Overlapping Operator-Splitting Methods with Higher-Order Splitting Methods and Applications in Stiff Differential Equations

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Abstract. In this article, we combine operator-splitting methods of an iterative and non-iterative type to problems for stiff differential equations. The time-splitting is performed with operator-splitting methods and the spatial splitting is done with an overlapping Schwarz waveform relaxation, see [2] and [4]. We discuss the iterative and non-iterative operator-splitting method in the context of achieving higher-order accuracy and with respect to stiff matrices. We discuss the stability of each decomposition method and influence of the higher-order approach via Richardson extrapolation. The stability analysis is presented and the benefit of the iterative method is discussed. At least we discuss the future work and the conclusions to our work.

Keywords: operator splitting, Schwarz waveform relaxation method, higher order methods, stability analysis, iterative methods.
AMS subject classifications. 35J60, 35J65, 65M99, 65N12, 65Z05, 74S10, 76R50, 80A20, 80M25

1 Introduction

Our motivation for this paper arose from different models for computational gas and fluid dynamics. For such models, we often obtain stiff partial differential equations because of their different physical effects, e.g. reaction and diffusion scales. Therefore it is essential to construct higher-order methods for the solvers, since often analytical solutions of the partial differential equations are not available, such that we want to investigate a more general prediction of the numerical models. Here, the iterative and non-iterative solver methods are important to solve linear systems of equations after a proper spatial discretization. We propose operator-splitting methods to decouple the different spatially discretized operators, with respect to their time scales, into simpler ones. Each underlying matrix can be solved with the adapted time step and the stiffness can be overcome. Therefore we discuss the time-splitting methods as non-iterative as well as iterative methods and present their stability analysis. To control also the spatial influence of the underlying domain, we propose a Schwarz waveform relaxation method to decouple the different domains. Further we discuss the Richardson extrapolation as a benefit to obtain higher-order methods for the standard splitting schemes. Taking into account the overlapping Schwarz waveform relaxation as an iterative method, we can improve the stability of the methods by overlapping range. The combined time-space iterative and non-iterative operator-splitting methods are discussed.

The outline of the paper is as follows. For our mathematical model we describe the convection-diffusion-reaction equation in Section 2. The time- and space-splitting methods are introduced in Section 3. For the iterative methods, we discuss the stability analysis in Section 4. In section 5 we present the numerical results from the solution of selective model problems. Section 6 forms the end of the article with conclusions and comments.

2 Mathematical Model

The motivation for the study presented below is coming from a computational simulation of heat transfer [9] and convection-diffusion-reaction equations [5], [12], [13], and [14].

In our paper we concentrate on a one-dimensional convection-diffusion-reaction equation as model problem, which is given as

$$\begin{aligned} & u_t \ - D_1 u_{xx} - D_2 u_{yy} = -\lambda u \ , \text{ in } \Omega \times (0,T), \qquad (1) \\ & u(x,0) = u_0, \text{ (initial condition)}, \\ & u(x,t) = u_1, \text{ on } \partial\Omega \times (0,T) \ , \text{ (Dirichlet boundary condition)}. \end{aligned}$$

The unknown $u = u(x,t)$ is considered in $\Omega \times (0,T) \subset \mathbb{R} \times \mathbb{R}$, where $\Omega = [0,L]$. The parameters $u_0, u_1 \in \mathbb{R}^+$ are constants and used as initial and boundary parameter, respectively. The parameter λ is a constant factor, for example a decay rate of a chemical reaction. $D1, D_2$ are constant factors, for example the diffusion factor of a transport process.

The aim of this paper is to present a new method based on a mixed discretization method with a fractional splitting and domain decomposition method for an effective solver method of strong coupled parabolic differential equations.

In the next subsection we discuss the decoupling of the time scales and space scales with decomposition methods.

3 Space- and Time-Splitting Methods

In this section, we discuss the space- and time-splitting methods, which can reduce the amount of computational work by decoupling into simpler subparts.

3.1 Overlapping Schwarz Waveform Relaxation for Differential Equations

We discuss the spatial decomposition methods, based on overlapping Schwarz waveform relaxation methods, for the solution of the convection-reaction-diffusion equation with constant coefficients. We will utilize the convergence analysis for the solution of the decoupled and coupled system of convection-reaction-diffusion equations to elaborate the impact of the coupling on the convergence of the overlapping Schwarz waveform relaxation.

Given is the following model problem,

$$\begin{aligned} & u_t + Lu = f \ , \text{in } \Omega \times (0,T), \overline{\Omega} \times (0,T) : \overline{\Omega}_1 \times (0,T) \cup \overline{\Omega}_2 \times (0,T), \\ & u(x,0) = u_0, \ \text{(initial condition)}, \\ & u = g, \ \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{2}$$

where L denotes for each time t a second-order partial differential operator $Lu = -\nabla D \nabla u + v \nabla u + cu$ for the given coefficients n denotes the dimension of the space. Each iteration step consists of two half steps, associated with the two subdomains, and we solve 2 subproblems

$$\begin{aligned} & u_1 t + L u_1^n = f, \text{in } \Omega_1 \times (0,T), \\ & u_1(x,0) = u_{10}, \ \text{(initial condition)}, \\ & u_1^n = g, \text{on } L_0 = \partial\Omega \times (0,T) \cap \partial\Omega_1 \times (0,T), \\ & u_1^n = u_2^{n-1}, \text{on } L_2 = \partial\Omega_1 \times (0,T) \backslash \partial\Omega \times (0,T), \end{aligned} \tag{3}$$

$$\begin{aligned} & u_{2t} + L u_2^n = f, \text{in } \Omega_2 \times (0,T), \\ & u_2(x,0) = u_{20}, \ \text{(initial condition)}, \\ & u_2^n = g, \text{on } L_3 = \partial\Omega \times (0,T) \cap \partial\Omega_2 \times (0,T), \\ & u_2^n = u_1^n, \text{on } L_1 = \partial\Omega_2 \times (0,T) \backslash \partial\Omega \ \times (0,T), \end{aligned} \tag{4}$$

where $L = A + B$ and $A = v\nabla + c$, $B = -\nabla D \nabla$.

In the following we discuss the time-splitting methods.

3.2 The Operator-Splitting Method

We classify non-iterative and iterative splitting methods. The non-iterative methods are direct methods and we obtain the results immediately after the definite steps. On the other hand, the iterative methods are indirect methods and we obtain the results based on fixed-point iterations, which are finished after reaching an specific error bound.

The non-iterative splitting methods are often based on higher-order reconstructions of Strang splitting methods, while the iterative splitting methods are based on the fixed-point iterations, that reconstruct the solution step by step, e.g. as Taylor expansion for each partial term.

The non-iterative splitting method

We deal with the following semi-discretized method. Our operators are derived by space-discretization methods.

The considered systems of ordinary differential equations are given as:

$$u_t + (A_1 + A_2)u = 0, \tag{5}$$
$$u(0) = u_0, \text{ (initial condition)}.$$

The higher-order splitting method based on the Richardson extrapolation, as discussed in [1] and [16], is given as:

$$D_4(\Delta t) = 4/3 \; S_2(\Delta t/2) \; S_2(\Delta t/2) - 1/3 \; S_2(\Delta t), \tag{6}$$

where $S_2(\Delta t) = \exp(A_2 \Delta t) \exp(A_1 2\Delta t) \exp(A_2 \Delta t)$ is the Strang splitting operator [17].

The higher order is reached after applying three times the Strang splitting method in a proper way.

In the next subsection we discuss the iterative splitting methods.

The iterative splitting method

The following algorithm is based on the iteration with fixed splitting discretization step size τ. On the time interval $[t^n, t^{n+1}]$ we solve the following subproblems consecutively for $i = 0, 2, \dots 2m$.

The iterative method is given as, see also [4],

$$\frac{\partial c_i(x,t)}{\partial t} = Ac_i(x,t) \; + \; Bc_{i-1}(x,t), \tag{7}$$
$$\text{with } c_i(t^n) = c^n, \; c_0(t^n) = c^n, \; c_{-1} = 0.0,$$
$$\text{and } c_i(x,t) = c_{i-1}(x,t) = c_1, \text{ on } \partial\Omega \times (0,T),$$

$$\frac{\partial c_{i+1}(x,t)}{\partial t} = \; Ac_i(x,t) \; + \; Bc_{i+1}(x,t), \tag{8}$$
$$\text{with } c_{i+1}(t^n) = c^n$$
$$\text{and } c_i(x,t) = c_{i-1}(x,t) = c_1, \text{ on } \partial\Omega \times (0,T),$$

where c^n is the known split approximation at time level $t = t^n$ [4].

The higher order is obtained by applying recursively the fixed-point iteration to reconstruct the analytical solution of the coupled operators, see [8].

The overlapping iterative operator-splitting method

The influence of the different stiffnesses of the operators are taken into account by the overlapping iterative method. Here, the idea is to use overlapped operators, that can balance the critical eigenvalues of the different operators by weighting, see [7].

The overlapping method is given as

$$\frac{\partial c_i(x,t)}{\partial t} = (1-\omega_1)Ac_i(x,t) \; + \; \omega_1(A+B)c_i(x,t) + (1-\omega_1)Bc_{i-1}(x,t),$$
$$\text{with } c_i(t^n) = c^n, \; c_0(t^n) = c^n, \text{ and } c_{-1} = c^n, \tag{9}$$

$$\frac{\partial c_{i+1}(x,t)}{\partial t} = (1-\omega_1)Ac_i(x,t) \ + \ \omega_1(A+B)c_i(x,t) + (1-\omega_1)Bc_{i+1}(x,t),$$
$$\text{with } c_{i+1}(t^n) = c^n, \tag{10}$$

where c^n is the known split approximation at time level $t = t^n$ [7].

Here, the higher order is obtained by balancing between each operator and applying recursively the fixed-point iteration.

4 Stability Analysis

In this section we present the stability analysis of the splitting methods.

4.1 Consistency and Stability Analysis of the Non-Iterative Methods

Let us consider the linear operator equation in a Banach space $\mathbf{X}$,

$$\begin{aligned} \partial_t c(t) &= A_1 c(t) + A_2 c(t), \quad 0 < t \leq T, \\ c(0) &= c_0, \end{aligned} \tag{11}$$

where $A_1, A_2, B_1, B_2, A_1 + A_2 + B_1 + B_2 : \mathbf{X} \to \mathbf{X}$ are given linear operators being generators of the C_0-semigroup and $c_0 \in \mathbf{X}$ is a given element.

The Strang formula can be constructed as

$$S_2(t)c_0 = \exp(tA_1/2)\exp(tA_2)\exp(tA_1/2). \tag{12}$$

Theorem 1. *We have the following error estimates:*

$$\begin{aligned} \| \exp(t(A_1 + A_2)) - S_2(t) \| &\leq c_0 \frac{t^3}{12} + c_1 \frac{t^2 \sqrt{t}}{15\sqrt{2\exp(1)}} \\ &\quad + c_2 \alpha \frac{t\sqrt{t}}{4}, \end{aligned} \tag{13}$$

$$\begin{aligned} \| \exp(t(A_1 + A_2)) - D_4(t) \| &\leq \tilde{c}_0 t^5 + \tilde{c}_1 t^4 \sqrt{t} + \tilde{c}_1 t^4 \sqrt{t} \\ &\quad + \tilde{c}_2 t^3 \sqrt{t} + \tilde{c}_3 t^2 \sqrt{t} + \tilde{c}_4 t \sqrt{t}, \end{aligned} \tag{14}$$

where the constants bound each operator A_1, A_2 and the Lie bracket.

Proof. The proof for the Strang formula is given in [1].

To prove the formula D_4 we have to apply the combination, given in equation (6).

$$\begin{aligned} &\| \exp(t(A_1 + A_2)) - D_4(t) \| = \\ &= \| \exp(t(A_1 + A_2)) - (4/3 S_2(t/2) S_2(t/2) - 1/3 S_2(t)) \|. \end{aligned}$$

By applying the result of each Strang formula and the Richardson extrapolation, which shifts the factors by t^2, we obtain

$$
\begin{aligned}
&= \| \exp(t(A_1 + A_2)) - 4/3 S_2(t/2) S_2(t/2) - 1/3 S_2(t) \| \\
&\leq 4/3 (c_0 \frac{t^3}{12} + c_1 \frac{t^2 \sqrt{t}}{15 \sqrt{2 \exp(1)}})(c_0 \frac{t^3}{12} + c_1 \frac{t^2 \sqrt{t}}{15 \sqrt{2 \exp(1)}}) \\
&- 1/3 (c_0 \frac{t^3}{12} + c_1 \frac{t^2 \sqrt{t}}{15 \sqrt{2 \exp(1)}}) \\
&\leq \tilde{c}_0 t^5 + \tilde{c}_1 t^4 \sqrt{t} + \tilde{c}_1 t^4 \sqrt{t} + \tilde{c}_2 t^3 \sqrt{t} + \tilde{c}_3 t^2 \sqrt{t} + \tilde{c}_4 t \sqrt{t},
\end{aligned}
$$

where we skip the factors higher $O(t^6)$ and obtain the results.

4.2 Stability Analysis of the Iterative Methods

Theorem 2. *Let us consider the nonlinear operator equation in a Banach space* $\mathbf{X}$*,*

$$
\begin{aligned}
\partial_t c(t) &= A_1(c(t)) + A_2(c(t)) + B_1(c(t)) + B_2(c(t)), \quad 0 < t \leq T, \\
c(0) &= c_0,
\end{aligned} \tag{15}
$$

where $A_1, A_2, B_1, B_2, A_1 + A_2 + B_1 + B_2 : \mathbf{X} \to \mathbf{X}$ *are given linear operators being generators of the* C_0*-semigroup and* $c_0 \in \mathbf{X}$ *is a given element. Then the iteration process (7)–(8) is convergent and the rate of the convergence is of second order.*

Proof. Let us consider the iteration (7)–(8) on the subinterval $[t^n, t^{n+1}]$. For the error function $e_i(t) = c(t) - c_i(t)$ we have the relations

$$
\begin{aligned}
&\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\
&t \in (t^n, t^{n+1}], \; e_{i,j}(t^n) = 0,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\partial_t e_{i+1,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j-1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\
&t \in (t^n, t^{n+1}], \; e_{i+1,j}(t^n) = 0,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\partial_t e_{i,j+1}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i-1,j-1}(t)), \\
&t \in (t^n, t^{n+1}], \; e_{i,j+1}(t^n) = 0 \;,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&\partial_t e_{i,j}(t) = A_1(e_{i,j}(t)) + A_2(e_{i,j+1}(t)) + B_1(e_{i+1,j}(t)) + B_2(e_{i+1,j+1}(t)), \\
&t \in (t^n, t^{n+1}], \; e_{i,j}(t^n) = 0 \;,
\end{aligned} \tag{19}
$$

for $i, j = 0, 2, 4, \ldots$, with $e_{0,0}(0) = 0$ and $e_{-1,0} = e_{0,-1} = e_{-1,-1}(t) = c(t)$.

In the following we derive the linear system of equations. We use the notations $\mathbf{X}^2$ for the product space $\mathbf{X} \times \mathbf{X}$ enabled with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$

$(u, v \in \mathbf{X})$. The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ are defined as follows.

$$\mathcal{E}_{i,j}(t) = \begin{bmatrix} e_{i,j}(t) \\ e_{i+1,j}(t) \\ e_{i,j+1}(t) \\ e_{i+1,j+1}(t) \end{bmatrix}; \quad \mathcal{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_1 & A_2 & 0 & 0 \\ A_1 & A_2 & B_1 & 0 \\ A_1 & A_2 & B_1 & B_2 \end{bmatrix}, \tag{20}$$

$$\mathcal{F}_{i,j}(t) = \begin{bmatrix} A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1}) \\ B_2(e_{i-1,j-1}) \\ 0 \end{bmatrix}. \tag{21}$$

Then, using the notations (20)–(21), the relations (16)–(19) can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_{i,j}(t) &= \mathcal{A}\mathcal{E}_{i,j}(t) + \mathcal{F}_{i,j}(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_{i,j}(t^n) &= 0. \end{aligned} \tag{22}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0-semigroup $(\mathcal{A}(t))_{t \geq 0}$. We also assume the estimation of our term $\mathcal{F}_i(t)$ with the growth conditions.

We estimate the right-hand side $\mathcal{F}_i(t)$ in the following lemma.

Lemma 1. *Let us consider the bounded Jacobian of $A(u)$ and $B(u)$. We then estimate $\mathcal{F}_i(t)$ as*

$$\|\mathcal{F}_{i,j}(t)\| \leq C \|e_{i-1,j-1}\|. \tag{23}$$

Proof. We have the following norm,

$$\|\mathcal{F}_{i,j}(t)\| = \max\{\mathcal{F}_{i,j,1}(t), \mathcal{F}_{i,j,2}(t), \mathcal{F}_{i,j,3}(t), \mathcal{F}_{i,j,4}(t)\}.$$

We have to estimate each term:

$$\begin{aligned} &\|\mathcal{F}_{i,j,1}(t)\| \leq \|A_2(e_{i,j-1}(t)) + B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1})\| \leq C_1 \|(e_{i-1,j-1})\|, \\ &\|\mathcal{F}_{i,j,2}(t)\| \leq \|B_1(e_{i-1,j}(t)) + B_2(e_{i-1,j-1})\| \leq C_2 \|(e_{i-1,j-1})\|, \\ &\|\mathcal{F}_{i,j,3}(t)\| \leq \|B_2(e_{i-1,j-1})\| \leq C_3 \|(e_{i-1,j-1})\|. \end{aligned}$$

So we obtain the estimation:

$$\|\mathcal{F}_{i,j}(t)\| \leq \tilde{C} \|e_{i-1,j-1}(t)\|,$$

where $\tilde{C}$ is the maximum value of C_1, C_2 and C_3.

Hence using the variations of constants formula, the solution of the abstract Cauchy problem (22) with homogeneous initial condition can be written as

$$\mathcal{E}_{i,j}(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s)) \mathcal{F}_{i,j}(s) ds, \quad t \in [t^n, t^{n+1}].$$

(See, e.g. [3].) Hence, using the denotation

$$\|\mathcal{E}_{i,j}\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_{i,j}(t)\|,$$

we have

$$\|\mathcal{E}_{i,j}\|(t) \leq \|\mathcal{F}_{i,j}\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds$$
$$\leq C\ \|e_{i-1,j-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds, \quad t \in [t^n, t^{n+1}]. \tag{24}$$

We have used the estimate $\|\mathcal{F}_{i,j}\| \leq C \|e_{i-1,j-1}\|$, where C is a constant that bounds the nonlinear terms of $\mathcal{F}_{i,j}(t)$.

Since $(\mathcal{A}(t))_{t\geq 0}$ is a semigroup, therefore the so-called *growth estimation*

$$\|\exp(\mathcal{A}t)\| \leq K \exp(\omega t); \quad t \geq 0, \tag{25}$$

holds with some numbers $K \geq 0$ and $\omega \in \mathbb{R}$, see [3].

– Assume that $(\mathcal{A}(t))_{t\geq 0}$ is a bounded or exponentially stable semigroup, i.e. (25) holds with some $\omega \leq 0$. Then obviously the estimate

$$\|\exp(\mathcal{A}t)\| \leq K; \quad t \geq 0,$$

holds, and, hence on base of (24), we have the relation

$$\|\mathcal{E}_{i,j}\|(t) \leq K\tau_n \|e_{i-1,j-1}\|, \quad t \in (0, \tau_n). \tag{26}$$

– Assume that $(\mathcal{A}(t))_{t\geq 0}$ has an exponential growth with some $\omega > 0$. Using (24) we have

$$\int_{t^n}^{t^{n+1}} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}],$$

where

$$K_\omega(t) = \frac{K}{\omega} \left(\exp(\omega(t - t^n)) - 1\right), \quad t \in [t^n, t^{n+1}],$$

and hence

$$K_\omega(t) \leq \frac{K}{\omega} \left(\exp(\omega \tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2). \tag{27}$$

The estimations (26) and (27) result in

$$\|\mathcal{E}_{i,j}\|_\infty = K\tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2).$$

Taking into account the definition of $\mathcal{E}_i$ and the norm $\|\cdot\|_\infty$, we obtain

$$\|e_{i,j}\| = K\tau_n \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^2),$$

and hence

$$\|e_{i+1,j+1}\| = K_1 \tau_n^2 \|e_{i-1,j-1}\| + \mathcal{O}(\tau_n^3), \tag{28}$$

which proves our statement.

5 Numerical experiments

In the next test example we use the analyzed operator decomposition methods and solve the initial value problem given as a reaction-diffusion equation:

$$\partial_t u = \partial_x (xu) + \partial_{xx} u = Au, \text{ with } (x,t) \in (0,1] \times (0,T), \quad (29)$$
$$u(x,0) = u_0(x) = -x \exp(-1/2x^2) \text{ with } x \in (0,1], \quad (30)$$
$$u(x=0,t) = g_0(t) = 0,\ u(x=1,t) = g_L(t) = -\exp(-t)\exp(-1/2), \quad (31)$$

where the time interval is given as $[0,T] = [0,1]$.

This problem has the exact solution

$$u_{analy}(x,t) = -x \exp(-t) \exp(-1/2x^2). \quad (32)$$

For the approximation error we choose the L_1-norm.
The L_1-norm is given by

$$err_{L_1} := |u(t^n) - u_{analy}(t^n)|. \quad (33)$$

The numerical convergence rate is given as

$$err_\rho := \frac{ln(err_{L_1(\Delta t_1)}/err_{L_1(\Delta t_2)})}{ln(\Delta t_1/\Delta t_2)}. \quad (34)$$

The result for the experiment is given in the following table.

time step	ρ	err	err	err	err	err
Dt = 0.5		$8.500 \cdot 10^{-3}$	$8.400 \cdot 10^{-3}$	$8.100 \cdot 10^{-3}$	$7.000 \cdot 10^{-3}$	$4.200 \cdot 10^{-3}$
Dt = 0.1	1.49	$8.986 \cdot 10^{-4}$	$8.845 \cdot 10^{-4}$	$6.470 \cdot 10^{-4}$	$3.964 \cdot 10^{-4}$	$2.659 \cdot 10^{-4}$
Dt = 0.02	1.508	$7.892 \cdot 10^{-5}$	$7.215 \cdot 10^{-5}$	$3.935 \cdot 10^{-5}$	$5.831 \cdot 10^{-5}$	$2.177 \cdot 10^{-4}$
final time		2	2	2	2	2
space step		h = 0.005	h = 0.01	h = 0.05	h = 0.1	h = 0.2

Table 1. Numerical results for the higher splitting method.

Remark 1. By using the higher splitting method, we can obtain higher-order convergence rates. The critical reaction-diffusion equation with the stiff influence lowers our fourth-order method to at least a maximum of second order. By the way, the higher order at least benefits, while the order reduction needs at least sufficient accuracy in the method.

6 Conclusions and Discussions

We present decomposition methods for differential equations based on classical methods, e.g. overlapping Schwarz waveform relaxation method in space and non-iterative splitting methods in time as well as modern methods, e.g. iterative operator-splitting methods. The mixture of such methods can benefit the accuracy and the stability by the use of more iteration steps. Combined methods have more freedom degrees and can benefit in a proper way the results. For achieving a competitive method, the optimization between time steps, iterative steps and overlapping have to be taken into account in varying degrees to reduce the amount of additional work. In future such methods will take an important role in decomposing complicated problems into simpler parts and reduce the computational time. Using higher-order decomposition methods, the critical decomposition errors can nearly be skipped compared to the discretization and solver errors.

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