Legendrian knot L now looks like L' in Figure 5.7 on the right. So in the exteriors of $U_1 \sqcup U_2$ and $U_1' \sqcup U_2'$ the Legendrian knots L and L' are the same.

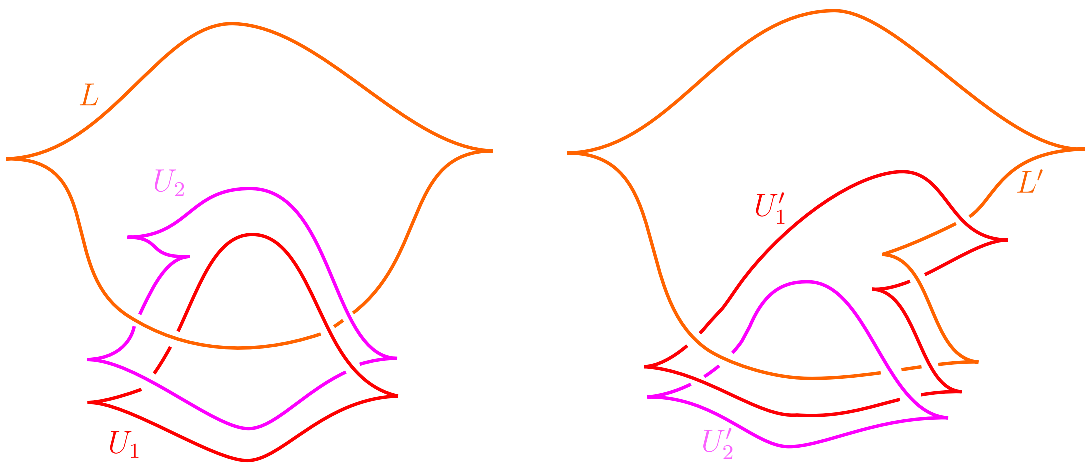


Figure 5.7: Two different Legendrian links with contactomorphic exteriors

(2) One can also get examples with different topological types and the same tb-numbers. For that consider the Legendrian links in Figure 5.8 similar to the Whitehead links as in the example from Section 1.1. In the left Legendrian link in Figure 5.8 all three knots are unknots, but in the right link the knot L' is non-trivial, so they cannot be equivalent. But their exteriors are contactomorphic by the same argument as in the foregoing example.

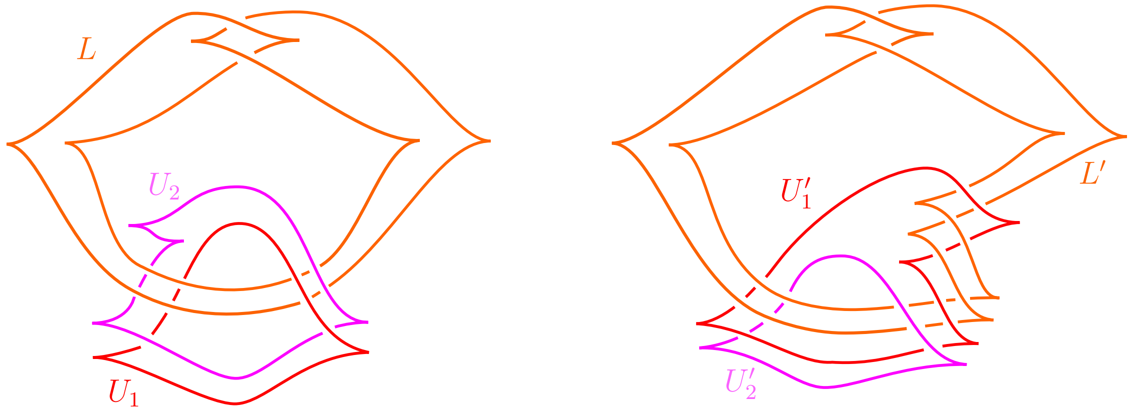


Figure 5.8: Two different Legendrian links with contactomorphic exteriors

(3) By doing contact Dehn surgeries corresponding to an n-fold Rolfsen twist for $n > 0$, one gets in both foregoing examples infinitely many different pairs of Legendrian links such that each pair has contactomorphic exteriors. But it is not clear if there also exist infinitely many different Legendrian links whose exteriors are all contactomorphic to each other.

5.3 The Legendrian knot complement problem in general manifolds

As far as I know, nothing is known about this in the contact setting. The question is which results from the topological setting generalize to the contact setting and where the differences are. In Section 1.3 I presented examples of non-equivalent knots in lens spaces with homeomorphic exteriors. The first interesting question is if one can generalize these examples to the contact setting.

Example 5.3.1 (Non-equivalent Legendrian knots with the same complements) Consider a Legendrian knot K with $\mathrm{tb}(K) = n$. A standard neighborhood of K is given by

$$\Big(S^1 \times D^2, \ker(\cos n\theta\, dx - \sin n\theta\, dy)\Big).$$

The Seifert longitude is given by $\lambda = S^1 \times \{p\}$ and the contact longitude by $\lambda + n\mu$. Take two copies (V_1, ξ_1) and (V_2, ξ_2) of this standard neighborhood and glue them together along their boundaries to obtain the lens space $L(p,q)$ as follows:

$$\begin{array}{rcccc} L(p,q) & = & V_1 & + & V_2 \quad \big/_{\sim} \\ & & \mu_1 & \longmapsto & -q\mu_2 + p\lambda_2, \\ & & \lambda_1 & \longmapsto & r\mu_2 + s\lambda_2, \end{array}$$

where $-qs - pr = 1$. The contact structures on the solid tori fit together to a contact structure on the new lens space $L(p,q)$ if the gluing map sends the contact longitude of V_1 to the contact longitude of V_2. (Since the contact longitude represents the dividing curves of the boundary convex surface.) This leads to the conditions $r - qn = n$ and $s + pn = 1$. Putting this together it follows that $q = -pn - 1$.

In particular, it follows that if a contact lens space is obtained by gluing together two copies of a standard neighborhood of a Legendrian knots then the lens space is of the form $L(p, p-1)$. But in this case the spines K_1 and K_2 of the Heegaard tori are topologically equivalent, as was explained in Section 1.3.

To distinguish the Legendrian knots K_1 and K_2 one can compute their rational Thurston–Bennequin invariants.

Therefore consider the following gluing:

$$\begin{array}{rcccc} L(p,p-1) & = & V_1 & + & V_2 \quad \big/_{\sim} \\ & & \mu_1 & \longmapsto & (1+pn)\mu_2 + p\lambda_2, \\ & & \lambda_1 & \longmapsto & -pn^2\mu_2 + (1-pn)\lambda_2. \end{array}$$

This gluing map sends the contact longitude of V_1 to the contact longitude of V_2 and leads therefore to a contact structure on the lens space. The Legendrian knots K_1 and K_2 represent the spines of the Heegaard tori in this lens space. Therefore they both have order p in $H_1(L(p,p-1);\mathbb{Z})$. The extended Seifert longitude r_2 of K_2 is given by

$$r_2 = (1+pn)\mu_2 + p\lambda_2.$$

(It is nullhomologous in the exterior of K_2 and $\mu_2 \bullet r_2 = p$.) So one computes

$$\operatorname{tb}_{\mathbb{Q}}(K_2) = \frac{1}{p}\big(\lambda_2 + n\mu_2\big) \bullet \big((1+pn)\mu_2 + p\lambda_2\big) = -\frac{1}{p}.$$

To compute $\operatorname{tb}_{\mathbb{Q}}(K_1)$ one first computes the inverse gluing map as:

$$\begin{array}{rcccc} L(p,p-1) & = & V_2 & + & V_1 \quad \big/_{\sim} \\ & & \mu_2 & \longmapsto & (1-pn)\mu_1 - p\lambda_1, \\ & & \lambda_2 & \longmapsto & pn^2\mu_1 + (1+pn)\lambda_1. \end{array}$$

Then one sees that the extended Seifert longitude of K_1 is given by

$$r_1 = (pn-1)\mu_1 + p\lambda_1,$$

and so

$$\operatorname{tb}_{\mathbb{Q}}(K_1) = \frac{1}{p}\Big(\lambda_1 + n\mu_1\Big) \bullet \Big((pn-1)\mu_1 + p\lambda_1\Big) = \frac{1}{p}.$$

So the rational Thurston–Bennequin invariants are different, therefore K_1 and K_2 are not coarse equivalent.

So in contrast to the topological case one gets obvious counterexamples to the Legendrian knot exterior conjectures exactly in the lens spaces $L(p, p-1)$. It would be interesting if one can generalize the examples from this section to a general classification result in lens spaces.

Problem 5.3.2 (The Legendrian knot exterior problem in lens spaces)
Can one classify all Legendrian knots in contact lens spaces (or general Seifert fibered spaces) that are not determined by their exteriors?

A first step to study the Legendrian knot exterior problem in general contact manifolds would be a generalization of Theorem 1.3.3. The generalization of the implication (2) $\Rightarrow$ (1) one can prove similarly to the proof of Theorem 2.2.10.

Lemma 5.3.3 (Criterion for the Legendrian knot exterior problem)
Let K be a Legendrian knot in a contact manifold (M,ξ) such that there is no non-trivial contact Dehn surgery along K resulting again in (M,ξ). Then the equivalence type of K is determined by the contactomorphism type of its exterior.

But the other implication does not generalize. The problem is again that the new glued-in solid torus is in general not a standard neighborhood of a Legendrian knot, as explained in Example 5.2.3.

So it is not directly clear if exotic cosmetic contact surgeries give negative answers to the Legendrian knot exterior problem. But cosmetic contact surgeries are interesting on their own and studied in the next section.

Moreover, in Section 5.2 we saw that all topological examples of links not determined by their complements, where one does a composition of (-1)-Rolfsen twist along unknot components, works also for the contact case. But in the topological category there are also examples that do not arise in this way.

Teragaito [78] and Ichihara [48] construct, building up on unpublished work of Berge, links in S^3 with no unknot components such that non-trivial Dehn surgery along that link leads again to S^3. Similar to the arguments in Theorem 1.3.3 this leads to links not determined by their complements.

Gordon proves in [42] that for every such link (not determined by its complement, but without unknot components) there exist only finitely many other links with homeomorphic complements. Finally in [61] the reverse question is considered. They study links that **are** determined by their complements.

Problem 5.3.4 (The Legendrian link exterior problem)
Which of these statements also hold for Legendrian links in contact manifolds?

5.4 Cosmetic contact surgeries

In this section I want to give examples of exotic cosmetic contact surgeries and classify all contact surgeries from (S^3, ξ_{st}) along a single Legendrian knot resulting again in a contact S^3. This will give another prove of the contact Dehn surgery theorem 5.1.1.

The following example was obtained earlier by Geiges and Onaran (see also [39]).

Example 5.4.1 (Exotic contact surgeries)
Consider the two different contact Dehn surgeries along the Legendrian unknot U with $\mathrm{tb} = -1$ in (S^3, ξ_{st}) with respect to the contact surgery coefficients $r_1 = -3/2$ and $r_2 = -2/3$ (see Figure 5.9).

By expressing the surgery coefficients with respect to the surface longitude one sees that the resulting manifolds are topologically the manifolds from Example 1.3.2, i.e. the homeomorphic lens spaces $L(5, 2)$ and $L(5, 3)$.

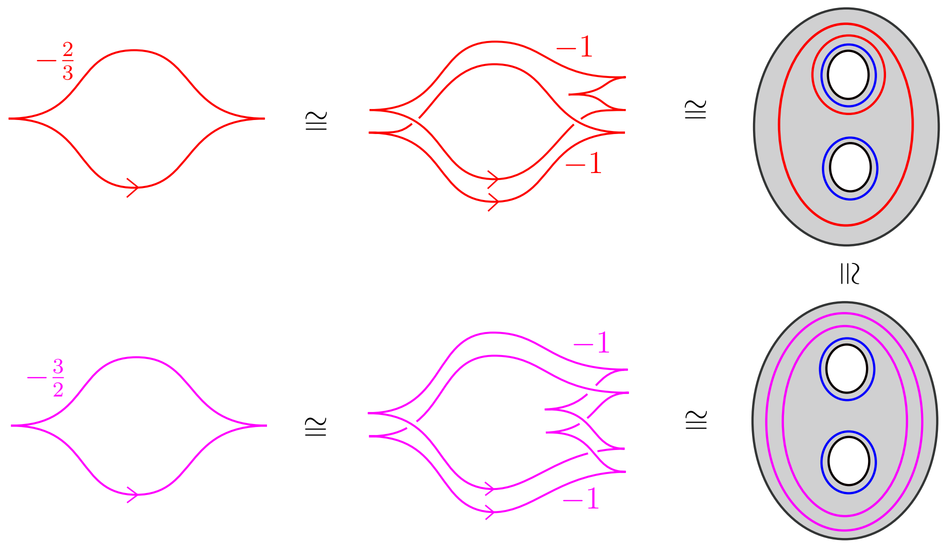


Figure 5.9: Two surgery diagrams of the same tight contact structure ξ on $L(5, 2)$

The next thing I want to show is that the two contact surgeries are unique and represent the same contact manifold. To this end, one uses the same approach as in Example 3.2.7. The same argument as in that example shows that the two surgery diagrams both represent a unique contact structure ξ on $L(5,2)$. The (± 1)-contact surgery diagrams resulting from the transformation lemma 3.2.4 are shown in Figure 5.9 in the middle (for both contact surgeries only one of the two possible stabilizations is drawn). Both surgery diagrams contain only (-1)-contact surgeries, so the resulting contact structures have to be tight. But the classification of tight contact structures on lens spaces [46, Theorem 2.1] says that on $L(5,2)$ there are two non-contactomorphic tight contact structures, so in this case this is not enough to conclude that these surgery diagrams represent the same contact manifold.

To show that the contact structures of the two different surgeries really are contactomorphic one changes the contact Dehn surgery diagrams into compatible open book decompositions (for details see for example [66]). The resulting open books are shown in Figure 5.9 on the right. All colored curves represent right-handed Dehn twists, the blue ones correspond to the monodromy of the open book decomposition of (S^3, ξ_{st}) and the red and purple curves represent the Dehn twists corresponding to the same colored Legendrian links. All Dehn twists together represent the monodromy of $(L(5,2), \xi)$. By interchanging the holes, these two open books are the same, and therefore represent the same contact manifolds.

These cosmetic contact surgeries are exotic because the corresponding topological cosmetic surgeries are exotic (i.e. there is no homeomorphism of the exterior of the unknot that maps one slope to the other).

With exactly the same methods one can get many other examples of this kind. For example, by contact Dehn surgery along U with contact surgery coefficients $r_1 = -2/5$ and $r_2 = -3/4$ one gets contactomorphic contact structures on the lens spaces $L(7,5)$ and $L(7,3)$.

An easy consequence of the contact Dehn surgery theorem 5.1.1 is that the only possible candidates for Legendrian knots in (S^3, ξ_{st}) admitting a (non-trivial) contact surgery resulting again in (S^3, ξ_{st}) are the Legendrian unknots of type U_n.

That cosmetic contact surgeries along such knots really exist was shown in Example 3.2.7. In the following I will classify all contact Dehn surgeries along a single Legendrian knot in (S^3, ξ_{st}) resulting again in a contact 3-sphere. The main tool for doing this will be the formula for computing the d_3-invariant from Section 4.8. As a corollary it will turn out that every Legendrian unknot of type U_n admits (infinitely many) cosmetic contact surgeries resulting again in (S^3, ξ_{st}) and that infinitely many overtwisted contact structures on S^3 cannot be obtained by a single contact Dehn surgery from (S^3, ξ_{st}).

Let K be a Legendrian knot in (S^3, ξ_{st}) such that the result of a contact r-surgery along K is again a contact 3-sphere. From the topological surgery theorem by Gordon–Luecke 1.2.4 and Remark 1.2.3 it follows that K is a Legendrian unknot and that the contact surgery coefficient has to be of the form $1/q$ (for $q \in \mathbb{Z} \setminus \{0\}$). If one denotes by $t \geq 1$ the negative Thurston–Bennequin invariant of K, then it follows that the contact surgery coefficient is of the form

$$r = \frac{1+qt}{q}.$$

Here I chose t to be the negative of tb because the formulas later are easier to read.

The following results fit naturally into a more general framework. Thurston proposed in an unpublished note a graph called the **Dehn surgery graph** (see for example [45]). This graph has a vertex for every closed oriented 3-manifold. Two vertices M and N are connected by an edge if and only if there exists a Dehn-surgery along a knot in M yielding N. A similar graph can be defined for contact manifolds. Here it is natural to orient the edges and to define different types of edges.

Definition 5.4.2 (Contact Dehn surgery graph)
Define the **Contact Dehn surgery graph** to consist of a vertex for every closed contact 3-manifold. There exists an oriented edge from the vertex (M, ξ_M) to the vertex (N, ξ_N) if there exists a contact r-surgery $r \in \mathbb{Q} \setminus \{0\}$ along a Legendrian knot in (M, ξ_M) yielding (N, ξ_N). Add the corresponding sign of the contact surgery coefficient to the edge. Finally one uses different colors for the edges depending on whether the surgery coefficient is of the form $r = \pm 1$, $r = \pm 1/n$ (and not of the form ± 1), $r \in \mathbb{Z} \setminus \{0, \pm 1\}$ and general r (not of the form $1/n$ or integer).

Observe that by the cancellation lemma 3.2.2 contact $(-1/n)$-surgery can reversed by a contact $(1/n)$-surgery. For simplicity one only adds the edge consisting of the negative contact surgery coefficient. A general contact r-surgery cannot be reversed by a single surgery as explained in Example 5.2.3.

Remark 5.4.3 (Properties of the contact Dehn surgery graph)
Observe that several theorems obtained earlier immediately translate into properties of the contact Dehn surgery graph. For example the theorem of Ding–Geiges 3.2.8 is equivalent to the fact that the contact Dehn surgery graph is connected (even if one uses only edges of one special type).

To determine if a given contact manifold (M, ξ) is tight (or fillable) it is enough to find a directed path to (M, ξ) along directed negative edges in the contact Dehn surgery graph starting at a tight (or fillable) contact manifold.

Cosmetic contact surgeries correspond exactly to edges starting and ending at the same contact manifold.

Here I only want to consider the subgraph of the contact Dehn surgery graph consisting of contact 3-spheres, while it would be interesting to study the contact

Dehn surgery graph in full generality. The goal here is to find all edges starting from (S^3, ξ_{st}) and ending in a contact 3-sphere.

First I consider only edges corresponding to contact surgery coefficients of the form ± 1. If one requires the contact surgery coefficient $r = \frac{1+qt}{q}$ to be of the form ± 1, it follows that the only possibility is a contact $(+1)$-surgery along a Legendrian unknot with $\mathrm{tb} = -2$ resulting in the contact structure ξ_1 by Example 4.8.4. (Observe that since I am interested only in the resulting contact structure I consider unoriented knots. And if one considers unoriented knots there is only one Legendrian unknot with $\mathrm{tb} = -2$.)

The case where one looks only at contact surgery coefficients of the form $r = \pm 1/n$ (and not of the form ± 1) works similar. In this case there also exists only one contact surgery, the contact $(1/2)$-surgery along the Legendrian unknot with $\mathrm{tb} = -1$ yielding also the contact structure ξ_1.

Therefore, the contact Dehn surgery graph consisting only of edges starting at (S^3, ξ_{st}) and corresponding to contact surgery coefficients of the form ± 1 and $1/n$ is pictured in Figure 5.10.

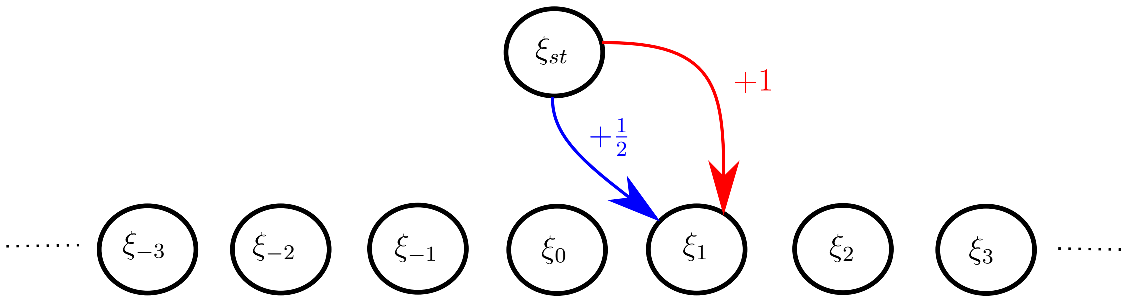


Figure 5.10: A part of the contact Dehn surgery graph

Next I want to look at integer contact surgeries (not of the form ± 1). Given a Legendrian knot K in (S^3, ξ_{st}) with $\mathrm{tb} = -t$, then the only integer contact surgeries resulting again in a contact 3-sphere are the contact $(t+1)$- and the contact $(t-1)$-surgeries.

Consider first the case of contact $(t-1)$-surgery for $t \geq 3$. (The case $t = 1$ corresponds to a 0-surgery which is excluded and the case $t = 2$ was already considered in the (± 1)-surgeries.) The transformation lemma 3.2.4 yields

$$K(t-1) = K(+1) \times K_1\left(-\frac{1}{t-2}\right).$$

Therefore, the generalized linking matrix computes for both stabilizations as

$$Q = \begin{pmatrix} 1-t & -t^2+2t \\ -t & -t^2+t+1 \end{pmatrix}$$

with determinant 1 and trace $2 - t^2 < 0$. It follows that the signature of Q is -2. Next, one solves the equation $Q\mathbf{b} = \mathbf{rot}$ and plugs the solution $\mathbf{b}$ into the formula for the d_3-invariant, yielding

$$d_3 = \frac{1}{4}\Big((-t^2+t+1)\operatorname{rot}_1^2 + (2t^2-4t)\operatorname{rot}_1\operatorname{rot}_2 + (-t^2+3t-2)\operatorname{rot}_2^2 + t + 3\Big) - \frac{1}{2}.$$

Now the possible values for rot_1 are $\operatorname{rot}_1 = 1 - t + 2l$ for $l = 0, \ldots, t-1$ and rot_2 is either $\operatorname{rot}_1 + 1$ or $\operatorname{rot}_1 - 1$. Plugging this in yields for $\operatorname{rot}_1 = 1 - t + 2l$ the solution

$$d_3 = \begin{cases} \Big(-(t-l)^2 + 3(t-l) - 1\Big) - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = \operatorname{rot}_1 + 1, \\ \Big(-l^2 + l + 1\Big) - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = \operatorname{rot}_1 - 1. \end{cases}$$

The case of a contact $(t+1)$-surgery works similar. First the transformation lemma 3.2.4 yields

$$K(t+1) = K(+1) \times K_1\left(-\frac{1}{t}\right).$$

Then with similar computations as before one gets

$$d_3 = \begin{cases} \Big(l^2 + l\Big) - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = \operatorname{rot}_1 + 1, \\ \Big((t-l)^2 - (t-l)\Big) - \frac{1}{2}; & \text{for } \operatorname{rot}_2 = \operatorname{rot}_1 - 1, \end{cases}$$

where $\operatorname{rot}_1 = 1 - t + 2l$ for $l = 0, \ldots, t-1$.

In particular, there are infinitely many overtwisted contact structures on S^3 that cannot be obtained by a single surgery along a Legendrian knot in (S^3, ξ_{st}) with integer contact surgery coefficient.

Recall from Section 4.8 that there are exactly two contact structures on S^3 with $d_3 = 0 - 1/2$, the tight one ξ_{st} and an overtwisted one ξ_0. From the above results it follows immediately that one cannot get these contact structures (by integer contact surgery) along a Legendrian knot not of type U_n.

Next I want to show that every contact structure (resulting in this way) with $d_3 = 0 - 1/2$ is in fact the tight contact structure, i.e. it is not possible to get ξ_0 by integer contact surgery along a Legendrian knot in (S^3, ξ_{st}).

Lemma 5.4.4 (Classification of integer cosmetic contact surgeries)
Let K be a Legendrian unknot in (S^3, ξ_{st}) with $\operatorname{tb} = -t$ and maximal rotation number $|\operatorname{rot}| = t - 1$. Then the surgery diagram from Figure 5.11 represents again (S^3, ξ_{st}).

Moreover, all surgery diagrams of contact structures on S^3 with $d_3 = 0 - 1/2$ coming from a single integer contact surgery along a Legendrian knot in (S^3, ξ_{st}) are of this form.

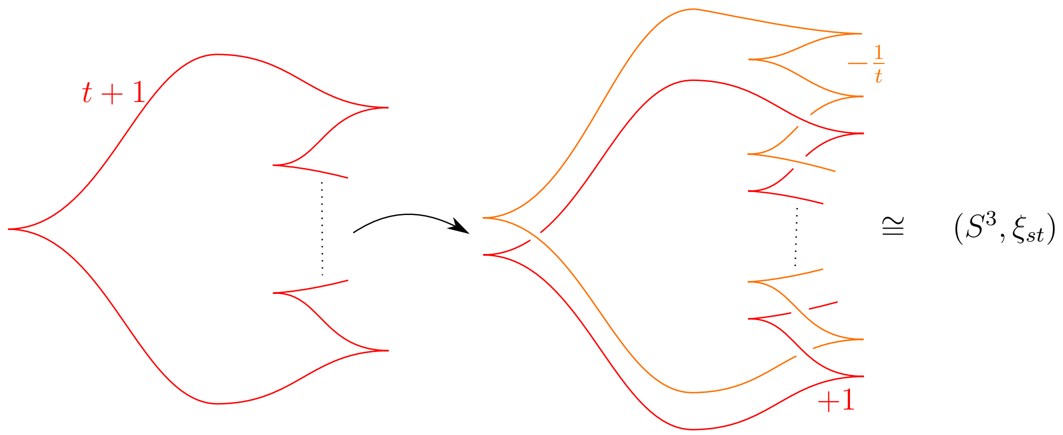


Figure 5.11: Integer cosmetic contact Dehn surgeries

Proof.
The idea of the proof is due to Lisca and Stipsicz (see proof of Proposition 2.4 in [56] where they obtain a more general statement about integer surgeries). The idea is to translate the surgery diagram from Figure 5.11 into an open book (see [66] for more information) and then show that this open book represents (S^3, ξ_{st}).

Therefore, first recall the lantern relation for open books (see for example [67, Lemma 15.1.9]). Consider the open book with page an annulus with two holes with monodromy given by Dehn twists along the curves as in Figure 5.12 on the left. The lantern relation states that this is the same as the open book in the middle of Figure 5.12 and by destabilizing this open book along the green curve one gets the open book on the right.

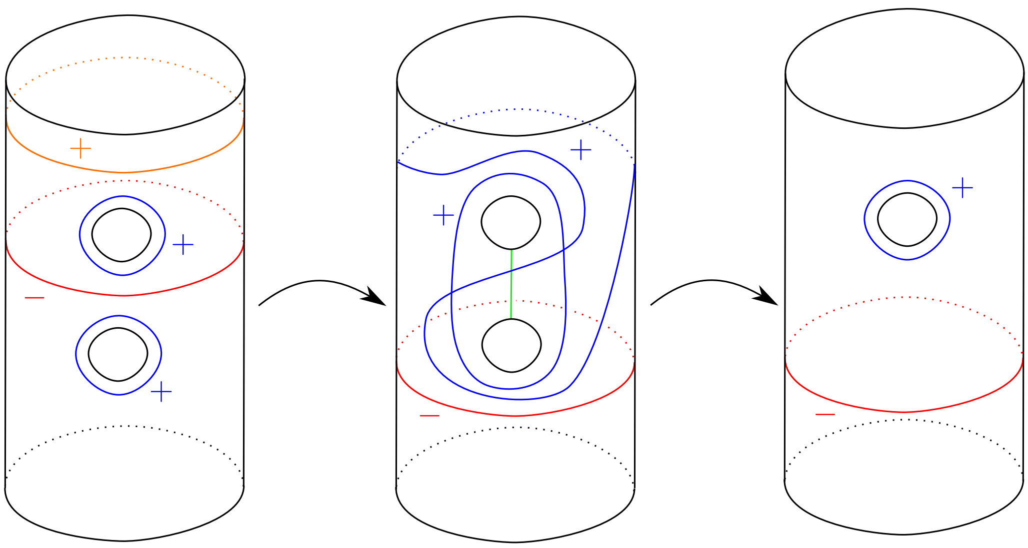


Figure 5.12: The lantern relation followed by a destabilization

The open book corresponding to the integer contact surgeries from Figure 5.11

is depicted in the upper line of Figure 5.13. The lighter shaded region is a two-holed annulus with monodromy as in Figure 5.12. By using the lantern relation several times one ends up with the open book in the lower row of Figure 5.13. This represents the surgery link from Example 3.2.7(6) representing (S^3, ξ_{st}). One could also conclude that this open book represents (S^3, ξ_{st}) by observing that the resulting open book only consists of positive Dehn twists and therefore represents a fillable contact structure on S^3.

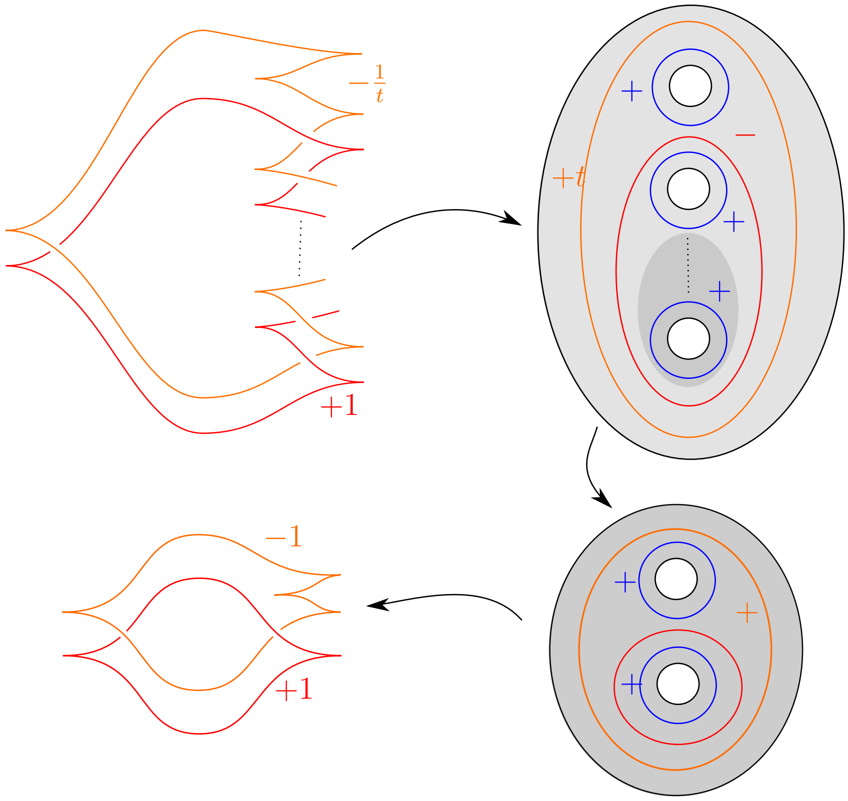


Figure 5.13: Contact surgeries with the same open books

That all surgery diagrams yielding a contact structure on S^3 with $d_3 = 0 - 1/2$ are of this form follows from the computations above. □

For general surgery coefficients this works very similar. It makes sense to distinguish the following two cases.

If $q \geq 2$ it follows that

$$K\left(\frac{1+qt}{q}\right) = K(+1) \times K_1\left(-\frac{1}{t-1}\right) \times K_{1,q-1}(-1),$$

and one computes similarly as before

$$d_3 = \Big(l(1-t+2l) + l^2 q - l\,\mathrm{rot}_3\Big) - \frac{1}{2},$$

for $\operatorname{rot}_2 = \operatorname{rot}_1 + 1$ and

$$d_3 = \Big(-(t-l-1)(1-t+2l) + (t-l-1)^2 q + (t-l-1)\operatorname{rot}_3 \Big) - \frac{1}{2},$$

for $\operatorname{rot}_2 = \operatorname{rot}_1 - 1$, where in both cases $\operatorname{rot}_1 = 1-t+2l$ for $l = 0, \dots, t-1$ and $\operatorname{rot}_3 = 1-t-q+2i$ for $i = 0, \dots, t+q-1$.

If $q \leq 2$ it follows that

$$K\left(\frac{1+qt}{q}\right) = K(+1) \times K_1\left(-\frac{1}{t-2}\right) \times K_{1,1}\left(-\frac{1}{-q-1}\right),$$

and one computes similar as before

$$d_3 = \begin{cases} (t-l-1)(t-l)q + 2(t-l-1) + 1 - \frac{1}{2}; & \text{for } r_2 = r_1 + 1 \text{ and } r_3 = r_2 + 1, \\ (t-l-1)(t-l-2)q + 1 - \frac{1}{2}; & \text{for } r_2 = r_1 + 1 \text{ and } r_3 = r_2 - 1, \\ l(l-1)q + 1 - \frac{1}{2}; & \text{for } r_2 = r_1 - 1 \text{ and } r_3 = r_2 + 1, \\ (l-1)(l-2)q + 2(l-2) + 1 - \frac{1}{2}; & \text{for } r_2 = r_1 - 1 \text{ and } r_3 = r_2 - 1, \end{cases}$$

where r_i is the short notation for rot_i and $\operatorname{rot}_1 = 1-t+2l$ for $l = 0, \dots, t-1$.

Observe that these formulas for the d_3-invariants are also correct for the degenerated cases $t = 1$ and $t = 2$.

The next lemma can be proven very similarly to Lemma 5.4.4. This lemma together with the computations of the d_3-invariants give the complete information about contact surgeries along single Legendrian knots yielding contact 3-spheres.

Lemma 5.4.5 (Classification of rationally cosmetic contact surgeries)
Let K be a Legendrian unknot in (S^3, ξ_{st}) with $\operatorname{tb} = -t$ and maximal rotation number $|\operatorname{rot}| = t-1$. Then the rational contact surgery diagram from Figure 5.11 represents again (S^3, ξ_{st}).

Moreover, all surgery diagrams of contact structures on S^3 with $d_3 = 0 - 1/2$ coming from a single rational contact surgery along a Legendrian knot in (S^3, ξ_{st}) are of this form.

In particular, ξ_0 cannot be obtained by a single contact Dehn surgery from (S^3, ξ_{st}). Note also that this gives another proof of the contact Dehn surgery theorem 5.1.1.

Remark 5.4.6 (Distance of contact 3-spheres in the contact Dehn surgery graph)
Observe also that from these results it follows that the subgraph consisting only of contact 3-spheres is also connected. Since there are rational contact surgeries resulting in ξ_1 and ξ_{-1}, this follows again by taking connected sums. But it is not directly clear what the minimal distance between two contact 3-spheres in this subgraph is.

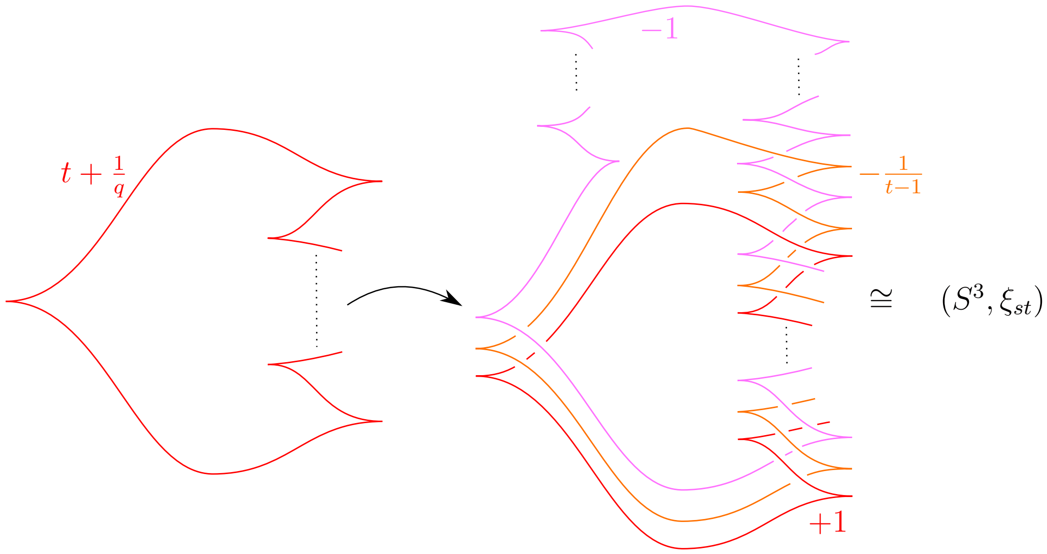


Figure 5.14: Rationally cosmetic contact Dehn surgeries

However, if one considers the whole contact Dehn surgery graph, then in [17, Section 4] it is shown that the distance between (S^3, ξ_{st}) and an arbitrary contact 3-sphere is bounded by 3 (even if one only allows contact (± 1)-surgeries).

It would be nice to study the contact Dehn surgery graph in a more systematic way and in particular to look at cosmetic contact surgeries.

Problem 5.4.7 (Cosmetic contact surgeries)
Is it possible to classify cosmetic contact surgeries in other contact manifolds? Can one find examples of exotic cosmetic contact surgeries not resulting in lens spaces?

6

The transverse knot complement problem

In this chapter the transverse knot complement problem is discussed. In Section 6.1 I will prove the transverse knot exterior theorem in (S^3, ξ_{st}). The methods in the proof are the same as those in the Legendrian case. In this section I will also explain where I use that the neighborhoods of the transverse knots are sufficiently small.

Examples of transverse links in (S^3, ξ_{st}) and transverse knots in general contact manifolds, that are not determined by the contactomorphism types of their exteriors, are discussed in Sections 6.2 and 6.3. But in both cases it remains unclear how these examples depend on the chosen exteriors.

Finally, in Section 6.4 I will discuss in more detail the relationship between equivalence of topological links as well as Legendrian and transverse links to their complements, their open exteriors and their closed exteriors.

6.1 The transverse knot exterior theorem

In this section I want to give a proof of the transverse knot exterior theorem 2.2.11, namely that two transverse knots in (S^3, ξ_{st}) are equivalent if and only if their sufficiently large exteriors are contactomorphic. The strategy of the proof is exactly the same as for the Legendrian case (compare Section 5.1). So first I generalize again the surgery theorem by Gordon–Luecke 1.2.4 to transverse knots.

Theorem 6.1.1 (Transverse contact surgery theorem)
Let T be a transverse knot in (S^3, ξ_{st}). If $(S^3_T(r), \xi_T(r))$, constructed using a sufficiently small tubular neighborhood νT, is contactomorphic to (S^3, ξ_{st}) for some $r \neq \infty$, then T is equivalent to the transverse unknot with self-linking number -1.

Proof.
From the topologically surgery theorem 1.2.4 by Gordon–Luecke and Remark 1.2.3 it follows that T is topological equivalent to the unknot and the topological surgery coefficient has to be of the form $r = 1/n$. Therefore by the classification of transverse unknots 2.3.11, T is equivalent to the unique transverse unknot U_{sl} with self-linking number sl.

The idea of the proof is now to consider, for every transverse unknot U_{sl} (other than U_{-1}) and every surgery coefficient of the form $r = 1/n$, a Legendrian knot L in the exterior $(S^3 \setminus \nu\mathring{U}_{\mathrm{sl}}, \xi_{st})$ of U_{sl}. This Legendrian knot can also be seen as a Legendrian knot in the new surgered contact manifold $(S^3_{U_{\mathrm{sl}}}(r), \xi_{U_{\mathrm{sl}}}(r))$. In this situation the formula for computing the new Thurston–Bennequin invariant tb_{new} from Lemma 4.2.1 simplifies by Example 4.3.4 to

$$\mathrm{tb}_{new} = \mathrm{tb}_{old} - n\,\mathrm{lk}^2(L, T).$$

Consider now the front projection in Figure 6.1 of a transverse unknot T of type U_{sl} and a Legendrian unknot L in the exterior of T.

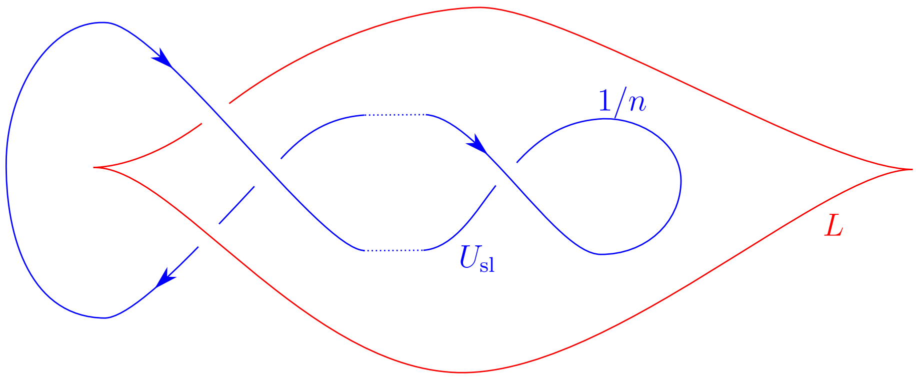


Figure 6.1: For $n < 0$ this contact surgery diagram yields an overtwisted 3-sphere.

If one does a $(1/n)$-surgery along U_{sl} then the knot L is (topologically) again an unknot in some new contact S^3 (this can be seen by doing a Rolfsen twist along U_{sl}). If the resulting manifold is again (S^3, ξ_{st}) then the Bennequin inequality holds, i.e. $\mathrm{tb}_{new} \leq -1$. On the other hand one can compute $\mathrm{tb}_{new} = -1 - n$ with the above formula. So it follows that for $n < 0$ the resulting contact 3-sphere has to be overtwisted.

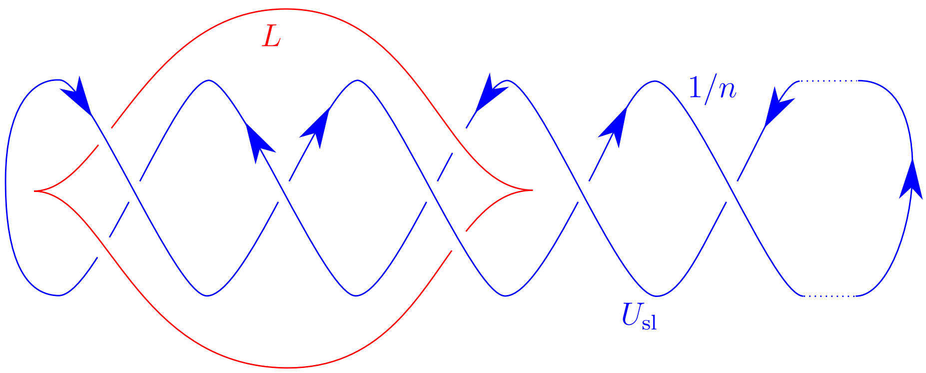


Figure 6.2: For $n > 0$ this contact surgery diagram yields an overtwisted 3-sphere.

For $n > 0$ one considers the diagram from Figure 6.2. Observe that this does not work for the transverse unknot with self-linking number -1, but for all other transverse unknots. The linking number is $\mathrm{lk}(T, L) = 0$ and therefore the Thurston–Bennequin number of L stays the same, i.e. $\mathrm{tb}_{new} = \mathrm{tb}_{old} = -1$.

To show that the resulting contact 3-sphere is always overtwisted one determines the new knot type L_n of L in the new S^3. Therefore, observe that topologically the link $T \sqcup L$ is the Whitehead link, which can also be pictured like in Figure 6.3(i). By doing a Rolfsen twist along T one gets the new knot L_n in the new S^3 pictured in Figure 6.3(ii).

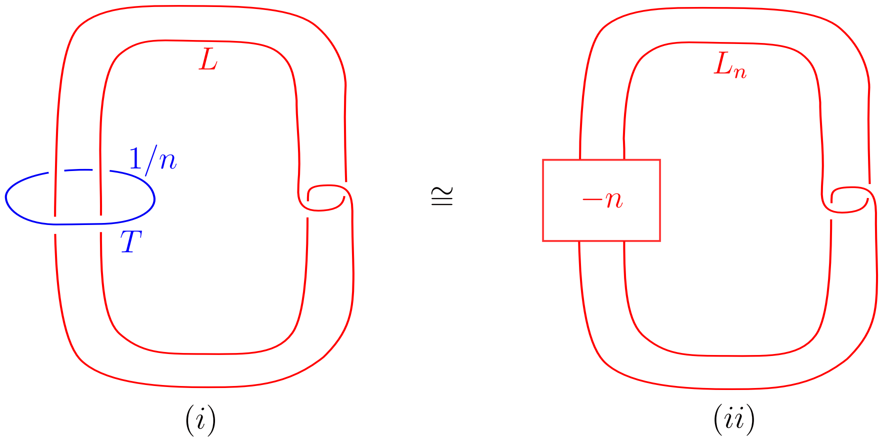


Figure 6.3: A Rolfsen twist along T yields L_n.

Of course here the Bennequin inequality cannot help because the minimal bound in the Bennequin inequality is -1. Therefore, one has to use a finer bound for the Thurston–Bennequin number. One such bound is the so-called Kauffman bound [73] (see also [31]), which says that for a Legendrian knot L in (S^3, ξ_{st}) the following inequality holds

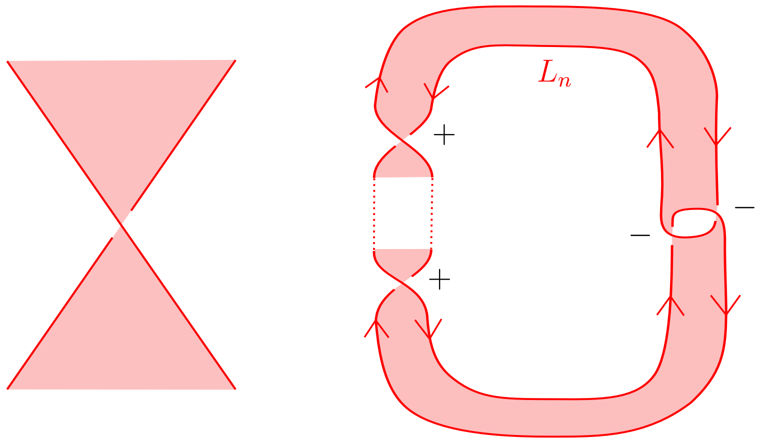
$$\operatorname{tb}(L) \leq \min\Big\{\deg_x\Big(F_L(x,y)\Big)\Big\} - 1,$$

where $F_L(x,y)$ is the Kauffman polynomial of L. From the work of Tanaka and Yokota (see [77] and [87]) it follows that in the case of a reduced alternating knot diagram D_n of L_n (as is the case here) this inequality transforms to

$$\operatorname{tb}(L_n) \leq \operatorname{sl}(D_n) - \operatorname{r}(D_n)$$

where $\operatorname{sl}(D_n)$ is the self-linking number of the knot diagram D_n and $\operatorname{r}(D_n)$ is the number of regions in the knot diagram D_n. By coloring the complement of an alternating knot diagram according to the rule in Figure 6.4(i) one gets the regions as the colored areas. (Moreover, in [77] and [87] it is shown that this bound is sharp, but this is not important for the argument here.)

In Figure 6.4(ii) one can count $\operatorname{sl}(D_n) = 2n - 2$ and $\operatorname{r}(D_n) = 2n + 1$. Therefore, every Legendrian realization of L_n in (S^3, ξ_{st}) has $\operatorname{tb}(L_n) \leq -3$. But earlier I computed $\operatorname{tb}_{new} = \operatorname{tb}_{old} = -1$ in the new surgered contact 3-sphere. Therefore this new contact 3-sphere has to be overtwisted. $\square$



(i) (ii)

Figure 6.4: Computing the Kauffman bound for L_n

The proof of the transverse knot exterior theorem 2.2.11 works now as in the Legendrian case.

Proof of the transverse knot exterior theorem 2.2.11.
Pick a contactomorphism between the exteriors

$$h\colon \left(S^3 \setminus \nu\mathring{T}_1, \xi_{st}\right) \longrightarrow \left(S^3 \setminus \nu\mathring{T}_2, \xi_{st}\right),$$

and then consider the following commutative diagram:

$$\begin{tikzcd}
\left(S^3,\xi_{st}\right) \cong \left(S^3_{T_1}(\mu_1),\xi_{T_1}(\mu_1)\right) := \left(S^1\times D^2,\xi'\right) \arrow[d, dashed, "f"'] & + & \left(S^3\setminus\nu\mathring{T}_1,\xi_{st}\right) \Big/_{\sim} \\
\mu_0 \arrow[r, mapsto, "\varphi_1"] \arrow[d, bend right, "\mathrm{Id}"'] & \arrow[d, phantom, "\circlearrowright"] & \mu_1 \arrow[d, bend left, "h"] \\
\mu_0 \arrow[r, mapsto, "h\circ\varphi_1"] & & r_2 := h(\mu_1) \\
\left(S^3_{T_2}(r_2),\xi_{T_2}(r_2)\right) := \left(S^1\times D^2,\xi'\right) & + & \left(S^3\setminus\nu\mathring{T}_2,\xi_{st}\right) \Big/_{\sim}
\end{tikzcd}$$

Here the contact solid torus $(S^1 \times D^2, \xi')$ is chosen to be equal to $(\nu T_1, \xi_{st})$. Because Id and h on the two factors are contactomorphisms and because they both send the characteristic foliations of the boundaries to each other, these two maps glue together to a contactomorphism f of the whole contact manifolds (see [38, Section 2.5.4]). From the transverse contact Dehn surgery theorem 6.1.1 it follows that r_2 is equal to μ_2, or T_2 is equivalent to the transverse unknot U_{-1} with self-linking number -1.

If $r_2 = \mu_2$ then this is a trivial contact Dehn surgery, and so the contactomorphism f maps T_1 to T_2.

In the other case the same argument with T_1 and T_2 reversed shows that T_1 is also equivalent to the transverse unknot U_{-1} with self-linking number -1. □

Remark 6.1.2 (Size of the exteriors)
Maybe it is not clear where in the proofs of the theorems it is important that the neighborhoods of the transverse knots are sufficiently small. To see why it is, observe that the proof of the transverse Dehn surgery theorem shows a bit more. It is not only shown that if there exists a cosmetic contact Dehn surgery along a transverse knot in (S^3, ξ_{st}), then this knot has to be the transverse unknot U_{-1}, but also the surgery coefficient has to be of the form $r = 1/n$ for $n > 0$ (so negative integers are excluded). At a first glance it looks as if this is a contradiction to the result of Etnyre–Ghrist (Theorem 3.1.3) about the existence of tight transverse knots.

But the surgeries in this theorem are done with respect to a sufficiently big tubular neighborhood of the transverse knot T to ensure tightness of the resulting manifold as explained in [28, Remark 2.1]. By contrast, in the proof of the transverse Dehn surgery theorem one has to choose sufficiently small neighborhoods of the transverse knot, in fact so small that one has enough space in the exterior for the Legendrian knot L. Both results together show that the Legendrian knot L from Figure 6.1 cannot realized in a sufficiently small exterior of T.

It would be interesting to compare this with the notion of fat and skinny framed transverse knots from [36] and [37] and how big the neighborhoods of the transverse knots can be chosen.

The following also remains open.

Problem 6.1.3 (Transverse cosmetic contact surgeries from (S^3, ξ_{st}) to itself.)
Does Theorem 2.2.11 also holds for arbitrary neighborhoods? Are there cosmetic contact surgeries along U_{-1} with small neighborhoods?

6.2 The transverse link complement problem

In this Section the transverse link complement problem is discussed. First a transverse link in (S^3, ξ_{st}) is presented which is not determined by the contactomorphism type of one of its exteriors.

Example 6.2.1 (Counterexample to the transverse link complement problem)
Consider the transverse unknot T and the Legendrian unknot K in its complement in Figure 6.5.

Denote the transverse link given by T and the transverse push-off of K by T_1. Choose the transverse push-off of K so close to K that the standard tubular neighborhood νK of K is also a (possible non-standard) tubular neighborhood of its

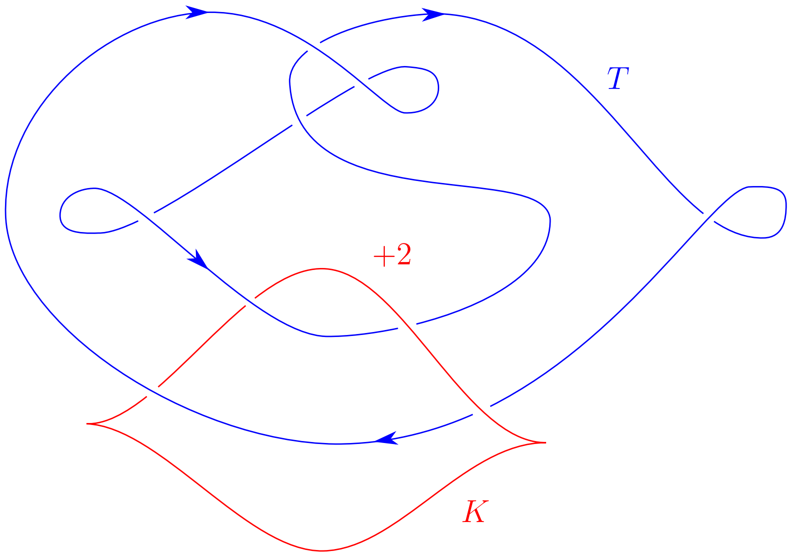


Figure 6.5: Construction of different transverse links with contactomorphic exteriors

push-off. Then the result of a contact $(+2)$-surgery along K can also be obtained by a transverse surgery along the push-off of K.

Next, approximate the spine of the new glued-in solid torus by a transverse knot such that the new glued-in solid torus is a (possible non-standard) tubular neighborhood of its transverse spine. Denote the transverse link given by this transverse spine and the transverse knot T in the surgered manifold by T_2. Then the exteriors of T_1 and T_2 are contactomorphic (if one chooses a tubular neighborhood of T so small that it does not intersect the other neighborhoods).

Since a contact $(+2)$-surgery along a Legendrian unknot yields again (S^3, ξ_{st}), both links T_1 and T_2 represent transverse knots in (S^3, ξ_{st}). It remains to show that they are not equivalent. But this follows immediately, since by Example 1.1.4 they are not even topologically equivalent. This construction will be used again in Section 7.3.

But note that it is not clear if the neighborhoods of the transverse push-off of L and of the transverse approximation of the spine are standard neighborhoods in the sense of Theorem 2.2.5(2). It remains open if the same works also for standard neighborhoods of transverse links.

6.3 The transverse knot complement problem in general manifolds

With a similar construction one can construct different transverse knots in lens spaces with the same exteriors. One takes an exotic cosmetic contact surgery and approximates the spines of the newly glued-in solid tori by transverse knots such that these solid tori are tubular neighborhoods of the transverse knots, but again it is not clear if the neighborhoods are standard ones.

Example 6.3.1 (Counterexamples for transverse knots in general manifolds)
(1) Consider the exotic cosmetic contact surgery along Legendrian unknots from Example 5.4.1 yielding a tight lens space. In both surgeries one approximates the spines of the new glued-in solid tori by transverse knots such that these solid tori are neighborhoods of the transverse approximations. Denote these transverse knots in the same tight lens space by T_1 and and T_2.

By construction they have the same exteriors, but they are not equivalent since the underlying topological knots are not equivalent by Example 1.3.2.

The same works for every other exotic cosmetic contact surgery along Legendrian knots. In this examples it remains open if the same works also for standard neighborhoods of transverse knots.

For standard neighborhoods of transverse knots one can make a similar construction as in Section 5.3.
(2) To that end, look at a standard neighborhood of νT of a transverse knot T with size $\varepsilon = 1/3$. The characteristic foliation on the boundary $\partial(\nu T)$ is given by the vector field

$$X = \partial_\varphi - \frac{1}{3}\partial_\theta.$$

If one denotes the longitude $S^1 \times \{p\}$ of $\partial(\nu T)$ by λ, then the characteristic foliation of $\partial(\nu T)$ is given by the linear curves $-\mu + 3\lambda$.

Now take two copies V_1 and V_2 of this standard tubular neighborhood and glue them together along their boundaries as follows

$$\begin{array}{rcccccl} L(9,7) & = & L(9,-2) & = & V_1 & + & V_2 \Big/\sim \\ & & & & \mu_1 & \longmapsto & -2\mu_2 + 9\lambda_2, \\ & & & & \lambda_1 & \longmapsto & -\mu_2 + 4\lambda_2. \end{array}$$

The characteristic foliation given by $-\mu_1 + 3\lambda_1$ of ∂V_1 is mapped to the characteristic foliation given by $-\mu_2 + 3\lambda_2$ of ∂V_2. Therefore, the contact structures on the two solid tori glue together to a global contact structure on the lens space $L(9,-2) = L(9,7)$. If one chooses the tubular neighborhoods of T_i as V_i, then by construction the corresponding exteriors are the same.

But the knots T_1 and T_2 are not even topologically equivalent by the discussion in Example 1.3.2.

It remains open how such examples depend on the chosen size of the tubular neighborhoods of the knots.

6.4 Complements and Exteriors

In this section I want to explain more on the difference between complements and exteriors in the topological case, as well as in the Legendrian and the transverse case.

Let K be a link with tubular neighborhood νK in some 3-manifold M. Then one can study the relation of the equivalence class of this link to the homeomorphism type of its complement, its open exterior and its closed exterior. These relations are shown in Figure 6.6.

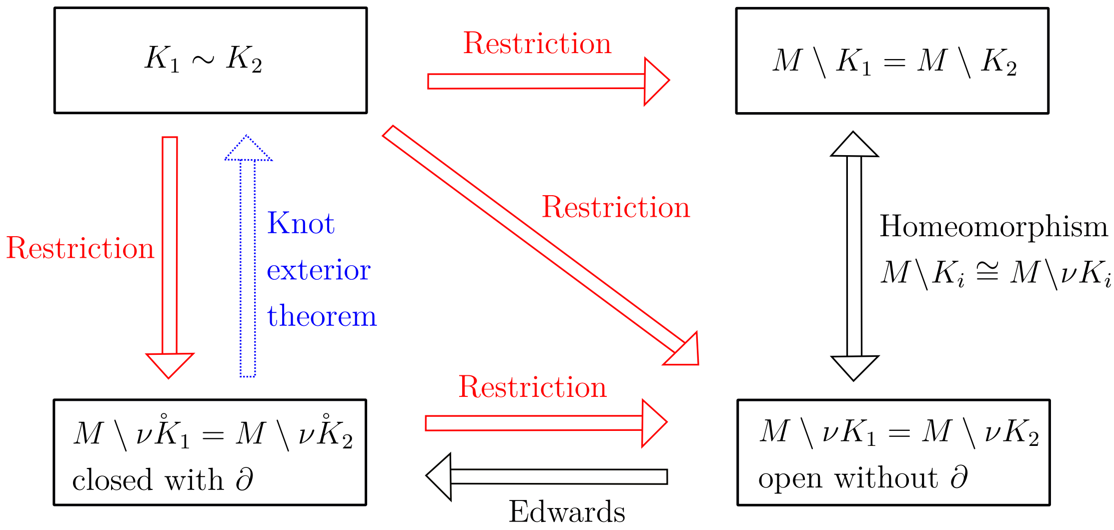


Figure 6.6: Relation between different types of classifications

If two links K_1 and K_2 are equivalent then one can restrict the homeomorphism of M mapping one link to the other to subsets of M. Since such a homeomorphism has to map a tubular neighborhood of one link to a tubular neighborhood of the other link one gets the red implications in Figure 6.6 by restriction.

Moreover, it is easy to construct a homeomorphism of the complement of a link to the open exterior of the same link. (Take a homeomorphism of a punctured disk to a half open annulus in every D^2-slice of the tubular neighborhood.)

A non-trivial theorem of Edwards [22] states that two 3-manifolds with boundary are homeomorphic if and only if their interiors are homeomorphic. Therefore, it is also equivalent to consider closed or open link exteriors.

And finally the blue implication is exactly the statement of the link exterior problem, which is in general not true, but holds for knots in S^3.

Now let K_1 and K_2 be Legendrian or transverse links in some contact 3-manifold (M, ξ). First recall from Remark 2.2.6 that all standard neighborhoods of Legendrian links are contactomorphic, but for transverse links this is not true. Therefore, for

Legendrian links the exteriors are independent of the tubular neighborhoods as in the topological case, but for a transverse link there exist in general more than one exterior.

If two Legendrian or transverse links are equivalent then one can again restrict the contactomorphism to subsets and gets as in the topological case the red implications in Figure 6.6 by restriction.

But whether the black implications from Figure 6.6 also hold in contact geometry is not clear. Contact structures on manifolds which are interiors of compact 3-manifolds with boundary are studied by Eliashberg [23], Makar-Limanov [57] and Tripp [82]. With the methods developed there it should be possible to study if the theorem of Edwards also holds for Legendrian and transverse links.

But the relation between the complements of Legendrian or transverse links and their open exteriors remains mysterious. The topological diffeomorphism between these sets does not preserve the contact structures, it remains open if such a contactomorphism exists or not.

Finally, the blue implication is the Legendrian or transverse link exterior problem, which is again in general not true, but holds for Legendrian and transverse knots in (S^3, ξ_{st}).

7

Surgery description of Legendrian and transverse knots

A classical result in knot theory states that every knot admits a surgery description to the unknot. That means there exists a link in the complement of the knot such that a Dehn surgery along this link yields the 3-sphere but changes the knot to the unknot. The key observation is that one can change a crossing in a knot diagram by a topological (±1)-Dehn surgery along an unknot in the complement of the crossing. I will recall the details of this construction in Section 7.1.

The main result of this chapter is that this is also possible for Legendrian and transverse knots. By doing this crossing change for a Legendrian knot via the contact Rolfsen twist from Lemma 5.2.1 one gets a similar result for Legendrian knots in Section 7.2. By approximating these Legendrian knots via transverse push-offs one also obtains similar result for transverse knots (see Section 7.3).

The main difference is that there are infinitely many different Legendrian and transverse unknots and it is not clear which of these unknots one gets with the surgery description. However, by computing the classical invariants of the resulting unknots one gets connections between the classical invariants of Legendrian and transverse knots in (S^3, ξ_{st}) and the unknotting information of the underlying topological knot type. This is discussed in Section 7.4.

7.1 Surgery description of knots

In this Section I will recall the topological surgery description of (smooth) knots following [73, Sections 6.C and 6.D]. The idea behind this is best explained in an example.

Example 7.1.1 (Surgery description of the trefoil)
Consider the right-handed trefoil knot K in S^3. Observe that changing the type of the marked crossing in Figure 7.1 changes the trefoil knot to the unknot.

Next consider the surgery diagram in Figure 7.2. Depicted is again the right-handed trefoil knot K and in the exterior around the marked crossing of K an unknot L with $+1$ surgery coefficient. (Here the surgery coefficient is measured with respect to the Seifert longitude of L.) By doing a Rolfsen twist along L (or in this case equivalently a blow-down, see [41, Page 151]) one sees that the resulting manifold is

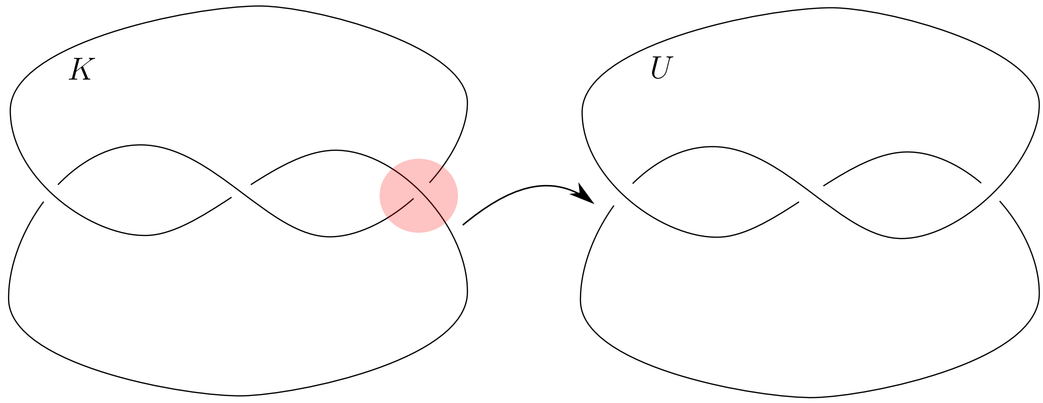


Figure 7.1: Transforming the trefoil knot to the unknot by changing a crossing

again S^3, but the type of the marked crossing changes. So this surgery changes the knot type of K to the unknot in S^3.

Such a diagram like in Figure 7.2 is called **surgery description of K to the unknot**.

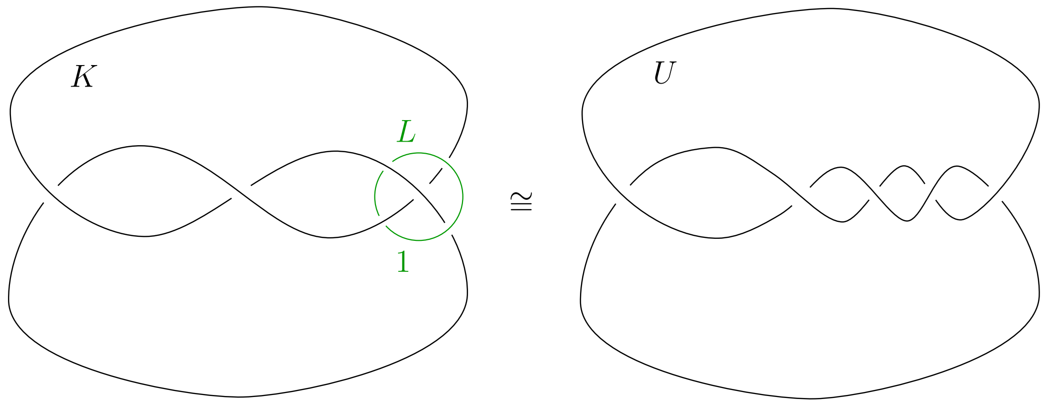


Figure 7.2: Transforming the trefoil knot to the unknot by a cosmetic surgery

On the other hand one can start with a diagram of the unknot U in S^3 and a link L in the exterior of U such that surgery along L does not change the manifold, but changes the unknot U to a given knot type K. This is called **surgery description of K from the unknot**.

By reversing the surgeries one can transform one description easily into the other, see for example Figure 7.3 for the surgery description of the right-handed trefoil from the unknot corresponding to the foregoing example. In fact, the methods from this example also work in general situations.

Theorem 7.1.2 (Surgery description of knots)
Any knot K in a 3-manifold M admits a surgery description from and to the unknot in S^3.

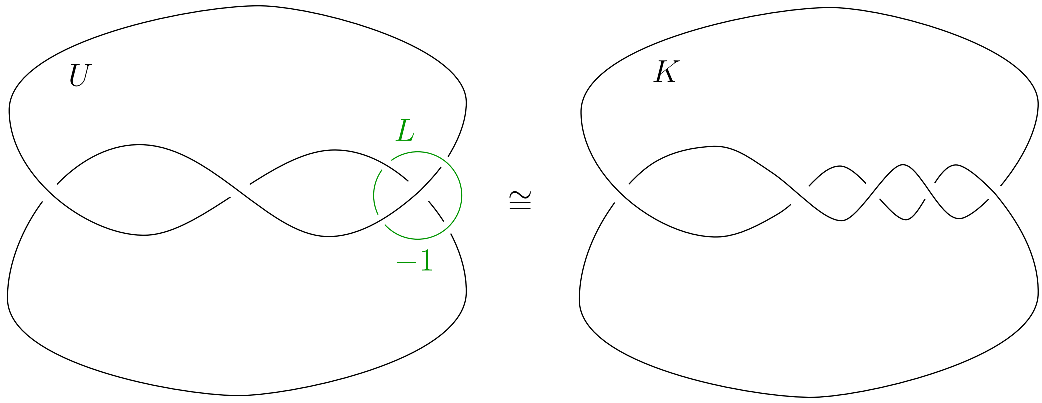


Figure 7.3: A surgery description of the trefoil K from the unknot U

Proof. Let L be a link in S^3 such that a Dehn surgery along L yields M. As explained in Section 4.7 one can assume that K lies completely in the exterior of L in S^3. Therefore, it is enough to consider the case of a knot K in S^3.

It is a standard fact that every knot in S^3 can be unknotted by finitely many crossing changes (see [70, Theorem 3.8]). Exactly like in the example above this can be achieved by doing a Dehn surgery along an unknot surrounding the crossing. This leads to a surgery description of K to the unknot. By reversing the surgeries one also gets a surgery description of K from the unknot. □

Remark 7.1.3 (The linking number of K and L)
By looking at Figure 7.4 one sees that one can choose the link L such that all components of L have vanishing linking number with K (in both surgery descriptions from and to the unknot).

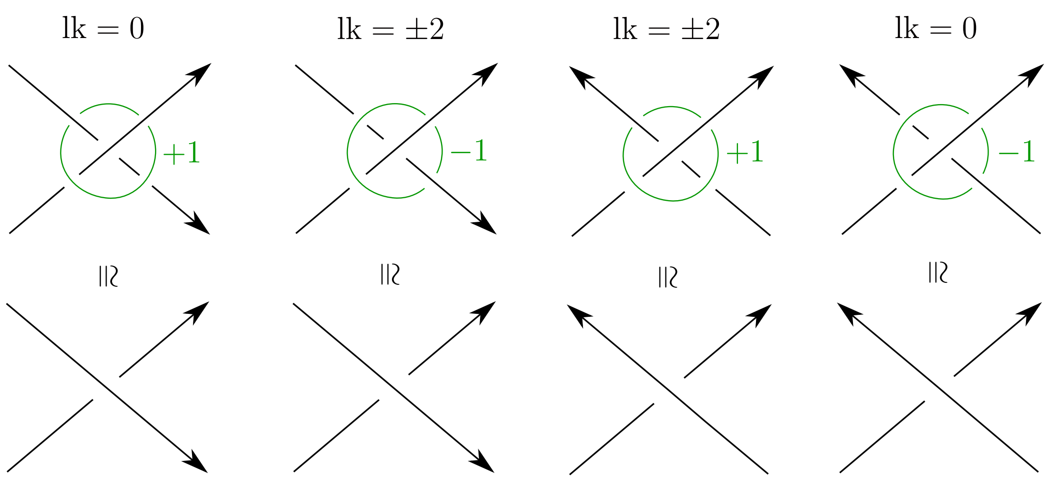


Figure 7.4: The linking numbers with the surgery knots

The main application of surgery descriptions is an easy and effective construction method for covering spaces of knot exteriors (see [73, Section 6.C and 7.C]) or more general branched coverings (see [70, Section 22] and [41, Section 6.3]). Because one can choose the linking numbers of K with all components of L to be zero, one can use the surgery description also to construct coverings of the corresponding 4-manifolds (see [41, Section 6.3]).

7.2 Surgery description of Legendrian knots

With the contact Rolfsen twist (Lemma 5.2.1) one can generalize the proof of Theorem 7.2.1 to Legendrian knots.

Theorem 7.2.1 (Surgery description of Legendrian knots)
Any Legendrian knot K in a contact 3-manifold (M,ξ) admits a contact surgery description along a Legendrian link L to and from a Legendrian unknot U in (S^3,ξ_{st}).

Proof.
First consider the case of a Legendrian knot K in (S^3,ξ_{st}). In a front projection of K every crossing is of the type in Figure 7.5(i). By a Legendrian isotopy one can change the local picture to look like in Figure 7.5(ii). Now one can insert a cosmetic contact surgery (see Figure 7.5(iii)) and using the contact Rolfsen twist one can change the type of the crossing. Observe that there are two different choices how to change the crossing, corresponding to the two different tight choices of the new contact structure on the new glued-in solid torus. By applying this to chosen crossings of the knot K one can unknot K to a Legendrian unknot.

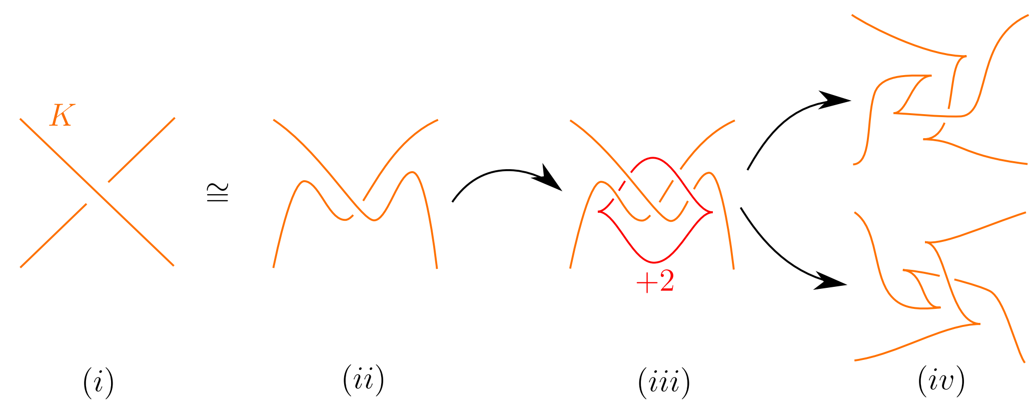


Figure 7.5: A Legendrian crossing change by a cosmetic contact surgery

The general case follows (like in the topological case) from the fact that for every knot K in a general contact manifold (M,ξ), there exists a surgery diagram L in (S^3,ξ_{st}) such that K can be presented in $(S^3\setminus \nu L,\xi_{st})$, see Lemma 4.7.1.

It remains to show that there also exists a surgery description from a Legendrian unknot. In general it is not possible to reverse a contact r-surgery by a contact surgery along a single Legendrian knot (see Example 5.2.3), but by the cancellation lemma 3.2.2 it is possible to reverse a contact $(\pm 1/n)$-surgery. So by the transformation lemma 3.2.4 one can reverse a general contact r-surgery by a (± 1)-surgery along a Legendrian **link** in the new glued-in solid torus. So it is also possible to reverse the cosmetic contact $(+2)$-surgeries as shown in Figure 5.6. And therefore one also gets a surgery description from a Legendrian unknot. □

Example 7.2.2 (Surgery description of a Legendrian trefoil)
Consider the front projection of a right-handed Legendrian trefoil in Figure 7.6. By inserting a cosmetic contact $(+2)$-surgery one can change a crossing such that this Legendrian trefoil transforms to a Legendrian unknot. There are two possibilities for doing it, corresponding to the two possibilities of doing the contact $(+2)$-surgery (compare Figure 5.3).

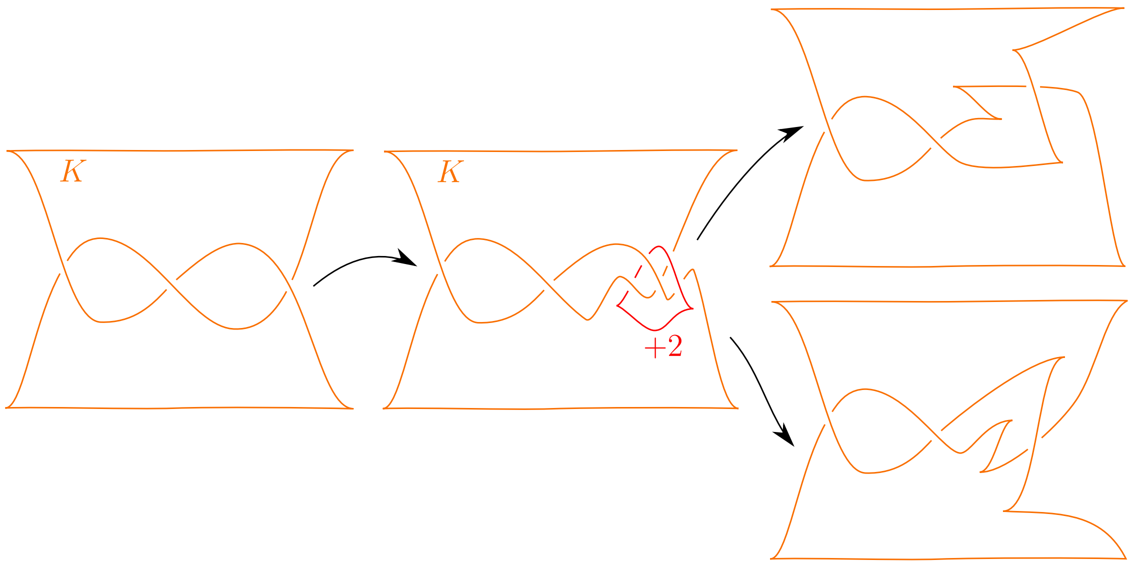


Figure 7.6: A surgery description of a Legendrian trefoil

In this example one sees that the resulting Legendrian unknot depends on the starting Legendrian knot and also on the chosen contact surgeries. Therefore, one can get information about Legendrian knots from the resulting Legendrian unknots. One aspect of this is discussed in Section 7.4.

In principle, one can now construct contact surgery diagrams of coverings of the exteriors of Legendrian knots with the surgery description of Legendrian knots, but in examples it turned out to be difficult. It would be interesting to see in examples how to do this explicitly.

Another main difference to the topological case is that in general one cannot choose the Legendrian link L to have vanishing linking number with K and therefore one cannot construct coverings of the corresponding 4-manifolds.

7.3 Surgery description of transverse knots

By writing a given transverse knot T as the transverse push-off of a Legendrian knot, one also gets a surgery description of the transverse knot T.

Theorem 7.3.1 (Surgery description of transverse knots)
Any transverse knot T in a contact 3-manifold (M, ξ) admits a contact surgery description along a Legendrian link L to and from a transverse unknot U in (S^3, ξ_{st}).

Proof.
Again by the same argument from Section 4.7 it is enough to consider transverse knots in (S^3, ξ_{st}).

First one approximates both sides of the Legendrian knot K in Figure 5.3 by a transverse-push off and obtains Figure 7.8. (And the same picture with all orientations reversed.)

And since every transverse knot is obtained as the push-off of a Legendrian knot one can change arbitrary crossings of this transverse knots similar to Figure 7.5. By changing the correct crossings this yields a transverse unknot. □

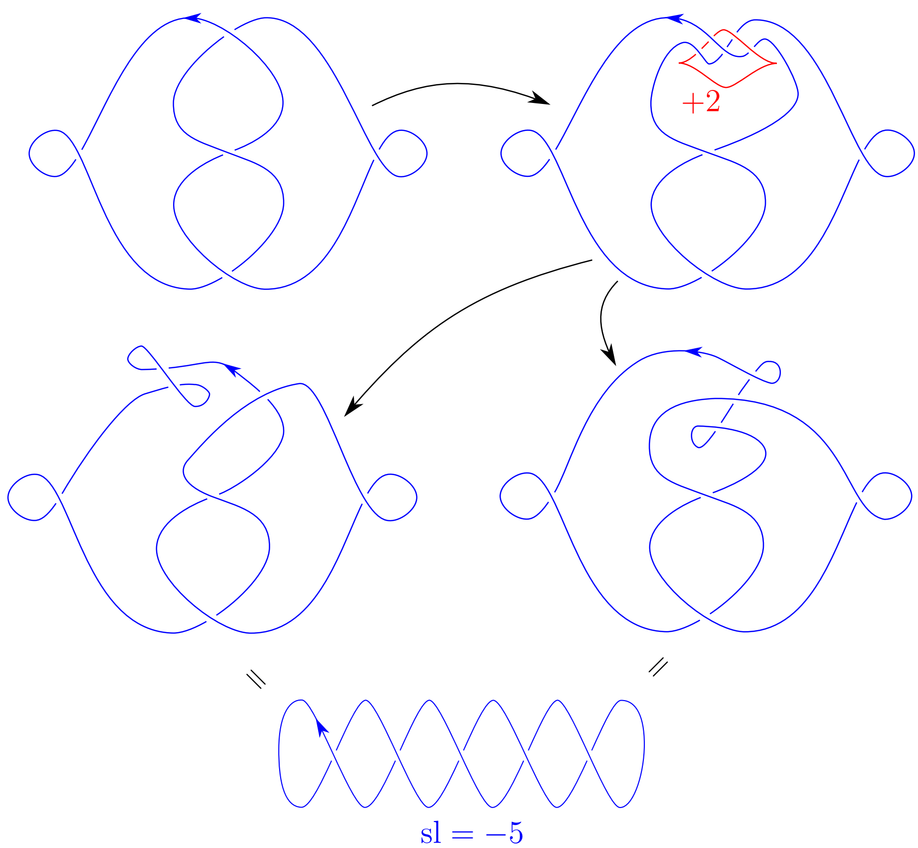


Figure 7.7: A surgery description of a transverse trefoil

Example 7.3.2 (Surgery description of a transverse trefoil)
Consider the front projection of a left-handed transverse trefoil in Figure 7.7. By inserting a cosmetic contact $(+2)$-surgery one can change a crossing such that this transverse trefoil transforms to a transverse unknot. There are again two possibilities for doing it, corresponding to the two possibilities of doing the contact $(+2)$-surgery. But the self-linking number in both cases is equal to -5. So by Theorem 2.3.11 both knots represent the same transverse unknot.

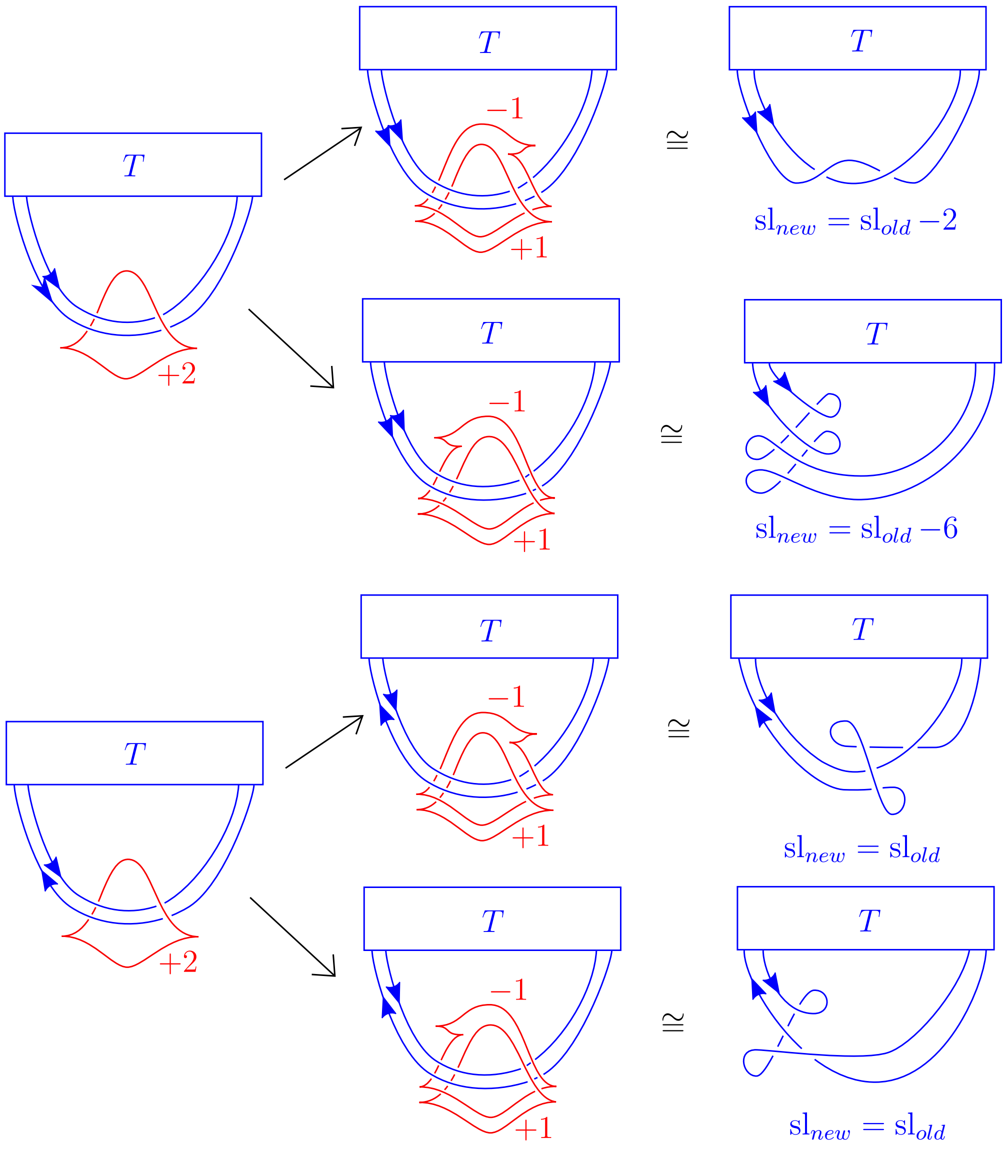


Figure 7.8: The effect of a contact Rolfsen twist on a transverse knot

With this theorem it should again be possible to explicitly construct contact surgery diagrams of coverings of the knot exteriors of transverse knots. And since there exist contact branched covers along transverse knots, it should also be possible to construct contact surgery diagrams of these (similar to the topological case, see [70, Section 22]).

7.4 Connection to the unknotting information

By taking a closer look at how the classical invariants sl of a transverse knot and tb and rot of a Legendrian knot in (S^3, ξ_{st}) change under a cosmetic contact surgery one can get restrictions on these contact invariants from the topological unknotting information of the knot.

The result depends on the sign of the crossing. In Figure 7.8 one can compute how the self-linking number of the transverse knot changes under a cosmetic contact (+2)-surgery that changes a crossing. The self-linking number does not change if one changes a positive to a negative crossing. If one changes a negative to a positive crossing the self-linking numbers reduces by 2 or 6 (depending on the chosen extension of the contact structure over the new glued-in solid tori).

In particular it is always possible to change crossings such that the following inequality is true

$$\mathrm{sl}_{new} \geq \mathrm{sl}_{old} - 2,$$

and after several crossing changes one gets a transverse unknot in (S^3, ξ_{st}) satisfying the Bennequin inequality

$$\mathrm{sl}_{new} \leq -1.$$

So one can hope to get new bounds for the classical invariants of a transverse knot in (S^3, ξ_{st}) in terms of the unknotting information of the topological knot type. But simply taking the **unknotting number** u, i.e. the minimal number of self-crossings required to transform a given knot to the unknot, does not lead to something new by the **slice Bennequin inequality** $\mathrm{sl} \leq 2g_* - 1$ (see [76]) and by $g_* \leq u$ where g_* denotes the slice 4-ball genus. (The inequality $g_* \leq u$ follows since the trace of a movie of finitely many crossing changes of a knot yields an immersed annulus in $S^3 \times I$. By resolving the self-intersections and by capping of the unknot component with a disc one gets a surface with genus u embedded in the 4-ball.)

Therefore, consider the so-called signed unknotting numbers defined as follows (see also [10] and [81]). First one defines the set $U(p, n)$ to consist of equivalence classes of (topological) knots in S^3 that can be unknotted by $p \in \mathbb{N}_0 \cup \{\infty\}$ crossing changes from a + crossing to a − crossing, and by $n \in \mathbb{N}_0 \cup \{\infty\}$ crossing changes from a − crossing to a + crossing. Then one can define the **positive unknotting number** $u_+(K)$ of a (topological) knot K in S^3 as

$$u_+(K) = \min_n \left\{ p \,\middle|\, K \in U(p, n) \right\}.$$

It follows immediately that $u_+ \leq u$, but in general both u_+ and u are hard to compute (see for example [10] and [81]).

In [8] Bohr shows that $\mathrm{sl} \leq \kappa_+$ where κ_+ is the so-called positive kinkiness of a knot K in S^3, which is smaller than the positive unknotting number u_+ of K. In particular, the positive unknotting Bennequin inequality $\mathrm{sl} \leq u_+$ is true. Here we get a weaker result.

Corollary 7.4.1 (Positive unknotting Bennequin inequality)
(1) Let K be a topological knot type in S^3 that admits a transverse realization in (S^3, ξ_{st}) such that in its front projection the maximal self-linking number SL *and the minimal positive unknotting number u_+ are realized. Then for all transverse realizations T_K and all Legendrian realizations L_K of K in (S^3, ξ_{st}) the following Bennequin-type inequalities hold:*

$$\mathrm{sl}(T_K) \leq 2u_+(K) - 1,$$
$$\mathrm{tb}(L_K) + |\operatorname{rot}(L_K)| \leq 2u_+(K) - 1.$$

(2) Let K be a topological knot type in S^3 that admits a Legendrian realization in (S^3, ξ_{st}) such that in its front projection the maximal Thurston–Bennequin invariant TB *and the minimal positive unknotting number u_+ are realized. Then for all Legendrian realizations L_K of K in (S^3, ξ_{st}) the following Bennequin-type inequality holds:*

$$\mathrm{tb}(L_K) \leq 4u_+(K) - 1.$$

Proof.
Take a front projection T_K (as in part (1)) with maximal self-linking number under all transverse realizations of K and change this knot by inserting cosmetic contact $(+2)$-surgeries to a transverse unknot U. By choosing the crossing changes optimal one obtains for the self-linking number the following inequality

$$\mathrm{sl}(T_K) - 2u_+(T_K) = \mathrm{sl}(U) \leq -1.$$

Because $\mathrm{sl}(T_K)$ is maximal and $u_+(T_K)$ is minimal the result follows.

The second inequality in (1) follows again because every transverse knot T is the push-off of a Legendrian knot L (see Remark 2.2.16) and by Lemma 2.3.8.

Part (2) follows by analyzing directly the classical invariants of a Legendrian knot (see Figure 7.9) and by using Theorem 7.2.1. □

Before saying something about the hypothesis of this corollary I want to present some examples and compare these inequalities with other known inequalities (see the introduction of [63] for a discussion of known bounds).

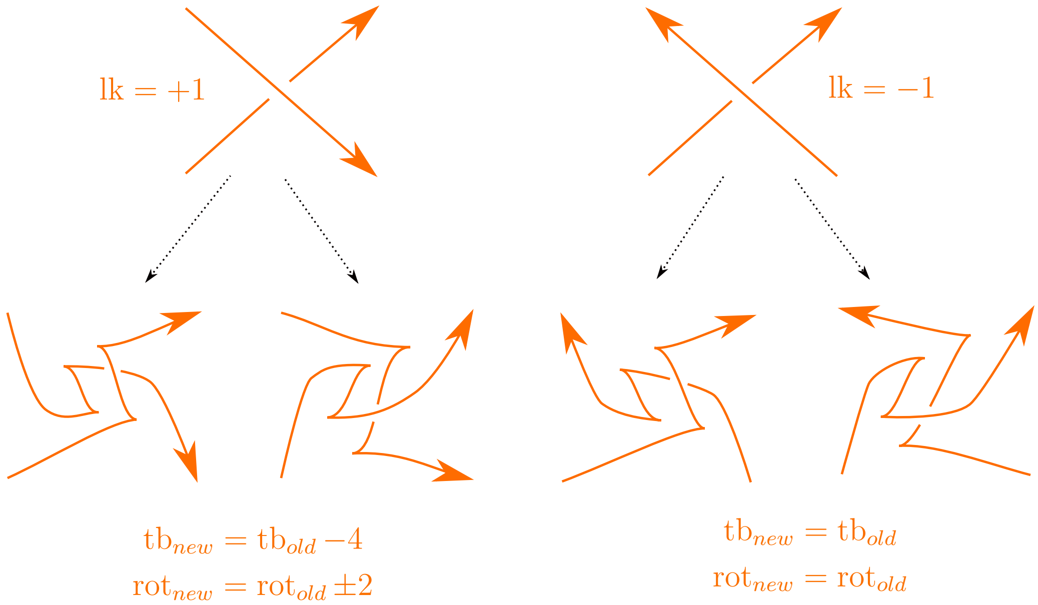


Figure 7.9: Behavior of the classical invariants under cosmetic contact surgeries

Example 7.4.2 (The right-handed trefoil)
Consider the front projection of a Legendrian right-handed trefoil R in Figure 7.5 on the left. By using the standard formulas one gets $\text{tb}(R) = 1$ and $\text{rot}(R) = 0$, and the Seifert algorithm leads to $g(R) = 1$. So the Bennequin bound is sharp in this case and $\text{TB}(R) = 1$. From the slice Bennequin inequality it follows that the slice genus g_* is also 1. From Figure 7.5 one sees that changing one crossing of R yields an unknot. The crossings are all positive, so $u_+(R) \leq 1$. In fact one can show that $u_+(R) = 1$ (see [81, Section 5]). Then it follows that this front projection is of the form as required in the corollary and the positive unknotting bound $2u_+(R)-1 = -1$ is also sharp.

Example 7.4.3 (The left-handed trefoil)
In Figure 7.10 one sees a front projection of a Legendrian left-handed trefoil L with $\text{tb}(L) = -6$ and $|\text{rot}(L)| = 1$. Because the genus and the slice genus do not change under mirroring a knot they are both 1 again and the Bennequin and slice Bennequin bound are both given by 1.

In this example the Kauffman bound $\text{tb}(L) + |\text{rot}(L)| \leq d_{\text{Kauffman}} - 1 = -5$ (see [75]) and the HOMFLY bound $\text{tb}(L) \leq d_{\text{HOMFLY}} - 1 = -6$ (see [32] and [60]) are both sharp. In Figure 7.10 one can change some negative crossing to unknot L, so $u_+(L) = 0$. So the front projection in Figure 7.10 is of the form as in the corollary and the positive unknotting bound $2u_+(L) - 1 = -1$ is stronger than the slice Bennequin bound, but not sharp. Observe that the positive unknotting bound is (like the slice Bennequin and the Bennequin bound) at least -1, so for a general knot far away from being sharp.

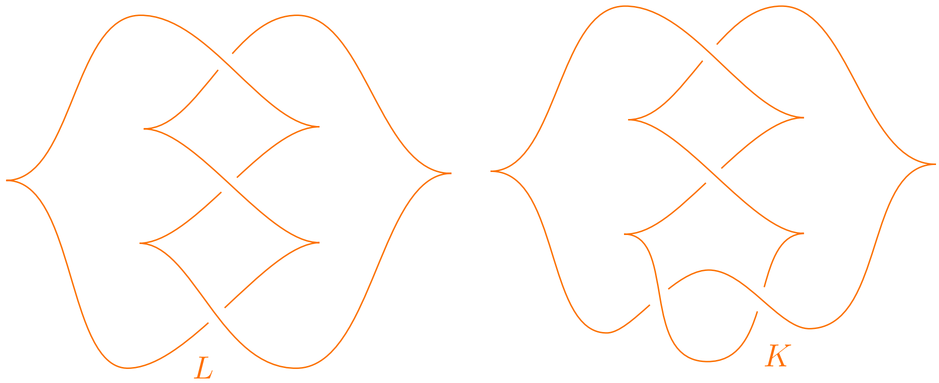


Figure 7.10: A Legendrian left-handed trefoil L and a Legendrian figure eight K

Example 7.4.4 (The figure eight)
A front projection of a Legendrian figure eight knot K with classical invariants $\mathrm{tb}(K) = 3$ and $\mathrm{rot}(K) = 0$ is shown in Figure 7.10. Both the Kauffman- and the HOMFLY bound are -3. So the other bounds are not sharp. Again it follows that $g(K) = g_*(K) = 1$, but $u_+(K) = 0$, so the positive unknotting bound is again better, but not sharp.

Observe, that by taking the transverse push-offs of these Legendrian knots, this also gives examples for transverse knots (with the self-linking number bounded instead of the sum $\mathrm{tb} + \mathrm{rot}$).

Some natural questions arise.

Question 7.4.5
For every (topological) knot, does there exist a front projection of the form as in the corollary? Does this hold if one replaces u_+ by the unknotting number u, the minimal crossing number c,...?

For many examples of knots with small crossing numbers this question admits an affirmative answer. Another indication that Question 7.4.5 admits an affirmative answer is the fact that if the so-called arc index of a front projection is minimal then the Thurston–Bennequin invariant in the same diagram is maximal (see [20, Corollary 3]). So it seems that a front projection that is in some way as simple as possible (here arc index minimal) also realizes the maximal Thurston–Bennequin invariant of that knot.

Also there is the well known example of the alternating knot $\overline{10.8}$ with $c = 10$ and $u = 2$. In [6] it is shown that in every knot diagram of $\overline{10.8}$ with $c = 10$ one needs more than 2 crossing changes to unknot this knot. In Figure 7.11(i) one sees a projection of the knot with 10 crossings and in (ii) a projection with 14 crossings. Changing the marked crossings yields the unknot, and one can show that this is not possible with fewer crossings changes. One can also look at the signed unknotting number and conclude then that $u_+(\overline{10.8}) = 0$.

However, there exist corresponding front projections depicted in Figure 7.11 both with $\mathrm{tb} = -11$. But the diagrams represent different Legendrian knots as one can see by computing the rotation numbers. (The rotation number in (i) is 4 and in (ii) it is 0.) Both the Kauffman and the HOMFLY bound lead to a sharp result. So the maximal Thurston–Bennequin invariant is -11 and the maximum value of $\mathrm{tb} + |\,\mathrm{rot}\,|$ is -7. So for this special knot there exists a front projection with maximal Thurston–Bennequin invariant and minimal positive unknotting number, but it is unclear if the same exists for the maximal value of $\mathrm{tb} + |\,\mathrm{rot}\,|$.

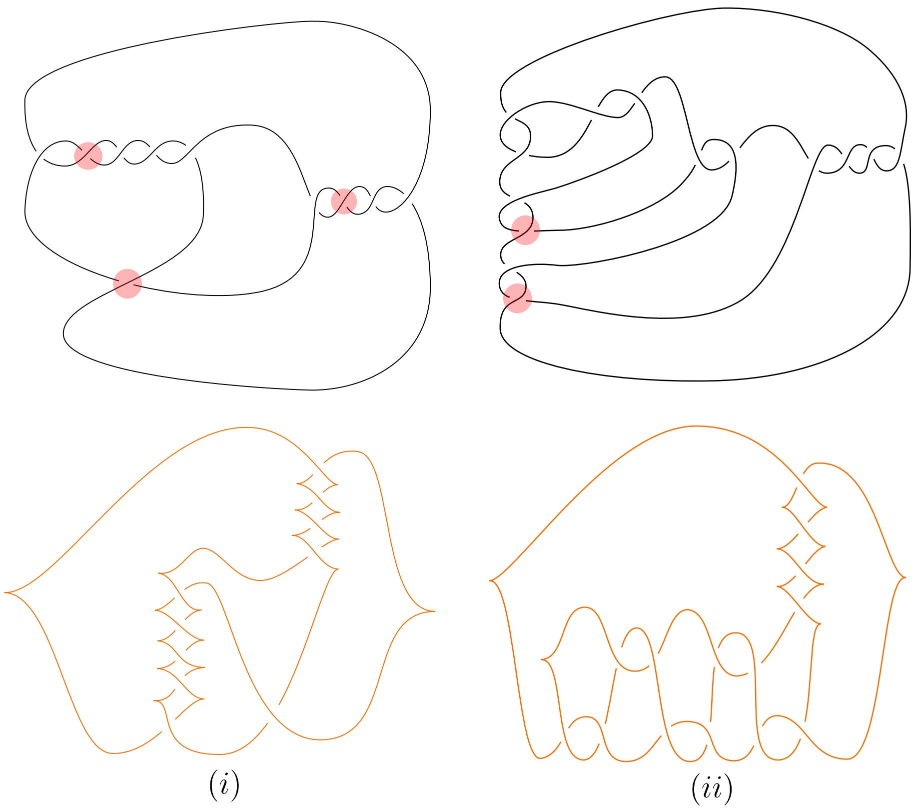


Figure 7.11: Projections of the knot $\overline{10.8}$ with extremal invariants

If the above question admits an affirmative answer, then this would yield another proof of the positive unknotting Bennequin inequality independent of the proof given in [8]. I expect the following two questions to have a negative answer.

Question 7.4.6
Are there examples where the inequalities from the corollary are sharp?

Question 7.4.7
Are there examples where the positive unknotting bound is smaller than the Kauffman or HOMFLY bound?

Acknowledgments

First of all, I would like to thank my advisor Prof. Hansjörg Geiges for the helpful discussions about mathematics and for his excellent lectures.

Special thanks also go to Maryam Beygmohammadi, Sebastian Durst and Christian Evers for proofreading this thesis and many useful suggestions.

Further thanks go to Sebastian Durst and Mirko Klukas for collaboration in our joint projects.

Moreover, I thank all my colleagues in the working group *Symplectic Topology* and in the *Graduiertenkolleg* and all others with whom I had interesting discussions about mathematics.

Funding

This thesis was partially supported by the *DFG-Graduiertenkolleg Globale Strukturen in Geometrie und Analysis*, Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50932 Köln, Germany.

Bibliography

[1] R. Avdek, Contact surgery and supporting open books, *Algebr. Geom. Topol.* **13** (2013), 1613–1660.

[2] K. Baker and J. Etnyre, Rational linking and contact geometry, in: *Perspectives in Analysis, Geometry, and Topology*, Progr. Math. **296**, Birkhäuser Verlag, Basel (2012), 19–37.

[3] K. Baker and J. Grigsby, Grid diagrams and Legendrian lens space links, *J. Symplectic Geom.* **7** (2009), 415–448.

[4] J. Baldwin and J. Etnyre, Admissible transverse surgery does not preserve tightness, *Math. Ann.* **37** (2013), 441–468.

[5] J. Berge, The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$, *Topology Appl.* **38** (1991), 1–19.

[6] S. Bleiler, A note on unknotting number, *Math. Proc. Cambridge Philos. Soc.* **96** (1984), 469–471.

[7] S. Bleiler, C. Hodgson and J. Weeks, Cosmetic surgery on knots, in: *Proceedings of the Kirbyfest*, Geometry and Topology Monographs **2**, Coventry (1999), 23–34.

[8] C. Bohr, On the kinkiness of closed braids, *J. Knot Theory Ramifications* **11** (2002), 1155–1163.

[9] Y. Chekanov, Differential algebra of Legendrian links, *Invent. Math.* **150** (2002), 441–483.

[10] T. Cochran and W. Lickorish, Unknotting information from 4-manifolds, *Trans. Amer. Math. Soc.* **297** (1986), 125–142.

[11] J. Conway, Transverse surgery on knots in contact 3-manifolds, `arXiv:1409.7077`.

[12] M. Dehn, Über die Topologie des dreidimensionalen Raumes, *Math. Ann.* **69** (1910), 137–168.

[13] F. Ding and H. Geiges, Symplectic fillability of tight contact structures on torus bundles, *Algebr. Geom. Topol.* **1** (2001), 153–172.

[14] F. Ding and H. Geiges, A Legendrian surgery presentation of contact 3-manifolds, *Math. Proc. Cambridge Philos. Soc.* **136** (2004), 583–598.

[15] F. Ding and H. Geiges, Handle moves in contact surgery diagrams, *J. Topol.* **2** (2009), 105–122.

[16] F. Ding and H. Geiges, The diffeotopy group of $S^1 \times S^2$ via contact topology, *Compos. Math.* **146** (2010), 1096–1112.

[17] F. Ding, H. Geiges and A. Stipsicz, Surgery diagrams for contact 3-manifolds, *Turkish J. Math.* **28** (2004), 41–74.

[18] S. Durst and M. Kegel, Computing rotation and self-linking numbers in contact surgery diagrams, *Acta Math. Hungar.* **150** (2016), 524–540.

[19] S. Durst, M. Kegel and M. Klukas, Computing the Thurston–Bennequin invariant in open books, *Acta Math. Hungar.* **150** (2016), 441–455.

[20] I. Dynnikov and M. Prasolov, Bypasses for rectangular diagrams. A proof of the Jones conjecture and related questions, *Trans. Moscow Math. Soc.* (2013), 97–144.

[21] I. Dynnikov, Recognition algorithms in knot theory, *Uspekhi Mat. Nauk* (2003), 45–92.

[22] J. Edwards, Concentricity in 3-manifolds, *Trans. Amer. Math. Soc.* **113** (1964), 406–423.

[23] Y. Eliashberg, New invariants of open symplectic and contact manifolds, *J. Amer. Math. Soc.* **4** (1991), 513–520.

[24] Y. Eliashberg, Legendrian and transverse knots in tight contact 3-manifolds, in: *Topological Methods in Modern Mathematics* (Stony Brook, 1991), Publish or Perish, Houston (1993), 171–193.

[25] Y. Eliashberg, Classification of contact structures on $\mathbb{R}^3$, *Internat. Math. Res. Notices* **3** (1993), 87–91.

[26] Y. Eliashberg and M. Fraser, Topologically trivial Legendrian knots, *J. Symplectic Geom.* **7** (2009), 77–127.

[27] J. Etnyre, Legendrian and transversal knots, in: *Handbook of Knot Theory* (W. Menasco and M. Thistlethwaite, eds.), Elsevier, Amsterdam (2005), 105–185

[28] J. Etnyre and R. Ghrist, Tight contact structures via dynamics, *Proc. Amer. Math. Soc.* **127** (1999), 3697–3706.

[29] J. Etnyre and K. Honda, Knots and contact geometry I: torus knots and the figure eight knot, *J. Symplectic Geom.* **1** (2001), 63–120.

[30] J. Etnyre and K. Honda, On the non-existence of tight contact structures, *Ann. of Math. (2)* **153** (2001), 749–766.

[31] E. Ferrand, On Legendrian knots and polynomial invariants, *Proc. Amer. Math. Soc.* **130** (2002), 1169–1176.

[32] J. Franks and R. Williams, Braids and the Jones polynomial, *Trans. Amer. Math. Soc.* **303** (1987), 97–108.

[33] D. Gabai, Foliations and the topology of 3-manifolds II, *J. Differential Geom.* **26** (1987), 461–478.

[34] D. Gabai, Surgery on knots in solid tori, *Topology* **28** (1989), 1–6.

[35] F. Gainullin, Heegaard Floer homology and knots determined by their complements, `arXiv:1504.06180`.

[36] D. Gay, Symplectic 4-dimensional 2-handles and contact surgery along transverse knots, Ph.D. Thesis, UC Berkeley (1999).

[37] D. Gay, Symplectic 2-handles and transverse links, *Trans. Amer. Math. Soc.* **354** (2002), 1027–1047.

[38] H. Geiges, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math. **109** (Cambridge University Press, 2008).

[39] H. Geiges and S. Onaran, Legendrian rational unknots in lens spaces, *J. Symplectic Geom.* **13** (2015), 17–50.

[40] R. Gompf, Handlebody construction of Stein surfaces, *Ann. of Math. (2)* **148** (1998), 619–693.

[41] R. Gompf and A. Stipsicz, 4-*Manifolds and Kirby Calculus*, Grad. Stud. Math. **20** (American Mathematical Society, Providence, RI, 1999).

[42] C. Gordon, Links and their complements, in: *Topology and Geometry: commemorating SISTAG*, Contemp. Math. **314** (American Mathematical Society, 2002), 71–82.

[43] C. Gordon and J. Luecke, Knots are determined by their complements, *J. Amer. Math. Soc.* **2** (1989), 371–415.

[44] W. Haken, Theorie der Normalflächen, *Acta Math.* **105** (1961), 245–375.

[45] N. Hoffman and G. Walsh, The big Dehn surgery graph and the link of S^3, *Proc. Amer. Math. Soc. Ser. B* **2** (2015), 17–34.

[46] K. Honda, On the classification of tight contact structures I, *Geom. Topol.* **4** (2000), 309–368.

[47] K. Honda, W. Kazez and G. Matić, Right-veering diffeomorphisms of compact surfaces with boundary, *Invent. Math.* **169** (2007), 427–449.

[48] K. Ichihara, On tunnel number one links with surgeries yielding the 3-sphere, *Osaka J. Math.* **47** (2010), 189–208.

[49] K. Ichihara and I. Jong, Cosmetic banding on knots and links, `arXiv:1602.01542`.

[50] Y. Kanda, The classification of tight contact structures on the 3-torus, *Comm. Anal. Geom.* **5** (1997), 413–438.

[51] Y. Kanda, On the Thurston–Bennequin invariant of Legendrian knots and non exactness of Bennequin's inequality, *Invent. Math.* **133** (1998), 227–242.

[52] M. Kegel, Kontakt-Dehn-Chirurgie entlang Legendre-Knoten, Bachelorarbeit, Universität zu Köln (2011), available online:
`http://www.mi.uni-koeln.de/~mkegel/downloads.html`

[53] M. Kegel, Symplektisches Füllen von Torusbündeln, Masterarbeit, Universität zu Köln (2014), available online:
`http://www.mi.uni-koeln.de/~mkegel/downloads.html`

[54] M. Kegel, The Legendrian knot complement problem, `arXiv:1604.05196`.

[55] P. Lisca, P. Ozsváth, A. Stipsicz and Z. Szabó, Heegaard Floer invariants of Legendrian knots in contact three-manifolds, *J. Eur. Math. Soc.* (JEMS) **11** (2009), 1307–1363.

[56] P. Lisca and A. Stipsicz, Contact surgery and transverse invariants, *J. Topol.* **4** (2011), 817–834.

[57] S. Makar-Limanov, Tight contact structures on solid tori, *Trans. Amer. Math. Soc.* **350** (1998), 1013–1044.

[58] Y. Mathieu, Closed 3-manifolds unchanged by Dehn surgery, *J. Knot Theory Ramifications* **1** (1992), 279–296.

[59] S. Matveev, *Algorithmic topology and classification of* 3*-manifolds*, Algorithms and Computation in Mathematics **9** (Springer-Verlag, Berlin, 2003).

[60] D. Matignon, On the knot complement problem for non-hyperbolic knots, *Topology Appl.* **157** (2010), 1900–1925.

[61] H. Matsuda, M. Ozawa and K. Shimokawa, On non-simple reflexive links, *J. Knot Theory Ramifications* **11** (2002), 787–791.

[62] H. Morton, Seifert circles and knot polynomials, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 107–109.

[63] L. Ng, A Legendrian Thurston–Bennequin bound from Khovanov homology, *Algebr. Geom. Topol.* **5** (2005), 1637–1653.

[64] K. Niederkrüger and F. Presas, Some remarks on the size of tubular neighborhoods in contact topology and fillability, *Geom. Topol.* **14** (2010), 719–754.

[65] B. Özbağci, A note on contact surgery diagrams, *Internat. J. Math.* **16** (2005), 87–99.

[66] B. Özbağci, An open book decomposition compatible with rational contact surgery, *Proceedings of Gökova Geometry-Topology Conference* (2005), 175–186.

[67] B. Özbağci and A. Stipsicz, *Surgery on Contact* 3*-Manifolds and Stein Surfaces*, Bolyai Society Mathematical Studies **13** (Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004).

[68] P. Ozsváth and A. Stipsicz, Contact surgeries and the transverse invariant in knot Floer homology, *J. Inst. Math. Jussieu* **9** (2010), 601–632.

[69] P. Ozsváth, A. Stipsicz and Z. Szabó, *Grid homology for knots and links*, Mathematical Surveys and Monographs **208** (American Mathematical Society, Providence, 2015).

[70] V. Prasolov and A. Sossinsky, *Knots, Links, Braids and* 3*-Manifolds*, Transl. Math. Monogr. **154** (American Mathematical Society, Providence, 1997).

[71] H. Ravelomanana, Exceptional Cosmetic surgeries on S^3, `arXiv:1505.00238`.

[72] H. Ravelomanana, Knot Complement Problem for L-space $\mathbb{Z}HS^3$, `arXiv:1505.00239`.

[73] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series **7** (Publish or Perish, Berkeley, 1976).

[74] Y. Rong, Some knots not determined by their complements, in: *Quantum topology*, Ser. Knots Everything **3**, (World Sci. Publishing, River Edge, NJ, 1993), 339–353.

[75] L. Rudolph, A congruence between link polynomials, *Math. Proc. Camb. Philos. Soc.* **107** (1990), 319–327.

[76] L. Rudolph, Quasipositivity as an obstruction to sliceness, *Bull. Amer. Math. Soc.* (N.S.) **29** (1993), 51–59.

[77] T. Tanaka, Maximal Thurston–Bennequin numbers of alternating links, *Topology Appl.* **153** (2006), 2476–2483.

[78] M. Teragaito, Links with surgery yielding the 3-sphere, *J. Knot Theory Ramifications* **11** (2002), 105–108.

[79] W. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 357–381.

[80] H. Tietze, Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten, *Monatsh. Math. Phys.* **19** (1908), 1–118.

[81] P. Traczyk, A criterion for signed unknotting number, *Contemp. Math.* **233** (1999), 212–220.

[82] J. Tripp, Contact structures on open 3-manifolds, *J. Symplectic Geom.* **4** (2006), 93–116.

[83] A. Wand, Tightness is preserved by Legendrian surgery, *Ann. of Math. (2)* **182** (2015), 723–738.

[84] J. Wang, Cosmetic surgeries on genus one knots, *Algebr. Geom. Topol.* **6** (2006), 1491–1517.

[85] J. Whitehead, On doubled knots, *J. London Math. Soc.* **12** (1937), 63–71.

[86] W. Whitten, MathSciNet review of *Nœuds qui ne sont pas déterminés par leur complément dans les* 3-*variétés à bord* by M. Domergue and Y. Mathieu, MR1125670.

[87] Y. Yokota, The Kauffman polynomial of alternating links, *Topology Appl.* **65** (1995), 229–236.

Erklärung gemäß §4 der Promotionsordung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Hansjörg Geiges betreut worden.

Teilpublikationen

1. S. Durst and M. Kegel, Computing rotation and self-linking numbers in contact surgery diagrams, *Acta Math. Hungar.* **150** (2016), 524–540.
2. S. Durst, M. Kegel and M. Klukas, Computing the Thurston–Bennequin invariant in open books, *Acta Math. Hungar.* **150** (2016), 441–455.
3. M. Kegel, The Legendrian knot complement problem, submitted.

Köln, 19. Januar 2017 Marc Kegel