New Duality Theorems in Stochastic Programming, Motivated by Financial Applications

Lisa A. Korf

Department of Mathematics, University of Washington

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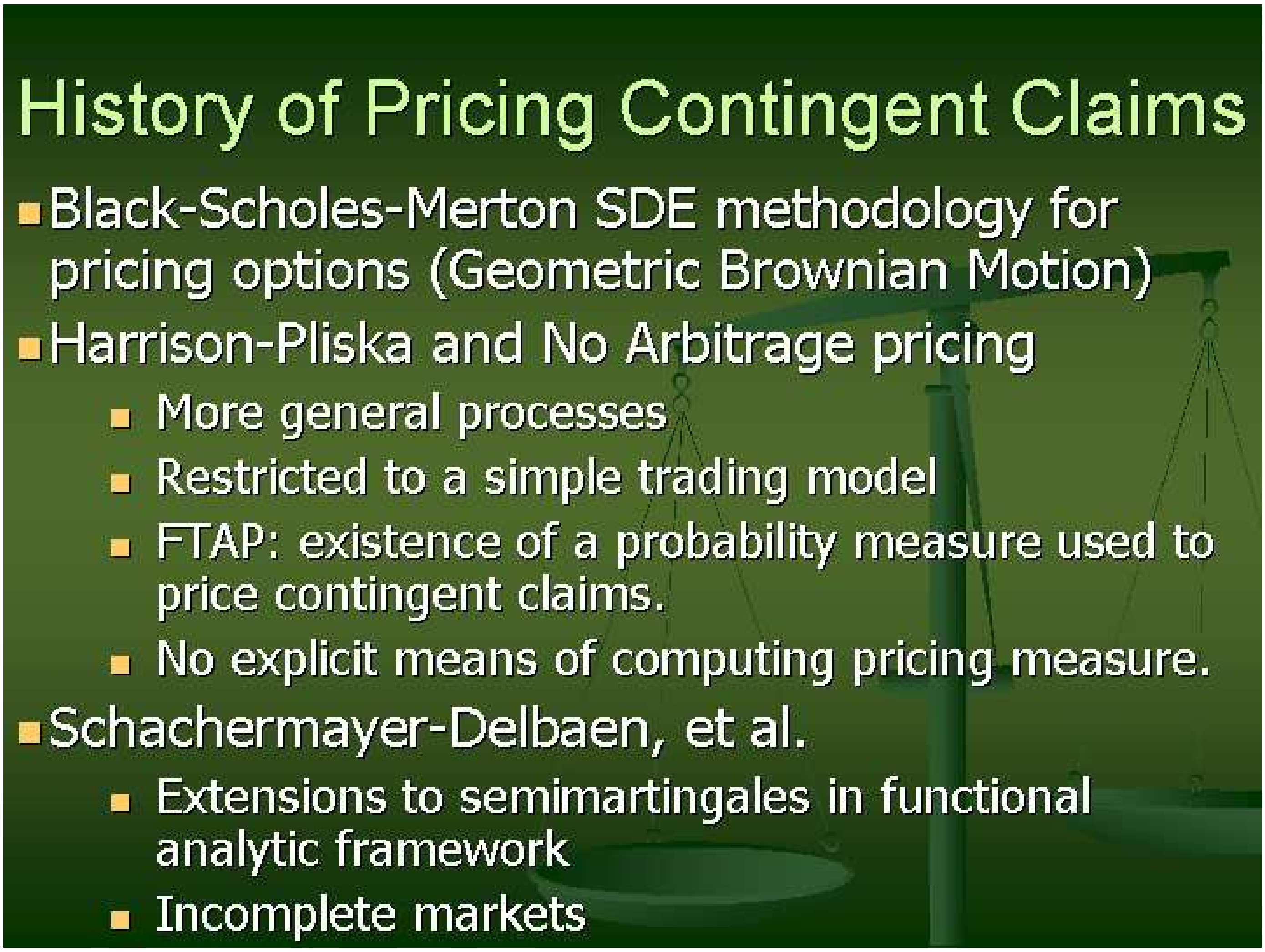
Outline

- Review of Contingent Claim Pricing Theory
- Why Standard S.P. Duality Doesn't Work
- No Arbitrage as a Constraint Qualification
- Geometric Interpretation: Direction-Free Feasibility and New Duality Theorems in S.P.
- Application to Pricing Problems
- Modeling "General" Pricing Problems in Stochastic Programming


History of Pricing Contingent Claims
Black-Scholes-Merton SDE methodology for pricing options (Geometric Brownian Motion)
Harrison-Pliska and No Arbitrage pricing
More general processes
Restricted to a simple trading model
FTAP: existence of a probability measure used to price contingent claims.
No explicit means of computing pricing measure.
Schachermayer-Delbaen, et al.
Extensions to semimartingales in functional analytic framework
Incomplete markets

Arbitrage: The free lunch condition

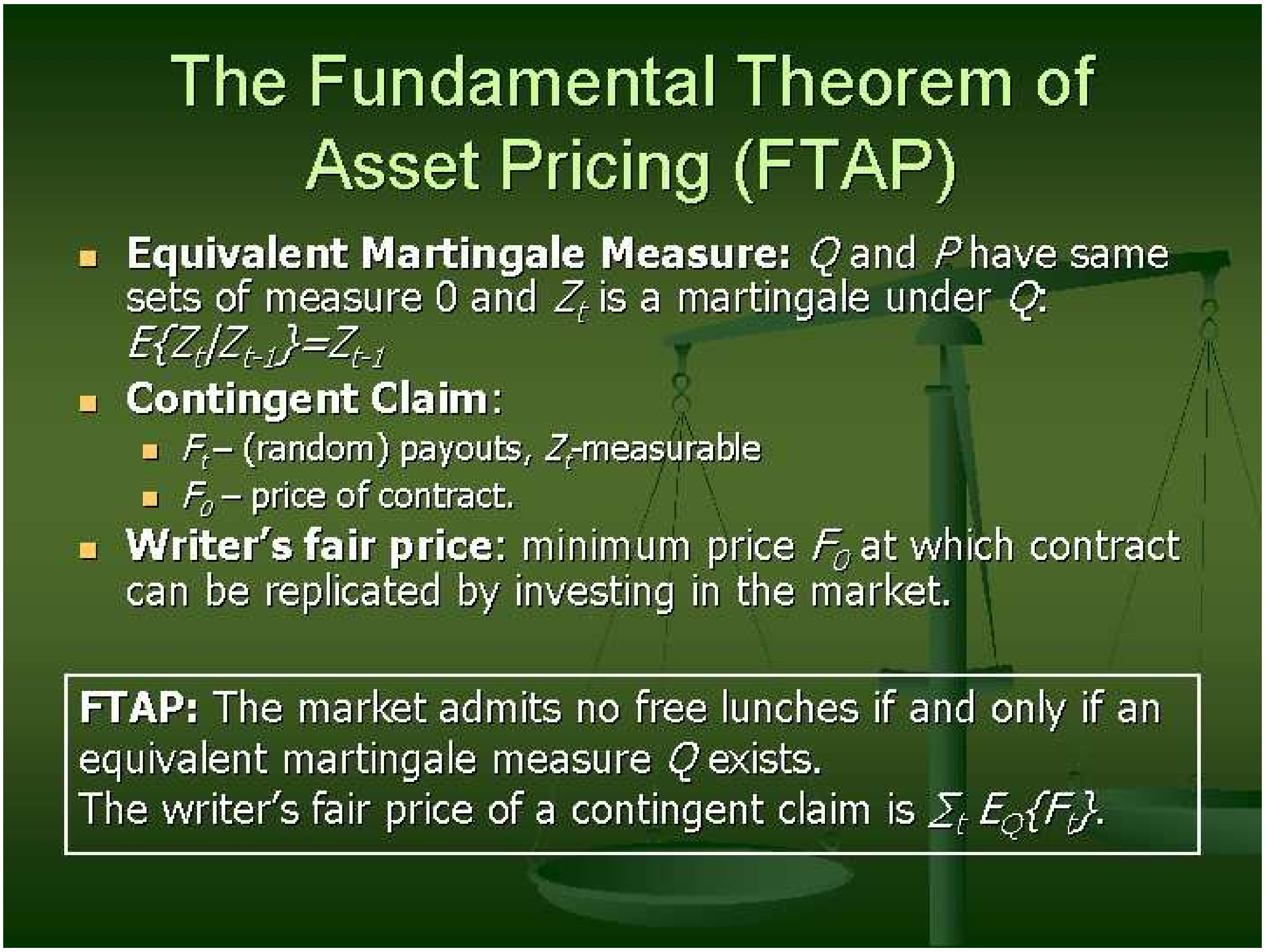
Free lunch: An investment strategy whereby starting with nothing, one is guaranteed a profit by time T, with a positive probability of positive profit.

Z_t - (random) vector of asset prices at time t

θ_t - (random) vector of investments at time t

Free lunch : $\exists\, \theta_t \;\; Z_t$ - measurable s.t.

Start with nothing.	$Z_0 \bullet \theta_0 \le 0$
Self-financing cond'n	$Z_t \bullet \Delta\theta_t \le 0 \quad P$ - a.s.
Guaranteed profit	$Z_T \bullet \theta_T \ge 0 \quad P$ - a.s.
Positive probability of positive profit	$E\{Z_T \bullet \theta_T\} > 0$


The Fundamental Theorem of Asset Pricing (FTAP)
Equivalent Martingale Measure: Q and P have same sets of measure 0 and Z_t is a martingale under Q: $E\{Z_t|Z_{t-1}\}=Z_{t-1}$
Contingent Claim:
F_t – (random) payouts, Z_t-measurable
F_0 – price of contract.
Writer's fair price: minimum price F_0 at which contract can be replicated by investing in the market.
FTAP: The market admits no free lunches if and only if an equivalent martingale measure Q exists.
The writer's fair price of a contingent claim is $\sum_t E_Q\{F_t\}$.

Duality: discrete probability

$$\mathrm{Max}_{\theta} \quad E\{Z_T \bullet \theta_T\}$$
$$\text{s.t.} \quad Z_0 \bullet \theta_0 \le F_0$$
$$Z_t \bullet [\theta_t - \theta_{t-1}] \le -F_t \quad P\text{-a.s.}$$
$$Z_T \bullet \theta_T \ge 0 \quad P\text{-a.s.}$$

Writer's Portfolio Optimization Problem

$$\mathrm{Min}_{y} \quad F_0 y_0 - \sum_{t=1}^{T} E\{F_t y_t\}$$
$$\text{s.t.} \; E\{Z_t y_t \mid Z_{t-1}\} = Z_{t-1} y_{t-1} \; P\text{-a.s.}$$
$$y_t \ge 0, \quad y_T \ge 1 \quad P\text{-a.s.}$$

Dual Portfolio Optimization Problem

Writer's Pricing Problem

$$\mathrm{Min}_{\theta, V} \quad V$$
$$\text{s.t.} \quad Z_0 \bullet \theta_0 - V \le 0$$
$$Z_t \bullet [\theta_t - \theta_{t-1}] \le -F_t \quad P\text{-a.s.}$$
$$Z_T \bullet \theta_T \ge 0 \quad P\text{-a.s.}$$

Dual Pricing Problem

$$\mathrm{Max}_{y} \quad \sum_{t=1}^{T} E\{F_t y_t\}$$
$$\text{s.t.} \; E\{Z_t y_t \mid Z_{t-1}\} = Z_{t-1} y_{t-1} \; P\text{-a.s.}$$
$$y_t \ge 0 \quad P\text{-a.s.}$$

Linear Programming Duality yields pricing results.

S.P. Duality: general probability

To obtain inf P = max D duality in L^∞ / L^1 setting, one needs

- Relatively complete recourse: constraints at a given time period do not impose further constraints on the past
- And, strict feasibility

OR

- Bounded constraint sets for min P = sup D

Pricing Problems Do Not Satisfy This!

Duality: general probability

$$\begin{aligned}
&\mathrm{Max}_{\theta}\ E\{Z_T \bullet \theta_T\}\\
&\text{s.t.}\quad Z_0 \bullet \theta_0 \le F_0\\
&Z_t \bullet [\theta_t - \theta_{t-1}] \le -F_t \quad P\text{-a.s.}\\
&\qquad Z_T \bullet \theta_T \ge 0 \quad P\text{-a.s.}
\end{aligned}$$

Writer's Portfolio Optimization Problem

$$\begin{aligned}
&\mathrm{Min}_{y}\ F_0 y_0 - \sum_{t=1}^{T} E\{F_t y_t\}\\
&\text{s.t. } E\{Z_t y_t \mid Z_{t-1}\} = Z_{t-1} y_{t-1} \ P\text{-a.s.}\\
&\qquad y_t \ge 0,\quad y_T \ge 1 \quad P\text{-a.s.}
\end{aligned}$$

Dual Portfolio Optimization Problem

Writer's Pricing Problem

$$\begin{aligned}
&\mathrm{Min}_{\theta,V}\quad V\\
&\text{s.t.}\quad Z_0 \bullet \theta_0 - V \le 0\\
&Z_t \bullet [\theta_t - \theta_{t-1}] \le -F_t \quad P\text{-a.s.}\\
&\qquad Z_T \bullet \theta_T \ge 0 \quad P\text{-a.s.}
\end{aligned}$$

Dual Pricing Problem

$$\begin{aligned}
&\mathrm{Max}_{y}\quad \sum_{t=1}^{T} E\{F_t y_t\}\\
&\text{s.t. } E\{Z_t y_t \mid Z_{t-1}\} = Z_{t-1} y_{t-1} \ P\text{-a.s.}\\
&\qquad y_t \ge 0 \quad P\text{-a.s.}
\end{aligned}$$

Do not obtain inf P = sup D in the L^{∞} / L^{1} setting!

Why doesn't it work?

- Do get extended duality: $L^\infty / L^{\infty *}$ yielding finitely additive martingale measures (King/Korf)
- But we don't want multipliers in $L^{\infty *}$
- Haven't used "no arbitrage" yet.
- Is there a constraint qualification hidden in the no free lunch condition?

No Arbitrage (No Free Lunch) as Constraint Qualification

No free lunch: $\nexists\, \theta_t \quad Z_t$ - measurable s.t.

$$Z_0 \bullet \theta_0 \le 0$$

$$Z_t \bullet \Delta\theta_t \le 0 \quad P\text{-a.s.}$$

$$Z_T \bullet \theta_T \ge 0 \quad P\text{-a.s.}$$

$$E\{Z_T \bullet \theta_T\} > 0$$

i.e. if $Z_0 \bullet \theta_0 \le 0, Z_t \bullet \Delta\theta_t \le 0 \quad P$-a.s., and $Z_T \bullet \theta_T \ge 0 \quad P$-a.s.,

then $Z_0 \bullet \theta_0 = 0, Z_t \bullet \Delta\theta_t = 0 \quad P$-a.s., and $Z_T \bullet \theta_T = 0 \quad P$-a.s.

Feasible Set under No Arbitrage

Feasible Set:

$$K = \{\theta_0 \mid Z_0 \bullet \theta_0 \le F_0 ,\ Z_T \bullet \theta_0 \ge F_T \ \ P\text{-a.s.}\}$$

Recession Cone of Feasible Set:

$$K^{\infty} = \{\theta_0 \mid Z_0 \bullet \theta_0 \le 0 ,\ Z_T \bullet \theta_0 \ge 0 \ \ P\text{-a.s.}\}$$

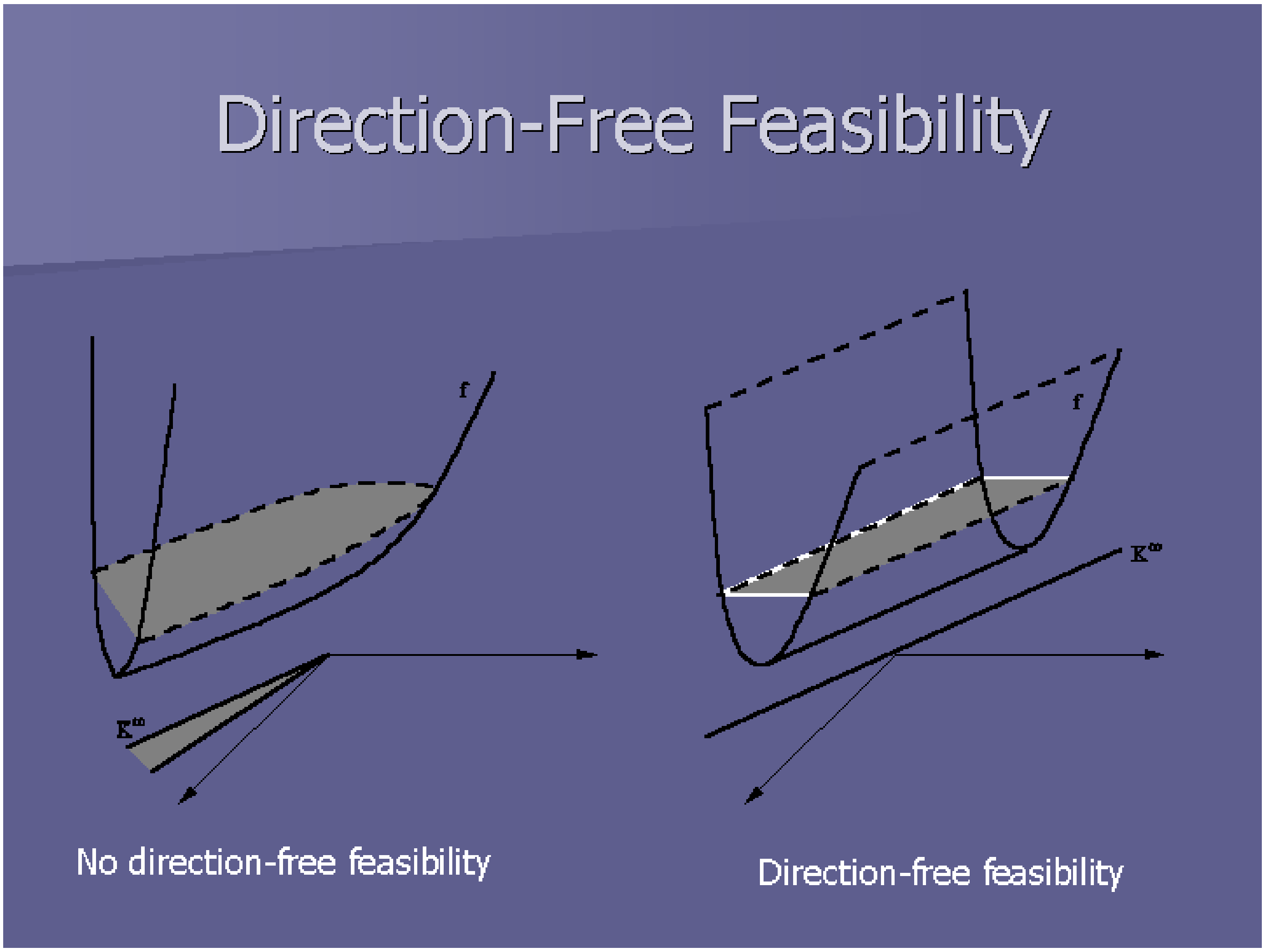
Under no free lunches:

$$K^{\infty} = \{\theta_0 \mid Z_0 \bullet \theta_0 = 0 ,\ Z_T \bullet \theta_0 = 0 \ \ P\text{-a.s.}\}$$

The Convex Stochastic Program and Direction-Free Feasibility

$$
\begin{aligned}
&\mathrm{Min}_{x\in \mathrm{R}^n} \quad f_{10}(x) + E\{f_{20}(\xi, x)\} \\
&\text{s.t.} \quad f_{1i}(x) \le 0 \quad i = 1,\ldots,m_1, \quad x \in C \qquad \textbf{(P)}\\
&\qquad\quad f_{2i}(\xi, x) \le 0 \;\; P\text{-a.s.} \quad i = 1,\ldots,m_2
\end{aligned}
$$

- $K=\{x \in C \mid f_{1i}(x)\le 0,\ f_{2i}(\xi,x)\le 0 \;\; P\text{-a.s.}\}$
- *P* has **direction-free feasibility** if *P is feasible and* $K^{\infty}=\{w \mid x+\lambda w \in K$ for all $\lambda\ge 0,\ x \in K\}$ *is a subspace of* $\boldsymbol{R}^n$


Direction-Free Feasibility
f
K∞
No direction-free feasibility
f
K∞
Direction-free feasibility

New Duality Theorem in S.P.

In an L^1/L^∞ setting…

Theorem: Suppose *P* has direction-free feasibility. Then inf *P*=sup *D*.

One can achieve attainment in the dual under `calmness' of a certain finite-dimensional value function:

$$\Phi(u,v)=\inf_C\{f_{10}(x) + E\{f_{20}(\xi,x)\} \mid f_{1i}(x)\leq u_i,\ E\{[f_{2j}(\xi,x)]^+\}\leq v_j,\ i=1,\ldots,m_1,\ j=1,\ldots,m_2\}$$

Current work: extension to many time stages.

Preliminary Pricing Results using New Duality

- $K = \{\theta_0 \mid Z_0 \bullet \theta_0 \leq F_0,\ Z_T \bullet \theta_0 \geq F_T \ \ P\text{-a.s.}\}$
- $K^\infty = \{\theta_0 \mid Z_0 \bullet \theta_0 \leq 0,\ Z_T \bullet \theta_0 \geq 0 \ \ P\text{-a.s.}\}$
- No arbitrage $\Longrightarrow$ $K^\infty = \{\theta_0 \mid Z_0 \bullet \theta_0 = 0,\ Z_T \bullet \theta_0 = 0 \ \ P\text{-a.s.}\}$, which is a subspace
- Hence, the writer's problems have direction-free feasibility and FTAP holds.

“No Arbitrage Unveiled”

- **More complex pricing problems** can be considered e.g. incomplete markets, bounded shortfall risk, inclusion of production decisions, other markets…
- **Boundedness of the S.P. plus direction-free feasibility** provides the generalization of no free lunch for convex S.P. problems.
- S.P. provides the **computational framework** that mathematical finance setting lacks.


Questions?