FORMULAS FOR NEGATIVE CONTINUED FRACTION EXPANSIONS

MARC KEGEL

ABSTRACT. This document collects formulas for negative continued fraction expansions.

Let $r \in \mathbb{Q}$ be a negative rational number. Then there exist a unique way to write r as

$$r = [r_1, \dots, r_n] := r_1 - \cfrac{1}{r_2 - \cfrac{1}{\cdots - \cfrac{1}{r_n}}}$$

with integers $r_1 \leq -1$ and $r_2, \dots, r_n \leq -2$. $[r_1, \dots, r_n]$ is called the **(negative) continued fraction expansion** of r.

For a given rational number its continued fraction expansion can be constructed algorithmically by a slight variation of the Euclidean algorithm. This algorithm is best understood in the following example for $r = -\frac{17}{10}$.

$$\begin{aligned} -17 &= -2 \cdot 10 + 3 \\ -10 &= -4 \cdot 3 + 2 \\ -3 &= -2 \cdot 2 + 1 \\ -2 &= -2 \cdot 1 + 0 \end{aligned}$$

$$\begin{aligned} -\frac{17}{10} &= -2 + 2 - \frac{17}{10} = -2 - \cfrac{1}{-\cfrac{10}{3}} = -2 - \cfrac{1}{-4 + 4 - \cfrac{10}{3}} \\ &= -2 - \cfrac{1}{-4 - \cfrac{1}{-\cfrac{3}{2}}} = -2 - \cfrac{1}{-4 - \cfrac{1}{-2 - \cfrac{1}{-2}}} \end{aligned}$$

Continued fractions appear for example in the transformation lemma in the study of contact surgery. That is why I am mainly interested in continued fractions.

All formulas below where obtained by using the above mentioned algorithm and can be proven easily by induction.

To find the correct pattern it is also often helpful to perform some computer experiments, which can be also used to confirm the below formulas for finitely

Date: January 22, 2022.

Key words and phrases. Negative continued fraction expansions, contact surgery.

many values. My python code for getting negative continued fraction expansions is here:

```
def negative_continued_fraction_expansion(p,q):
    '''
    Creates the ncfe of p/q, where p is negative and q positive.
    '''
    cont_fract=[]
    while q!=0:
        cont_fract.append(floor(p/q))
        (p,q)=(-q,p % q)
    return cont_fract
```

1. The formulas

All variables represent integers.

For $n \geq 1$ we have

$$-\frac{n}{n-1} = [\underbrace{-2,\dots,-2}_{n-1}]. \tag{1}$$

For $n \geq 1$ we have

$$-\frac{n}{n+1} = [-1,-(n+1)]. \tag{2}$$

For $t \leq -2$ and $q \leq -1$ we have

$$-\frac{qt-1}{q(t+1)-1} = \big[\underbrace{-2,\dots,-2}_{-t-2},-3,\underbrace{-2,\dots,-2}_{-q-2}\big]. \tag{3}$$

For $t \leq -1$ and $q \geq 1$ we have

$$-\frac{qt-1}{q(t+1)-1} = \big[\underbrace{-2,\dots,-2}_{-t-1},-q-1\big]. \tag{4}$$

For $t \leq -1$ and $k \geq 1$ we have

$$-\frac{k(2-t)+1}{k(1-t)+1} = \big[\underbrace{-2,\dots,-2}_{1-t},-k-1\big]. \tag{5}$$

For $u \leq -1$ and $k \geq 1$ we have

$$-\frac{-2k-u(k+1)+1}{k-u(k+1)} = \big[\underbrace{-2,\dots,-2}_{-u},-3,\underbrace{-2,\dots,-2}_{k-1}\big]. \tag{6}$$

For $u \leq -1$ and $k \geq 1$ we have

$$-\frac{3k(2-u)-u-1}{3k(1-u)-u-2} = \big[\underbrace{-2,\dots,-2}_{-u},-3,\underbrace{-2,\dots,-2}_{k-2},-4\big]. \tag{7}$$

For $n \geq 1$ and $m \geq 2$ we have

$$-\frac{nm+1}{(n-1)m+1} = \big[\underbrace{-2,\dots,-2}_{n-1},-m-1\big]. \tag{8}$$

For $k \geq 1$ we have

$$-\frac{6k+5}{3k+4} = \big[-2, -k-2, -2, -2\big]. \tag{9}$$

For $k \geq 1$ and $u \leq -3$ we have

$$-\frac{-6k-3ku+1}{-9k-3ku-u} = \big[\underbrace{-2, \dots, -2}_{-u-3}, -k-2, -2, -2\big]. \tag{10}$$

For $k \geq 2$, $n \geq 2$ and $x \leq -1$ we have

$$-\frac{(2n-1)k(2-x)-nx+1}{(2n-1)k(1-x)-n(x+1)+1} = \big[\underbrace{-2, \dots, -2}_{-x}, -3, \underbrace{-2, \dots, -2}_{k-2}, -3, -n\big]. \tag{11}$$

For $k \geq 2$ and $t \leq -1$ we have

$$-\frac{2k-kt-1}{k-kt-1} = \big[\underbrace{-2, \dots, -2}_{-t}, -3, \underbrace{-2, \dots, -2}_{k-2}\big]. \tag{12}$$

For $k \geq 2$ and $u \leq -1$ we have

$$-\frac{2k-ku+u-1}{k-kt+u} = \big[\underbrace{-2, \dots, -2}_{-u+1}, -k\big]. \tag{13}$$

For $p > q > 0$ let

$$-\frac{p}{p-q} = [r_1, \dots, r_n]. \tag{14}$$

Then we have for any $s \geq 0$

$$-\frac{p+qs}{(p+qs)-q} = \big[\underbrace{-2, \dots, -2}_{s}, r_1, \dots, r_n\big]. \tag{15}$$

Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany.
Email address: kegemarc@math.hu-berlin.de, kegelmarc87@gmail.com