GEOMETRIC BROWNIAN MOTION WITH DELAY: MEAN SQUARE CHARACTERISATION

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Abstract. A geometric Brownian motion with delay is the solution of a stochastic differential equation where the drift and diffusion coefficient depend linearly on the past of the solution, i.e. a linear stochastic functional differential equation. In this work the asymptotic behavior in mean square of a geometric Brownian motion with delay is completely characterized by a sufficient and necessary condition in terms of the drift and diffusion coefficients.

1. Introduction

Geometric Brownian motion is one of the stochastic processes most often used in applications, not least in financial mathematics for modelling the dynamics of security prices. More recently, however, modelling the price process by a geometric Brownian motion has been criticised because the past of the volatility is not taken into account.

The *geometric Brownian motion* is the strong solution of the stochastic differential equation

$$dX(t) = bX(t)\,dt + \sigma X(t)\,dW(t) \qquad \text{for } t \geqslant 0,$$

where b and σ are some real constants. If we wish that the dynamics of the process X at time t are to depend on its past, a natural generalisation involves replacing the constants b and σ by some linear functionals on an appropriate function space, say the space of continuous functions on a bounded interval. Then we are led by the Riesz' representation theorem to the following stochastic differential equation with delay:

$$dX(t) = \left(\int_{[-\alpha,0]} X(t+u)\mu(du)\right)dt + \left(\int_{[-\alpha,0]} X(t+u)\nu(du)\right)dW(t) \tag{1}$$

for all $t \geqslant 0$ and some measures μ, ν. We call the solution X of this stochastic differential equation *geometric Brownian motion with delay* and its asymptotic behavior in mean square will be characterised in this article. In contrast to the geometric Brownian motion without delay no explicit representation of X is known from which the asymptotic behaviour can be inferred directly.

Equation (1) is a stochastic functional differential equation with diffusion coefficient depending on the past. Such equations may exhibit most irregular asymptotic behaviour, see for example Mohammed and Scheutzow [7] for the noisy feedback equation. Nevertheless, for a much more general equation than (1) a wide variety of sufficient conditions have been established guaranteeing stability in some sense. An exhaustive

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list of researchers and papers are not quoted here, but a good selection of such results are collated in Mao [6] and Kolmanovskii and Myskhis [5]. Despite this activity over the last twenty–five years to the best of our knowledge no sufficient and necessary conditions are known for stability, even for the linear equation (1). By contrast, in this work we are able to find necessary and sufficient conditions which characterise completely the asymptotic behavior of the solution in mean square. An interesting by-product of this stability characterisation is the observation that a deterministic solution may transpire to be a solution of the stochastic equation. This feature cannot arise in linear non-delay stochastic equations.

The proof of our stability characterisation relies on the fact that a non–negative functional of the process has expected value which satisfies a deterministic linear renewal equation. The asymptotic behaviour of this functional is characterised by the renewal theorem; once this characterisation has been obtained, it is straightforward to characterise the asymptotic behaviour of the mean square.

2. Preliminaries

We first turn our attention to the deterministic delay equation underlying the stochastic differential equation (1). For a fixed constant $\alpha \geqslant 0$ we consider the deterministic linear delay differential equation

$$\begin{aligned} \dot{x}(t) &= \int_{[-\alpha,0]} x(t+u)\,\mu(du) \quad \text{for } t \geqslant 0, \\ x(t) &= \varphi(t) \quad \text{for } t \in [-\alpha,0], \end{aligned} \tag{2}$$

for a measure $\mu \in M = M[-\alpha,0]$, the space of signed Borel measure on $[-\alpha,0]$ with the total variation norm $\|\cdot\|_{TV}$. The initial function φ is assumed to be in the space $C[-\alpha,0] := \{\psi : [-\alpha,0] \to \mathbb{R} : \text{continuous}\}$. A function $x : [-\alpha,\infty) \to \mathbb{R}$ is called a *solution* of (2) if x is continuous on $[-\alpha,\infty)$, its restriction to $[0,\infty)$ is continuously differentiable, and x satisfies the first and second identity of (2) for all $t \geqslant 0$ and $t \in [-\alpha,0]$, respectively. It is well known that for every $\varphi \in C[-\alpha,0]$ the problem (2) admits a unique solution $x = x(\cdot,\varphi)$.

The *fundamental solution* or *resolvent* of (2) is the unique locally absolutely continuous function $r : [0,\infty) \to \mathbb{R}$ which satisfies

$$r(t) = 1 + \int_0^t \int_{[\max\{-\alpha,-s\},0]} r(s+u)\,d\mu(u)\,ds \quad \text{for } t \geqslant 0.$$

It plays a role which is analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations. Formally, it is the solution of (2) corresponding to the initial function $\varphi = \mathbb{1}_{\{0\}}$. For later convenience we set $r(u) = 0$ for $u \in [-\alpha,0)$.

The solution $x(\cdot,\varphi)$ of (2) for an arbitrary initial segment φ exists, is unique, and can be represented as

$$x(t,\varphi) = \varphi(0)r(t) + \int_{[-\alpha,0]} \int_s^0 r(t+s-u)\,\varphi(u)\,du\,\mu(ds) \quad \text{for } t \geqslant 0, \tag{3}$$

cf. Diekmann et al [3, Chapter I]. The fundamental solution converges for $t \to \infty$ to zero if and only if

$$v_0(\mu) := \sup \left\{ \Re(\lambda) : \lambda \in \mathbb{C},\ \lambda - \int_{[-\alpha,0]} e^{\lambda s}\, \mu(ds) = 0 \right\} < 0, \tag{4}$$

where $\Re(z)$ denotes the real part of a complex number z. In this case the decay is exponentially fast (see Diekmann et al [3, Thm. 5.4]) and the zero solution of (2) is uniformly asymptotically stable. In this situation we have for every solution $x(\cdot, \varphi)$ of (2):

$$|x(t,\varphi)| \leqslant c e^{v_0(\mu)t} \qquad \text{for all } t \geqslant 0,$$

and for a constant $c > 0$ depending only on the initial function φ.

Let us introduce equivalent notation for (2). For a function $x : [-\alpha, \infty) \to \mathbb{R}$ we define the *segment of* x at time $t \geqslant 0$ by the function

$$x_t : [-\alpha, 0] \to \mathbb{R}, \qquad x_t(u) := x(t+u).$$

If we equip the space $C[-\alpha, 0]$ of continuous functions with the supremum norm Riesz' representation theorem guarantees that every continuous functional $F : C[-\alpha, 0] \to \mathbb{R}$ is of the form

$$F(\psi) = \int_{[-\alpha,0]} \psi(u)\, \mu(du)$$

for a measure $\mu \in M$. Hence, we will write (2) in the form

$$\dot{x}(t) = F(x_t) \quad \text{for } t \geqslant 0, \qquad x_0 = \varphi$$

and assume F to be a continuous and linear functional on $C[-\alpha, 0]$.

Let us fix a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual conditions and let $(W(t) : t \geqslant 0)$ be a Wiener process on this space. We study the following stochastic differential equation with time delay:

$$\begin{aligned} dX(t) &= F(X_t)\, dt + G(X_t)\, dW(t) \quad \text{for } t \geqslant 0, \\ X(u) &= \varphi(u) \quad \text{for } u \in [-\alpha, 0], \end{aligned} \tag{5}$$

where F and G are continuous and linear functionals on $C[-\alpha, 0]$ for a constant $\alpha \geqslant 0$. As before, we can write the functional G in the form

$$G(\psi) = \int_{[-\alpha,0]} \psi(u)\, \nu(du) \qquad \text{for all } \psi \in C[-\beta, 0]$$

and for a measure $\nu \in M$. We note that assuming the same domain $[-\alpha, 0]$ for the arguments of the functionals F and G does not involve any restriction or loss of generality.

For every $\varphi \in C[-\alpha, 0]$ there exists a unique, adapted strong solution $(X(t, \varphi) : t \geqslant -\alpha)$ with finite second moments of (5) (cf., e.g., Mao [6]). The dependence of the solutions on the initial condition φ is neglected in our notation in what follows; that is, we will write $x(t) = x(t, \varphi)$ and $X(t) = X(t, \varphi)$ for the solutions of (2) and (5) respectively.

By Reiß et al [8, Lemma 6.1] the solution $(X(t) : t \geqslant -\alpha)$ of (5) obeys a variation of constants formula

$$X(t) = \begin{cases} x(t) + \int_0^t r(t-s)G(X_s)\,dW(s), & t \geqslant 0, \\ \varphi(u), & u \in [-\alpha, 0], \end{cases} \tag{6}$$

where r is the fundamental solution of (2). It is to be noted that this equation does not supply an explicit form of the solution.

3. Stability

The asymptotic behavior of the solution X relies on the stochastic convolution integral arising in the variation of constants formula (6). Let us define

$$Y(t) := G(X_t) \qquad \text{for } t \geqslant 0, \tag{7}$$

such that the stochastic convolution integral is the convolution of the stochastic process $Y = (Y(t) : t \geqslant 0)$ and the fundamental solution r. The following result shows that the functional $\mathbb{E}[Y^2]$ satisfies a linear convolution integral equation.

Theorem 3.1. *Let $(X(t) : t \geqslant -\alpha)$ be the solution of (5). Then we have for all $t \geqslant 0$*

$$E\,|X(t)|^2 = |x(t)|^2 + \int_0^t r^2(t-s)\,E\,|Y(s)|^2\,ds, \tag{8}$$

where Y, defined by (7), obeys for all $t \geqslant 0$:

$$E\,|Y(t)|^2 = G^2(x_t) + \int_0^t G^2(r_{t-s})E\,|Y(s)|^2\,ds. \tag{9}$$

Proof. The variation of constants formula (6) and Itô's isometry imply the first assertion.

Using again the variation of constants formula we obtain by Fubinis's theorem for stochastic integrals for $t \in [0, \alpha]$:

$$\begin{aligned} E\,|Y(t)|^2 &= E\,|G(X_t)|^2 \\ &= E\left|\int_{[-\alpha,-t)} X_t(u)\,\nu(du) + \int_{[-t,0]} X_t(u)\,\nu(du)\right|^2 \\ &= E\left|\int_{[-\alpha,-t)} \varphi(t+u)\,\nu(du) + \int_{[-t,0]} x(t+u)\,\nu(du)\right. \\ &\qquad \left. + \int_{[-t,0]} \left(\int_0^{t+u} r(t+u-s)Y(s)\,dW(s)\right)\,\nu(du)\right|^2 \\ &= E\left|G(x_t) + \int_0^t \left(\int_{[s-t,0]} r(t+u-s)\,\nu(du)\right) Y(s)\,dW(s)\right|^2 \\ &= |G(x_t)|^2 + \int_0^t G^2(r_{t-s})E\,|Y(s)|^2\,ds, \end{aligned}$$

where we used in the last line $r(u) = 0$ for $u < 0$. Setting $\nu([a,b)) = 0$ for all $a \leqslant b \leqslant -\alpha$ enables us to enlarge the integration domain for G such that we can write also for $t \geqslant \alpha$:

$$E\,|Y(t)|^2 = E\left|G(x_t) + \int_{[-t,0]} \left(\int_0^{t+u} r(t+u-s)Y(s)\,dW(s)\right)\,\nu(du)\right|^2.$$

We can proceed as above to verify that equation (9) is also satisfied for $t \geqslant \alpha$. □

We next turn to stating and proving our first stability result. In it, the hypothesis $r \in L^2(\mathbb{R}_+)$ is employed. We remark that this assumption is necessary if $E\left|X^2(t)\right| \to 0$ as $t \to \infty$. To see this, note by (8) that $E\left|X^2(t)\right|$ cannot tend to zero as $t \to \infty$ if $x(t)$ does not tend to zero. But the latter cannot occur if r is not in $L^2(\mathbb{R}_+)$.

The function $s \mapsto G(r_s)$, which we denote by $G(r_\bullet)$, is square integrable if $r \in L^2(\mathbb{R}_+)$ and its norm in $L^2(\mathbb{R}_+)$ is given by

$$\|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} = \left(\int_0^\infty (G(r_s))^2\,ds\right)^{1/2}.$$

This quantity allows to characterise the asymptotic behaviour of the solution for (5):

Theorem 3.2. *If the fundamental solution r is in $L^2(\mathbb{R}_+)$ then the solution $(X(t) : t \geqslant -\alpha)$ of (5) obeys the following trichotomy:*

(a) *if $\|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} < 1$, then there exists $\kappa > 0$ such*

$$\lim_{t\to\infty} e^{\kappa t} E\,|X(t)|^2 = 0.$$

(b) *if $\|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} = 1$, then*

$$\lim_{t\to\infty} E\,|X(t)|^2 = \frac{\left(\int_0^\infty G^2(x_s)\,ds\right)\left(\int_0^\infty r^2(s)\,ds\right)}{\int_0^\infty sG^2(r_s)\,ds} < \infty.$$

(c) *if $\|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} > 1$, then there exists $\kappa > 0$ such that*

$$\lim_{t\to\infty} e^{-\kappa t} E\,|X(t)|^2 = \frac{\left(\int_0^\infty e^{-\kappa s}G^2(x_s)\,ds\right)\left(\int_0^\infty e^{-\kappa s}r^2(s)\,ds\right)}{\int_0^\infty se^{-\kappa s}G^2(r_s)\,ds} < \infty.$$

Proof. Let us introduce the following functions and measures for $t \geqslant 0$:

$$y(t) := E\,|Y(t)|^2 \qquad f(t) := G^2(x_t)$$
$$g(t) := G^2(r_t) \qquad \zeta(dt) := g(t)\,dt.$$

Then we can rewrite (9) as the renewal equation

$$y(t) = f(t) + \int_0^t y(t-s)\,\zeta(ds) \quad \text{for all } t \geqslant 0 \tag{10}$$

and the three cases (a) to (c) correspond to whether the renewal equation (10) is defective, proper or excessive.

To give the main idea of the proof we establish (a) first without the convergence rate. In case (a) the renewal Theorem [1, Thm 3.1.4] implies

$$\lim_{t\to\infty} y(t) = \frac{f(\infty)}{1 - \|G(r_\bullet)\|_{L^2(\mathbb{R}_+)}} = 0,$$

as $f(\infty) := \lim_{t\to\infty} f(t) = 0$ due to $x(t) \to 0$ for $t \to \infty$. Using the notation for y introduced above, equation (8) reads

$$E\,|X(t)|^2 = x^2(t) + \int_0^t r^2(s)y(t-s)\,ds. \tag{11}$$

Consequently, as $x(t) \to 0$ for $t \to \infty$ and $r \in L^2(\mathbb{R}_+)$ we arrive at

$$\lim_{t\to\infty} E\,|X(t)|^2 = \lim_{t\to\infty} \int_0^\infty r^2(s)\,\mathbb{1}_{[0,t]}(s)y(t-s)\,ds = 0$$

by dominated convergence.

We now turn to prove the exponential decay. Because $\|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} < 1$ there exists $\vartheta > 0$ such that

$$\int_0^\infty e^{\vartheta s}\,\zeta(ds) = 1.$$

Moreover, $r \in L^2(\mathbb{R}_+)$ implies that there exists $\gamma > 0$ such that $r(t) = \mathcal{O}(\exp(-\gamma t))$ and $x(t) = \mathcal{O}(\exp(-\gamma t))$ as $t \to \infty$. Consequently, we have $f(t) = \mathcal{O}(\exp(-2\gamma t))$ and $g(t) = \mathcal{O}(\exp(-2\gamma t))$ as $t \to \infty$. Therefore, we can infer by standard methods in renewal theory that $y(t) = o(\exp(-\kappa t))$ for all $\kappa < (2\gamma \wedge \vartheta)$ which leads to

$$\int_0^t r^2(s)y(t-s)\,ds = o\big(\exp(-\kappa t)\big).$$

Consequently, equation (11) yields the assertion for all $\kappa < 2\gamma \wedge \vartheta$.

In case (b), the renewal Theorem [1, Thm 3.1.5] implies under appropriate conditions on f that

$$\lim_{t\to\infty} y(t) = (m(\zeta))^{-1} \int_0^\infty f(s)\,ds$$

with $m(\zeta) := \int_0^\infty sG^2(r_s)\,ds$. Note that $m(\zeta)$ is finite as r tends to zero exponentially fast and it is non-zero because $\int_0^\infty G^2(r_s)\,ds = 1$. Since the measure ζ is absolutely continuous with respect to the Lebesgue measure it is sufficient for the application of the renewal Theorem that the function f be in $(L^1 \cap L^\infty)(\mathbb{R}_+)$ and $f(t) \to 0$ for $t \to \infty$: both of these conditions are satisfied here. By proceeding as in case (a) we obtain

$$\lim_{t\to\infty} E\,|X(t)|^2 = \frac{1}{m(\zeta)} \left(\int_0^\infty f(s)\,ds\right)\left(\int_0^\infty r^2(s)\,ds\right).$$

In case (c), the renewal equation (10) is excessive. Then there exists a unique $\kappa > 0$ such that $\int_{\mathbb{R}_+} e^{-\kappa s}\,\zeta(ds) = 1$. Furthermore, $m_\kappa(\zeta) := \int_0^\infty se^{-\kappa s}\,\zeta(ds)$ is non-zero (see [1, Remark 3.1.8]) and $m_\kappa(\zeta)$ is finite as r decays exponentially fast. As in case (b) it is sufficient for the application of the renewal Theorem that the function f_κ with $f_\kappa(t) := e^{-\kappa t} f(t)$ for $t \geqslant 0$ tends to zero and is in $L^1(\mathbb{R}_+)$. These conditions are satisfied as $x(\cdot,\varphi)$ decays exponentially fast and hence, the renewal Theorem [1, Cor. 3.1.9] implies

$$\lim_{t\to\infty} e^{-\kappa t}y(t) = (m_\kappa(\zeta))^{-1} \int_0^\infty e^{-\kappa s} f(s)\,ds.$$

Finally, from (8) we have

$$e^{-\kappa t}E\,|X(t)|^2 = e^{-\kappa t}\,|x(t)|^2 + \int_0^t e^{-\kappa s}r^2(s)e^{-\kappa(t-s)}y(t-s)\,ds,$$

and so, because of the exponential decay of r, we conclude

$$\lim_{t\to\infty} e^{-\kappa t} E\,|X(t)|^2 = \frac{1}{m_\kappa(\zeta)} \left(\int_0^\infty e^{-\kappa s} f(s)\,ds\right)\left(\int_0^\infty e^{-\kappa s} r^2(s)\,ds\right).$$

□

Alsmeyer [1] contains a treatment of the renewal equation which covers equations with measures. Similar results may be found in Feller [4].

As a corollary of Theorem 3.2 we obtain an equivalence between the asymptotic behavior of the mean square of the solution X and a condition on the fundamental solution r and the diffusion coefficient G. Naturally, this requires that the solution X does not reduce to the solution of the deterministic equation (2), for in this case X would not provide any information on the diffusion coefficient. We argue below that this situation may occur, and must be excluded in the next corollary. This will also be illustrated presently in an example.

Corollary 3.3. *Let the fundamental solution r be in $L^2(\mathbb{R}_+)$ and assume that no version of the solution $X = X(\cdot,\varphi)$ of* (5) *coincides with the deterministic solution $x = x(\cdot,\varphi)$ of* (2)*. Then we have the following:*

$$\lim_{t\to\infty} E\,|X(t)|^2 = \begin{cases} 0 & \iff \quad \|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} < 1, \\ c > 0 & \iff \quad \|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} = 1, \\ \infty & \iff \quad \|G(r_\bullet)\|_{L^2(\mathbb{R}_+)} > 1. \end{cases}$$

Proof. We have to show that the constants in Theorem 3.2 in (b) and (c) describing the limiting behavior of X are non-zero, which is equivalent to

$$\int_0^\infty G^2(x_s(\cdot,\varphi))\,ds \neq 0. \tag{12}$$

Because $s \mapsto x_s(\cdot,\varphi)$ and G are continuous operators, equation (12) does not hold if and only if $G(x_t(\cdot,\varphi)) = 0$ for all $t \geqslant 0$. In this case $x(\cdot,\varphi)$ would be also a solution of the stochastic equation (5) which contradicts our assumption because of the uniqueness of the solution of (5). □

Remark 3.4. If the solution $x(\cdot,\varphi)$ of the deterministic equation (2) also solves the stochastic equation (5) it follows by taking expectation and Itô's isometry that

$$G(x_t(\cdot,\varphi)) = 0 \qquad \text{for all } t \geqslant 0. \tag{13}$$

Conversely, condition (13) implies that a version of the solution $X(\cdot,\varphi)$ of (5) coincides with $x(\cdot,\varphi)$.

Thus, a sufficient condition for the trichotomy only relying on the initial condition φ is $G(\varphi) \neq 0$.

A more abstract point of view shows that equation (13) holds true if a generalized eigenspace N of the deterministic equation (2) is a subset of the kernel $\ker(G)$ of the diffusion coefficient G. Then, for every $\varphi \in N$ the segment $x_t(\cdot,\varphi)$ is in N and consequently in $\ker(G)$, so that (13) is satisfied. For details on eigenspaces and related results for equation (2) we refer the reader to Diekmann et al [3] and Hale and Lunel [2]. A concrete example is given below.

We emphasise that the forgoing situation in which a deterministic solution may solve a non-trivial linear stochastic differential equation is a very specific feature of stochastic functional differential equations, and cannot occur in linear stochastic ordinary differential equations.

Example 3.5. Let us consider the solution $(X(t) : t \geqslant 0)$ of the simple equation

$$dX(t) = bX(t)\,dt + (cX(t) + dX(t-\alpha))\,dW(t) \qquad \text{for } t \geqslant 0, \tag{14}$$

with $b < 0$, $c, d \in \mathbb{R}$ and $\alpha > 0$. For the condition in Theorem 3.2 we calculate

$$\|G(r_\bullet)\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty G^2(r_s)\,ds = \frac{1}{-2b}(c^2 + d^2 + 2cde^{b\alpha}).$$

By using results on deterministic linear difference equations we obtain for a continuous function $y : [-\alpha, \infty) \to \mathbb{R}$:

$$\begin{aligned} G(y_t) = 0 \quad \text{for all } t \geqslant 0 \quad &\iff \quad cy(t) + dy(t-\alpha) = 0 \qquad \text{for all } t \geqslant 0 \\ &\iff \quad y(t) = y(0)e^{\gamma t} \quad \text{for all } t \geqslant -\alpha \quad \text{and} \quad cd < 0 \end{aligned}$$

with $\gamma := \frac{1}{\alpha} \ln \frac{-d}{c}$. In the case when $b \neq \gamma$ and $cd < 0$, we obtain that for every initial condition φ that the solution $X = X(\cdot, \varphi)$ obeys:

$$\lim_{t\to\infty} E\,|X(t)|^2 = 0 \iff c^2 + d^2 + 2cde^{b\alpha} < -2b. \tag{15}$$

In the case when $b < 0$ and $cd \geqslant 0$, the equivalence (15) also holds.

In the non-delay case $d = 0$ the solution X is the geometric Brownian motion and (15) reproduces the well-known and easily calculated fact that $E\,|X(t)|^2 \to 0$ if and only if $c^2 < -2b$. In the pure delay case $c = 0$ and $d \neq 0$ we find that $d^2 < -2b$ is necessary and sufficient to guarantee $E\,|X(t)|^2 \to 0$. However, although the condition on the noise intensities is the same as for geometric Brownian motion, the rate of decay to zero is different.

If $c \neq 0$ and $d \neq 0$ then the dependence of the stability region on the drift coefficient b is described by $b < b_0 < 0$ where b_0 is the largest real root of

$$c^2 + d^2 + 2cde^{b_0\alpha} + 2b_0 = 0.$$

In particular, we observe that while the stability region for (14) is symmetric in c and d, it is not symmetric in the sign of cd.

We finish by pointing out that equation (14) provides an example of the situation already mentioned in Remark 3.4 in which a solution of the deterministic equation (2) is also a solution of the stochastic equation (5). To see this take for example $c = -e$, $d = 1$ and $b = \gamma$ for some $\alpha > 0$. Then, for $\varphi(u) = e^{\gamma u}$, $u \in [-\alpha, 0]$, the solution $x(t, \varphi) = e^{\gamma t}$, $t \geqslant 0$, satisfies $G(x_t(\cdot, \varphi)) = 0$ for all $t \geqslant 0$. Thus, $x(\cdot, \varphi)$ is also a solution of the stochastic equation and $x(t, \varphi) \to 0$ for $t \to \infty$ even though $\|G(r_\bullet)\|^2_{L^2(\mathbb{R}_+)} > 1$.

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