Topological index calculation of DAEs in coupled electromagnetic field/circuit simulation

Lennart Jansen[a,*]

[a] *Humboldt University of Berlin, Institute of Mathematics, Rudower Chaussee 25, 12489 Berlin, Germany*

Abstract

The demand of combining circuit simulation directly with complex device models to refine critical circuit parts becomes more and more important, since the classical circuit simulation can no longer supply sufficiently accurate results. The simulation of such coupled problems leads to large systems and therefore to high computing times. We consider a set of differential-algebraic equations, which arise from an electric circuit modeled by the modified nodal analysis coupled with electromagnetic devices. While the normal circuit elements are zero dimensional elements the electromagnetic devices are given by a three dimensional model. Therefore the number of variables can easily go beyond millions, if we refine the spatial discretization. We analyze the structure of the discretized coupled system and present a way to transform it into a semi-explicit system of differential-algebraic equations. In the process we make use of a new decoupling method for DAEs which results from a mix of the strangeness index and the tractability index. After this remodeling the electromagnetic part of the equation will be a system of ordinary differential equations with sparse matrices only. It will be shown that the topological index conditions for this coupled system are analogous to the conditions for an ordinary electric circuit.

Keywords: Differential-Algebraic Equation, Index analysis, Electromagnetic Device, Electric Circuit

*Corresponding author

Email address: `lejansen@math.hu-berlin.de` (Lennart Jansen)

1. Introduction

Consider an electric circuit consisting of capacitors, resistors, inductors and independent voltage and current sources. We use the modified nodal analysis (MNA) to model this circuit with a set of differential-algebraic equation (DAE). There are many index concepts which classify DAE by assigning an integer to them which is called the index of the DAE. The index of a DAE measures its complexity in comparison to an ordinary differential equation. One of the well established index concepts is the tractability index. It was mainly developed by März, see [GM86, Mär02, LMT13]. The index of an electric circuit consisting of the basic elements mentioned before can be calculated by its topology only, see [Tis99]. In realistic applications the basic elements are not capable of simulating all needed effects. Therefore more complex elements like electromagnetic devices are added to an electric circuit, see [Gün01, Ben06, Sch11, Bau12]. To extend the topological index result to a circuit including electromagnetic devices we introduce a mixed index concept based on the tractability index and the strangeness index, which was firstly established by Kunkel and Mehrmann, see [KM06]. We call this mixed concept the tractability-strangeness index. The extension of the topological index conditions will be accomplished in three steps. First the known index result for electric circuits will be verified by the tractability-strangeness index. Afterwards the model of the electromagnetic device will be discussed. As the third step the index result will be extended to circuits including electromagnetic devices.

2. Basic Electrical Circuits

The first section is divided into three parts. First it introduces the standard notation of electrical circuits and it presents the modified nodal analysis. Second the tractability-strangeness index will be introduced as a mixed index concept based on the tractability index and the strangeness index. After these introductions the known topological index result for electrical circuits will be shown by the tractability-strangeness index.

2.1. Modified Nodal Analysis

A circuit is modeled by a directed graph $G := (N, E)$, where the edges are arbitrarily orientated. The quantities of an electric circuit are the currents over the edges and the electric potentials at the nodes. In order to get a unique solution we need to choose one node as a reference node. The potential of this reference node will be fixed, in general it can be chosen to be zero. We call this reference node the mass node. Then the network topology for elements with two contacts is retained by the incidence matrix $A \in \{-1, 0, 1\}^{(|N|-1)\times|E|}$. The matrix A describes the relation between all edges and all nodes except the mass node. The incidence matrix is defined by:

$$(A)_{ij} := \begin{cases} 1 & \text{, if the edge j leaves node i,} \\ -1 & \text{, if the edge j enters node i,} \\ 0 & \text{, else.} \end{cases}$$

The classical MNA deals with capacitors, resistors, inductors, voltage and current sources as electric elements, see [CL75, CDK87, DK84]. To put these elements in a network framework the network edges are sorted in such a way that the incidence matrix A forms a block matrix with blocks describing the different types of network elements, that is,

$$A = \begin{pmatrix} A_C & A_R & A_L & A_V & A_I \end{pmatrix}.$$

Then the well known MNA can be formulated based on Kirchhoff's current law, Kirchhoff's voltage law and the physical element relations, see [Tis99].

$$A_C q_C'(A_C^T e, t) + A_R g_R(A_R^T e, t) + A_L j_L + A_V j_V + A_I i_s(t) = 0 \tag{2.1}$$

$$\phi_L'(j_L, t) - A_L^T e = 0 \tag{2.2}$$

$$A_V^T e - v_s(t) = 0 \tag{2.3}$$

with $t \in \mathcal{I}$ and $\mathcal{I}$ a compact time interval. Here e are the node potentials while j_L are the currents over the inductors and j_V are the currents over the voltage sources. Further q_C, g_R and ϕ_L are the characteristic functions of the capacitors, resistors and inductors. The function q_C resembles the electric charges of the capacitors, ϕ_L stands for the magnetic flux of the inductors and g_R is the conductance of the resistors. The jacobians $\frac{\partial q_C}{\partial v_C}$, $\frac{\partial g_R}{\partial v_R}$ and $\frac{\partial \phi_L}{\partial j_L}$ are assumed to be positive definite. This assumption can be physically interpreted as the passivity of these elements. Passivity here means that these elements do not emit energy by themselves. We consider independent voltage and current sources, i.e. the source terms can be modeled by functions depending only on time. Furthermore we assume that the circuit is connected and not shorted, i.e. A_V has full column rank and $\begin{pmatrix} A_C & A_R & A_L & A_V \end{pmatrix}$ have full row rank.

2.2. Tractability-Strangeness Index

Now we will introduce the mixed index concept which will be used to analyze the MNA (2.1). The analysis of DAEs is strongly affected by the choice of the index concept. There are many different index concepts, for example the differentiation index [Cam87], the perturbation index [HLR89], geometrical index [RR90, RR02], the strangeness index [KM06] and the tractability index [LMT13]. All of these concepts have their own strength and drawbacks. We now introduce a mixed index concept based on the strangeness index and the tractability index. In contrast to the tractability index this mixed index will provide a forward decoupling of the variables. This mixed concept will later on allow us to extend the known result for the classical MNA with little effort to circuits which include electromagnetic devices. We restrict ourselves to the following class of differential-algebraic equations since we are dealing with circuit applications.

Definition 2.1. (Differential-algebraic equation with leading term)
Let $\mathcal{I} \subset \mathbb{R}$ be a compact interval and let $\mathcal{D} \subset \mathbb{R}^n$ be open and connected. For $(x,t) \in \mathcal{D} \times \mathcal{I}$ observe the following equation

$$Ad'(x,t) + b(x,t) = 0 \tag{2.4}$$

with $d \in C^1(\mathcal{D} \times \mathcal{I}, \mathbb{R}^m)$, $A \in \mathbb{R}^{n\times m}$ and $b \in C(\mathcal{D} \times \mathcal{I}, \mathbb{R}^n)$. Furthermore the partial derivative b_x exists and is continuous. We call (2.4) a semi-linear differential-algebraic equation with nonlinear leading term.

It is possible that parts of im d_x lie in ker A or that parts of coker A lie in coim d_x, with coker $A(x,t) :=$ im $A^T(x,t)$ and coim $A(x,t) :=$ ker $A^T(x,t)$ defined pointwise for a matrix function $A \in C(\mathcal{D} \times \mathcal{I}, \mathbb{R}^{m\times n})$. To avoid these unnecessary gaps and overlaps we only consider DAEs with properly stated leading term, see [LMT13].
To establish a mixed index concept based on the tractability and the strangeness index consider $M \in C(\mathcal{D} \times \mathcal{I}, \mathbb{R}^{m\times n})$ such that there are constant bases of the subspaces coker $M(x,t)$, im $M(x,t)$, ker $M(x,t)$ and coim $M(x,t)$. Thereby there are integers $n_x, n_y, n_v \in \mathbb{N}$ and $n_w \in \mathbb{N}$ such that

$$\begin{aligned} n_x &= \dim(\operatorname{coker} M(x,t)), & n_y &= \dim(\ker M(x,t)), \\ m_v &= \dim(\operatorname{im} M(x,t)), & m_w &= \dim(\operatorname{coim} M(x,t)) \quad \forall (x,t) \in \mathcal{D} \times \mathcal{I}. \end{aligned}$$

We choose four matrix functions

$$\begin{aligned} \mathrm{p} &: \mathcal{D} \times \mathcal{I} \to \mathbb{R}^{n\times n_x}, & \mathrm{q} &: \mathcal{D} \times \mathcal{I} \to \mathbb{R}^{n\times n_y}, \\ \mathrm{v} &: \mathcal{D} \times \mathcal{I} \to \mathbb{R}^{m\times m_v}, & \mathrm{w} &: \mathcal{D} \times \mathcal{I} \to \mathbb{R}^{m\times m_w} \end{aligned}$$

such that the columns of the matrix functions are bases of the subspaces coker $M(x,t)$, im $M(x,t)$, ker $M(x,t)$ and im $M(x,t)$, respectively. We call p, q, v and w the associated basis functions of M and we say M has constant subspaces.
We now construct the tractability-strangeness index up to an index of two under the assumption that all involved matrices have constant subspaces.
Similar to the tractability index we build a chain of matrices starting with $D(x,t) := d_x(x,t)$, $G_0(x,t) = AD(x,t)$ and $B(x,t) = b_x(x,t)$. Let p, q, v and w be the associated basis functions of G_0 and define with their help

$$\begin{aligned} G_1(x,t) := \mathrm{v}^T G_0(x,t)\mathrm{p}, \quad & B_x^{\mathrm{v}}(x,t) := \mathrm{v}^T B(x,t)\mathrm{p}, \quad B_y^{\mathrm{v}}(x,t) := \mathrm{v}^T B(x,t)\mathrm{q}, \\ & B_x^{\mathrm{w}}(x,t) := \mathrm{w}^T B(x,t)\mathrm{p}, \quad B_y^{\mathrm{w}}(x,t) := \mathrm{w}^T B(x,t)\mathrm{q}. \end{aligned}$$

Let $\mathrm{p}_y, \mathrm{q}_y, \mathrm{v}_y, \mathrm{w}_y$ be the four associated basis functions of $B_y^{\mathrm{w}}(x,t)$ and let $\mathrm{p}_x, \mathrm{q}_x$ be the associated basis functions of $\mathrm{w}_y^T B_x^{\mathrm{w}}(x,t)$ with respect to the kernel and the cokernel. Furthermore let $\mathrm{v}_x, \mathrm{w}_x$ be the associated basis functions of $B_y^{\mathrm{v}}(x,t)\mathrm{q}_y$ with respect to the image and the coimage.
With the help of these matrices and basis functions we define the tractability-strangeness index.

Definition 2.2. (Tractability-Strangeness Index)
Let $\mathcal{G} \subset \mathcal{D} \times \mathcal{I}$. We say (2.4) has tractability-strangeness index 0 on $\mathcal{G}$ if $G_0(x,t)$ is non-singular for all $(x,t) \in \mathcal{G}$. If the tractability-strangeness index is not 0 but $\mathrm{w}^T B(x,t)\mathrm{q}$ is non-singular for all $(x,t) \in \mathcal{G}$ the DAE (2.4) has tractability-strangeness index 1 on $\mathcal{G}$. If the tractability-strangeness index is neither 0 nor 1 but $\mathrm{w}_x^T G_1(x,t)\mathrm{q}_x$ is non-singular for all $(x,t) \in \mathcal{G}$ the DAE (2.4) has tractability-strangeness index 2 on $\mathcal{G}$.

2.3. Topological Index Conditions

The purpose of this section is to investigate the index behavior of the MNA equations. Therefore we define

$$A := \begin{pmatrix} A_C & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad d(x,t) := \begin{pmatrix} q_C(A_C^T e, t) \\ \phi_L(j_L, t) \end{pmatrix}$$

and

$$b(x,t) := \begin{pmatrix} A_R g_R(A_R^T e, t) & + & A_L j_L + A_V j_V & + & A_I i_s(t) \\ -A_L^T e & & & & \\ -A_V^T e & & & + & v_s(t) \end{pmatrix}$$

with the variables $x = \begin{pmatrix} e & j_L & j_V \end{pmatrix}^T$. With this notation the MNA can be written as a DAE in the form (2.4). Now we can construct the matrix chain of the tractability-strangeness index. We start by defining

$$D(x,t) := \begin{pmatrix} C(A_C^T e, t) A_C^T & 0 & 0 \\ 0 & L(j_L, t) & 0 \end{pmatrix},$$

$$B(x,t) := \begin{pmatrix} A_R G(A_R^T e, t) A_R^T & A_L & A_V \\ -A_L^T & 0 & 0 \\ -A_V^T & 0 & 0 \end{pmatrix}$$

and therefore we get

$$G_0(x,t) := AD(x,t) = \begin{pmatrix} A_C C(A_C^T e, t) A_C^T & 0 & 0 \\ 0 & L(j_L, t) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To continue the chain we need basis functions related to the incidence matrices. Let p_C and q_C be the basis function associated to the co-kernel and the kernel of A_C^T. Then we call

$$A_{\bar{C}X} \quad := \quad \mathrm{q}_C^T A_X, \quad X \in \{V, R, L, I\}$$

the C-reduced incidence matrix of the voltage sources, resistors, inductors or current sources, respectively. Further denote the full set of associated basis function of $A_{\bar{C}V}^T$ by $\mathrm{p}_{\bar{C}V}, \mathrm{q}_{\bar{C}V}, \mathrm{v}_{\bar{C}V}$ and $\mathrm{w}_{\bar{C}V}$. Analogously we call

$$A_{\bar{C}\bar{V}X} \quad := \quad \mathrm{q}_{CV}^T \mathrm{q}_C^T A_X, \quad X \in \{R, L, I\}$$

the CV-reduced incidence matrix of the resistors, inductors or current sources, respectively. At last we obtain the basis function $\mathrm{p}_{\bar{C}\bar{V}R}$ and $\mathrm{q}_{\bar{C}\bar{V}R}$ associated to the co-kernel and the kernel of $A^T_{\bar{C}\bar{V}R}$ and denote by

$$A_{\bar{C}\bar{V}\bar{R}X} \quad := \quad \mathrm{q}^T_{\bar{C}\bar{V}R}\mathrm{q}^T_{\bar{C}V}\mathrm{q}^T_{C}A_X, \quad X \in \{L, I\}$$

the CVR-reduced incidence matrix of the inductors or current sources, respectively.
We can now define the associated basis function of G_0 by

$$\mathrm{p} = \mathrm{v} := \begin{pmatrix} \mathrm{p}_C & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \mathrm{q} = \mathrm{w} := \begin{pmatrix} \mathrm{q}_C & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}$$

with the help of the basis functions of A^T_C. The basis functions for the kernel and the co-image and the basis functions for the image and the co-kernel are equal since G_0 is symmetric. Carrying on to the next stage of the matrix chain we get

$$\begin{aligned}
G_1(x,t) &= \mathrm{v}^T G_0(x,t)\mathrm{p} = \begin{pmatrix} \mathrm{p}^T_C A_C C(A^T_C e, t) A^T_C \mathrm{p}_C & 0 \\ 0 & L(j_L, t) \end{pmatrix}, \\
B^{\mathrm{v}}_x(x,t) &= \mathrm{v}^T B(x,t)\mathrm{p} = \begin{pmatrix} \mathrm{p}^T_C A_R G(A^T_R e, t) A^T_R \mathrm{p}_C & \mathrm{p}^T_C A_L \\ -A^T_L \mathrm{p}_C & 0 \end{pmatrix}, \\
B^{\mathrm{v}}_y(x,t) &= \mathrm{v}^T B(x,t)\mathrm{q} = \begin{pmatrix} \mathrm{p}^T_C A_R G(A^T_R e, t) A^T_{\bar{C}R} & \mathrm{p}^T_C A_V \\ -A^T_{\bar{C}L} & 0 \end{pmatrix}, \\
B^{\mathrm{w}}_x(x,t) &= \mathrm{w}^T B(x,t)\mathrm{p} = \begin{pmatrix} A_{\bar{C}R} G(A^T_R e, t) A^T_R \mathrm{p}_C & A_{\bar{C}L} \\ -A^T_V \mathrm{p}_C & 0 \end{pmatrix}, \\
B^{\mathrm{w}}_y(x,t) &= \mathrm{w}^T B(x,t)\mathrm{q} = \begin{pmatrix} A_{\bar{C}R} G(A^T_R e, t) A^T_{\bar{C}R} & A_{\bar{C}V} \\ -A^T_{\bar{C}V} & 0 \end{pmatrix}
\end{aligned}$$

and obtain the basis functions

$$\mathrm{p}_y = \mathrm{v}_y := \begin{pmatrix} \mathrm{p}_{\bar{C}V} & \mathrm{q}_{\bar{C}V}\mathrm{p}_{\bar{C}\bar{V}R} & 0 \\ 0 & 0 & \mathrm{v}_{\bar{C}V} \end{pmatrix}, \quad \mathrm{q}_y = \mathrm{w}_y := \begin{pmatrix} \mathrm{q}_{\bar{C}V}\mathrm{q}_{\bar{C}\bar{V}R} & 0 \\ 0 & \mathrm{w}_{\bar{C}V} \end{pmatrix}$$

associated to B^{w}_y. Write

$$\mathrm{w}^T_y B^{\mathrm{w}}_x(x,t) = \begin{pmatrix} 0 & A_{\bar{C}\bar{V}\bar{R}L} \\ -\mathrm{w}^T_{\bar{C}V} A^T_V \mathrm{p}_C & 0 \end{pmatrix},$$

$$B^{\mathrm{v}}_y(x,t)\mathrm{q}_y = \begin{pmatrix} 0 & \mathrm{p}^T_C A_V \mathrm{w}_{\bar{C}V} \\ -A^T_{\bar{C}\bar{V}\bar{R}L} & 0 \end{pmatrix}$$

and get with $\mathrm{p}_{LI}, \mathrm{q}_{LI}$ the associated basis functions of the co-kernel and kernel of $A_{\bar{C}\bar{V}\bar{R}L}$ and $\mathrm{p}_{CV}, \mathrm{q}_{CV}$ the associated basis functions of the co-kernel and kernel of $\mathrm{w}_{\bar{C}V}^T A_V^T \mathrm{p}_C$ the last basis functions of the chain:

$$\mathrm{p}_x = \mathrm{v}_x := \begin{pmatrix} \mathrm{p}_{CV} & 0 \\ 0 & \mathrm{p}_{LI} \end{pmatrix}, \quad \mathrm{q}_x = \mathrm{w}_x := \begin{pmatrix} \mathrm{q}_{CV} & 0 \\ 0 & \mathrm{q}_{LI} \end{pmatrix}.$$

Notice here again that $\mathrm{p}_x = \mathrm{v}_x$ and $\mathrm{q}_x = \mathrm{w}_x$ due to the symmetry of $\mathrm{w}_y^T B_x^{\mathrm{w}}(x,t)$ and $B_y^{\mathrm{v}}(x,t)\mathrm{q}_y$. For the further index investigation we only need the three matrices $G_0(x,t), B_y^{\mathrm{w}}(x,t)$ and

$$\mathrm{w}_x^T G_1(x,t)\mathrm{q}_x = \begin{pmatrix} \mathrm{q}_{CV}^T \mathrm{p}_C^T A_C C(A_C^T e,t) A_C^T \mathrm{p}_C \mathrm{q}_{CV} & 0 \\ 0 & \mathrm{q}_{LI}^T L(j_L,t)\mathrm{q}_{LI} \end{pmatrix}.$$

With the help of these three matrices we can prove the following topological index theorem for electrical circuits.

Theorem 2.3.
The MNA has tractability-strangeness index

(i) 0, if and only if there is a spanning tree in the circuit consisting only of capacitors and there are no voltage sources in the circuit.

(ii) 1, or lower if there are no loop consisting of capacitors and voltage sources with at least one voltage source and no cutsets consisting of inductors and current sources.

(iii) 2, else.

Proof. As long as $\mathrm{w}_x^T G_1(x,t)\mathrm{q}_x$ is non-singular the index is 2 at most. We know that $C(A_C^T e,t)$ and $L(j_L,t)$ are positive definite and that $A_C^T \mathrm{p}_C$, q_{CV} and q_{LI} have full column rank and thereby follows (iii).
The topological index-1 conditions yield $\mathrm{im}\ A_C \cap \mathrm{im}\ A_V = \{0\}$ and that $\begin{pmatrix} A_C & A_R & A_V \end{pmatrix}$ has full row rank. The first condition yields that $A_{\bar{C}V}$ has full column rank since A_V has full column rank and $\mathrm{im}\ A_C = \ker\ \mathrm{q}_C^T$. The second index-1 condition provides the full row rank of $A_{\bar{C}\bar{V}R}$ since q_C^T and $\mathrm{q}_{\bar{C}V}^T$ have full row rank.
Define the non-singular matrix

$$T(x,t) = \begin{pmatrix} \mathrm{p}_{\bar{C}V} & \mathrm{q}_{\bar{C}V} & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then $B_y^{\mathrm{w}}(x,t)$ is non-singular since

$$\begin{aligned} &T^T(x,t)B_y^{\mathrm{w}}(x,t)T(x,t) \\ &= \begin{pmatrix} \mathrm{p}_{\bar{C}V}^T A_{\bar{C}R} G(A_R^T e,t) A_{\bar{C}R}^T \mathrm{p}_{\bar{C}V} & \mathrm{p}_{\bar{C}V}^T A_{\bar{C}R} G(A_R^T e,t) A_{\bar{C}\bar{V}R}^T & \mathrm{p}_{\bar{C}V}^T A_{\bar{C}V} \\ A_{\bar{C}\bar{V}R} G(A_R^T e,t) A_{\bar{C}R}^T \mathrm{p}_{\bar{C}V} & A_{\bar{C}\bar{V}R} G(A_R^T e,t) A_{\bar{C}\bar{V}R}^T & 0 \\ -A_{\bar{C}V}^T \mathrm{p}_{\bar{C}V} & 0 & 0 \end{pmatrix} \end{aligned}$$

is non-singular because $G(A_R^T e, t)$ is positive definite, $A_{\bar{C}\bar{V}R}$ has full column rank and $\mathrm{p}_{\bar{C}V}^T A_{\bar{C}V}$ is non-singular. Therefore (ii) holds. Next observe G_0 under the assumption that there are no voltage sources

$$G_0(x,t) = \begin{pmatrix} A_C C(A_C^T e, t) A_C^T & 0 \\ 0 & L(j_L, t) \end{pmatrix}.$$

The index 0 condition also states that there is a spanning tree in the circuit consisting only of capacitors and therefore A_C^T has full column rank. With $C(A_C^T e, t)$ and $L(j_L, t)$ positive definite again G_0 is non-singular. □

The topological index result with respect to the tractability index can be found in [Tis99].

3. Electromagnetic Device

After verifying the topological index conditions for basic network elements we introduce a more advanced element: the electromagnetic device. This new element will be based on Maxwell's laws formulated on a three dimensional domain. As the first section this one is also split into three subsections. First the continuous three dimensional model will be discussed. Second we obtain a discrete version of the model with the help of the finite integration technique. And thirdly we will classify this discrete model in the classes of the basic elements.

3.1. Maxwell's Laws

Let $\Omega \subset \mathbb{R}^3$ be a connected domain and let $\mathcal{I} \subset \mathbb{R}$ be a compact time interval. The electric and magnetic field is given by $\mathcal{E}, \mathcal{H} : \Omega \times \mathcal{I} \to \mathbb{R}^3$ while the electric and magnetic induction is given by $\mathcal{D}, \mathcal{B} : \Omega \times \mathcal{I} \to \mathbb{R}^3$. The conduction current density is described by $\mathcal{J}_C : \Omega \times \mathcal{I} \to \mathbb{R}^3$.
With these quantities the Maxwell-Ampere law and Maxwell-Faraday law read:

$$\nabla \times \mathcal{H} = \mathcal{J}_C + \frac{\partial}{\partial t}\mathcal{D}$$
$$\nabla \times \mathcal{E} = -\frac{\partial}{\partial t}\mathcal{B}$$

Depending on the material there are three relations between the electromagnetic quantities. The electric permittivity is given by $\varepsilon : \Omega \to \mathbb{R}$, the electric conductivity $\sigma : \Omega \to \mathbb{R}$ and the magnetic reluctivity $\nu : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ describe these relations by

$$\mathcal{D} = \varepsilon \mathcal{E}, \quad \mathcal{J}_C = \sigma \mathcal{E}, \quad \mathcal{H} = \nu(\mathcal{B})\mathcal{B}$$

Inserting these material relations into the Maxwell-Ampere law we obtain a formulation of the Maxwell's equations in the electric field and the magnetic induction:

$$\nabla \times \nu(\mathcal{B})\mathcal{B} = \sigma \mathcal{E} + \varepsilon \frac{\partial}{\partial t}\mathcal{E}$$

$$\nabla \times \mathcal{E} = -\frac{\partial}{\partial t}\mathcal{B}.$$

Before adding boundary conditions to the system we need to reformulate the Maxwell equations in the potential formulation. Therefore we introduce the scalar potential $\varphi : \Omega \times \mathcal{I} \to \mathbb{R}$ and the vector potential $A : \Omega \times \mathcal{I} \to \mathbb{R}^3$ which are uniquely defined by the two potential equations

$$\begin{aligned} \mathcal{E} &= -\nabla\varphi - \frac{\partial}{\partial t}A \\ \mathcal{B} &= \nabla \times A \end{aligned}$$

and the Coulomb gauge equation $0 = \nabla \cdot A$. The gauge equation is only a auxiliary equation needed for the uniqueness of the vector and scalar potential. If we add these three equations to the Maxwell equations we end up with the potential formulation

$$\begin{aligned} \nabla \times \nu(\mathcal{B})\mathcal{B} &= \sigma\mathcal{E} + \varepsilon\frac{\partial}{\partial t}\mathcal{E} \\ \mathcal{E} &= -\nabla\varphi - \frac{\partial}{\partial t}A \\ \mathcal{B} &= \nabla \times A \\ 0 &= \nabla \cdot A. \end{aligned}$$

We dropped the Maxwell-Faraday law since it is implied by the potential equations. We separate the boundary of Ω into $\partial\Omega = \partial\bar{\Omega} \cup \Gamma_{j_E}$ with $\partial\bar{\Omega} \cap \Gamma_{j_E} = \emptyset$. We call Γ_{j_E} the contact areas of the electromagnetic device. The contact areas Γ_{j_E} may consist of several disjunct sets, such that $\Gamma_{j_E} = \bigcup_{i=1}^{n} \Gamma_{j_E}^{i}$ with n_Γ the number of contact areas. Now we are able to connect the Maxwell equations in an easy way to the quantities of an electric circuit. These quantities are the electric currents $j_E \in \mathbb{R}^{n_\Gamma}$ and voltages $u_E \in \mathbb{R}^{n_\Gamma}$ at the contact areas. First we couple the conduction current density to the current of the electric circuit with the help of the following current coupling equation:

$$j_E = \int_{\Gamma_{j_E}} \mathcal{J}_t \cdot \vec{n_\perp} dF$$

This means that the electric current is the sum of the conduction current density at the contact areas. With the help of the Maxwell-Faraday law and the material relations we can again express this part in terms of $\mathcal{B}$. Therefore we introduce the total and displacement current $\mathcal{J}_t$ and $\mathcal{J}_d$ with the properties $\mathcal{J}_d = \frac{\partial}{\partial t}\mathcal{D}$ and $\mathcal{J}_t = \mathcal{J}_C + \mathcal{J}_d$ and write

$$\mathcal{J}_t = \mathcal{J}_C + \mathcal{J}_d = \mathcal{J}_C + \frac{\partial}{\partial t}\mathcal{D} \underset{\text{Faraday}}{\overset{\text{Maxwell}}{=}} \nabla \times \mathcal{H} \underset{\text{relation}}{\overset{\text{Material}}{=}} \nabla \times \nu(\mathcal{B})\mathcal{B}.$$

So we can write the current coupling equation as

$$j_E = \int_{\Gamma_{j_E}} \nabla \times \nu(\mathcal{B})\mathcal{B} \cdot \vec{n_\perp} dF.$$

The boundary conditions are used to couple the electric potential to the Maxwell equations. The scalar potential is set equal to the electric potential at the contact areas $\varphi_{|\Gamma_{j_E}} = u_E$ and otherwise homogeneous Dirichlet boundary condition are used. Together we obtain the system

$$\begin{aligned} j_E &= \int_{\Gamma_{j_E}} \nabla \times \nu(\mathcal{B})\mathcal{B} \cdot \vec{n_\perp} dF \\ \varepsilon \frac{\partial}{\partial t}\mathcal{E} + \sigma\mathcal{E} &= \nabla \times \nu(\mathcal{B})\mathcal{B} \\ \frac{\partial}{\partial t} A &= -\nabla\varphi - \mathcal{E} \\ \mathcal{B} &= \nabla \times A \\ 0 &= \nabla \cdot A \end{aligned}$$

with $\varphi_{|\Gamma_{j_E}} = u_E$.

3.2. Spatial Discretization

For spatial discretization we choose the finite integration technique (FIT). The FIT discretization is an established tool to discretize electromagnetic devices which was was developed and formulated by Thomas Weiland [Wei77, TW96, CW01]. For detailed information on the FIT discretization we refer to pages 60–90 in [Bau12] or pages 5–14 in [Sch11]. We call the discretized electric field $E \in \mathbb{R}^{3n}$ and the discretized magnetic density $B \in \mathbb{R}^{3n}$ with n depending on the refinement of the FIT discretization. Further we call $a \in \mathbb{R}^{3n}$ and $\phi \in \mathbb{R}^n$ the discretized vector and scalar potential while $M_\varepsilon, M_\sigma, M_\nu \in \mathbb{R}^{3n\times 3n}$ represent for the three material properties, respectively. The discretized versions of the differential operators are notated with $G \in \mathbb{R}^{3n\times n}$ in the case of the gradient, $G^T \in \mathbb{R}^{3\times 3n}$ in the case of the divergence and $C \in \mathbb{R}^{3n\times 3n}$ in the case of the rotation operator. Last we define the excitation matrix $\Lambda \in \mathbb{R}^{3n\times n_\Gamma}$ which represents the boundary operator meaning Λ indicates if a point of the discretization grid belongs to the contact areas. Note that due to the FIT discretization the transposed excitation matrix $\Lambda^T \in \mathbb{R}^{n_\Gamma \times 3n}$ represents the integral over the contact areas.

The discretized operators and matrices of the FIT discretization fulfill a set of important properties, see [Bau12, Sch11]. The discretized material relations M_ε and M_ν are positive definite diagonal matrices while M_σ is a positive semidefinite diagonal matrices. Furthermore $C\Lambda$ has full column rank and the equality $\nabla \times \nabla = 0$ is inherited by the discretized operators $CG = 0$ and also it holds that the kernel of C and G^T are disjunct, i.e. $\ker C \cap \ker G^T = \{0\}$

With the FIT discretization we can write the Maxwell equations in the potential formulation like

$$j_E = \Lambda^T C^T M_\nu(B)B \tag{3.1}$$

$$M_\varepsilon E' + M_\sigma E = C^T M_\nu(B)B \tag{3.2}$$

$$a' = -G\phi + \Lambda u_E - E \tag{3.3}$$

$$B = Ca \tag{3.4}$$
$$0 = G^T a. \tag{3.5}$$

Previously we introduced the potential equation to connect the quantities of the electromagnetic device to the electric circuit via boundary conditions. After the spatial discretization these boundary conditions are wrapped up in the equations with the help of the excitation Λ. In the following we will show that the potential equation is not needed any more and that we are able to decouple them from the rest of the equations again.
For that matter multiply equation (3.3)

$$a' = -G\phi + \Lambda u_E - E$$

from the left by C and G^T, respectively. It holds that $\ker C \cap \ker G^T = \{0\}$ and hence we get the equivalent system

$$\begin{aligned} (Ca)' &= -CG\phi + C\Lambda u_E - CE \\ (G^T a)' &= -G^T G\phi + G^T \Lambda u_E - G^T E. \end{aligned}$$

Insert $B = Ca$, $CG = 0$ and $G^T a = 0$ into these equations and obtain

$$B' = C\Lambda u_E - CE \tag{3.6}$$
$$0 = -G^T G\phi + G^T \Lambda u_E - G^T E. \tag{3.7}$$

If we replace (3.3) in the discretized Maxwell equations with (3.6) we get

$$\begin{aligned} j_E &= \Lambda^T C^T M_\nu(B) B \\ M_\varepsilon E' + M_\sigma E &= C^T M_\nu(B) B \\ B' &= C\Lambda u_E - CE \\ 0 &= -G^T G\phi + G^T \Lambda u_E - G^T E \\ B &= Ca \\ 0 &= G^T a. \end{aligned}$$

The variables ϕ and a do not appear in the first three equations anymore. Therefore the potential equations can be dropped again.
We call

$$j_E = \Lambda^T C^T M_\nu(B) B \tag{3.8}$$
$$M_\varepsilon E' + M_\sigma E = C^T M_\nu(B) B \tag{3.9}$$
$$B' = C\Lambda u_E - CE \tag{3.10}$$

the discretized Maxwell equations in standard formulation. The discretized Maxwell equations include the discretized current coupling equation, the discretized Maxwell-Ampere's law with inserted material relations and the discretized Maxwell-Faraday's law with inserted potential boundary conditions.

3.3. Electromagnetic Inductor

In the following part the space discretized model will be classified as an inductor-like electric element. Therefore consider an electric inductor with a linear constant inductance L. If j_L is the current flowing through the inductor and u_L is the voltage over the inductor then an electric inductor describes a relation $Lj'_L = u_L$ between the derivative of its current and its voltage. Before the inductor-like behavior of the electromagnetic device will be verified with the model equations we motivate this objective by an example. Consider a simple coil made out of copper. We apply a voltage on this electromagnetic device and observe the resulting current. At the same time we apply the same voltage on a suitable linear resistor as a reference. While the resistor is influenced by a change in the voltage directly the electromagnetic device seems to be more inertial, see Figure 3.1. If we focus on the middle part we can clearly see a direct relation between the value of the voltage and the slope of the current. Next we also want to verify this behavior with the help of the model equa-

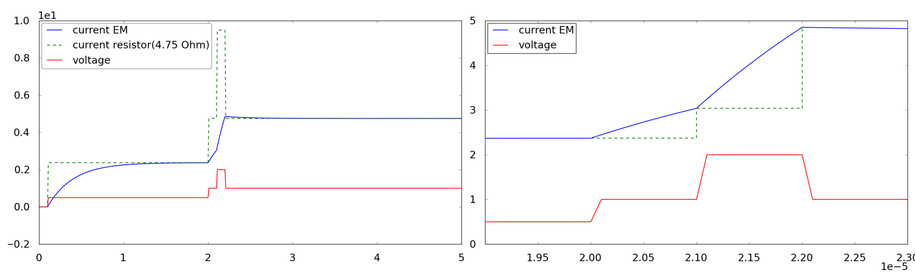


Figure 3.1: The electromagnetic device compared to a resistor

tions. To accomplish that we need to assume the reluctivity of the material in a neighborhood around the contact areas is constant, i.e.

$$\Lambda^T C^T M_\nu(B) = \Lambda^T C^T M_\nu.$$

In applications we can assume that we will find electrical wires at the contact areas since these are the places where the device is connected to the electric circuit. Electrical wires are made of copper or aluminum and these two material have in fact a constant reluctivity hence the assumption is reasonable.
With this assumption we consider the following system

$$\begin{aligned} j_E &= \Lambda^T C^T M_\nu B \\ M_\varepsilon E' + M_\sigma E &= C^T M_\nu(B) B \\ B' &= C\Lambda u_E - CE. \end{aligned}$$

For the classification we take a special interest in the current coupling equation

$$j_E = \Lambda^T C^T M_\nu B.$$

Differentiate the current coupling equation to work out the relation between j'_E and u_E:

$$j'_E = \Lambda^T C^T M_\nu B'$$

Insert the discretized Maxwell-Faraday's law into the derived current coupling equation and get

$$\begin{aligned} j'_E &= \Lambda^T C^T M_\nu B' \\ \Leftrightarrow j'_E &= \Lambda^T C^T M_\nu (C\Lambda u_E - CE) \\ \Leftrightarrow j'_E &= \Lambda^T C^T M_\nu C\Lambda u_E - \Lambda^T C^T M_\nu CE. \end{aligned}$$

Remember that $C\Lambda$ has full column rank and M_ν is positive definite. Therefore $\Lambda^T C^T M_\nu C\Lambda$ is also positive definite. Define $L_E := (\Lambda^T C^T M_\nu C\Lambda)^{-1}$ and $V_E := L_E \Lambda^T C^T M_\nu C$ and write

$$\begin{aligned} j'_E &= L_E^{-1} u_E - \Lambda^T C^T M_\nu CE \\ \Leftrightarrow L_E j'_E &= u_E - L_E \Lambda^T C^T M_\nu CE \\ \Leftrightarrow L_E j'_E &- u_E + V_E E = 0 \end{aligned}$$

This equation reminds us of the characteristic equation of a linear electric inductor

$$L j'_L - u_L = 0.$$

Hence we end up with the following set of equations

$$L_E j'_E - u_E + V_E E = 0 \tag{3.11}$$

$$M_\varepsilon E' + M_\sigma E - C^T M_\nu(B)B = 0 \tag{3.12}$$

$$B' + CE - C\Lambda u_E = 0, \tag{3.13}$$

which we call the electromagnetic inductor equations.

4. Coupled Circuit/Field Problem

In the last section we want to add the electromagnetic inductor to the electrical circuit and afterwards extend the topological index theorem to this more general case.

4.1. Extended Modified-Nodal-Analysis

Remember the equations of the MNA (2.1) which describe an electrical circuit with capacitors, resistors, inductors and current and voltage sources. We want to include the electromagnetic inductor to this framework. To illustrate this procedure consider the following example.
We have two interlocking copper half loops with two contact areas per half loop. There is a resistor connected directly to the contact areas of the first

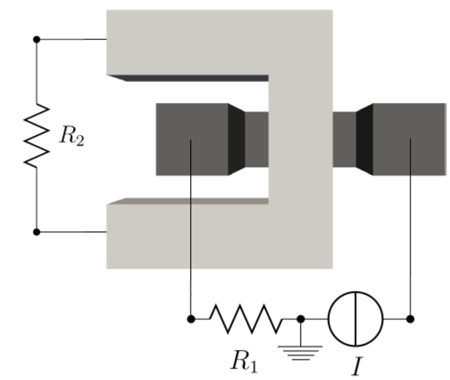
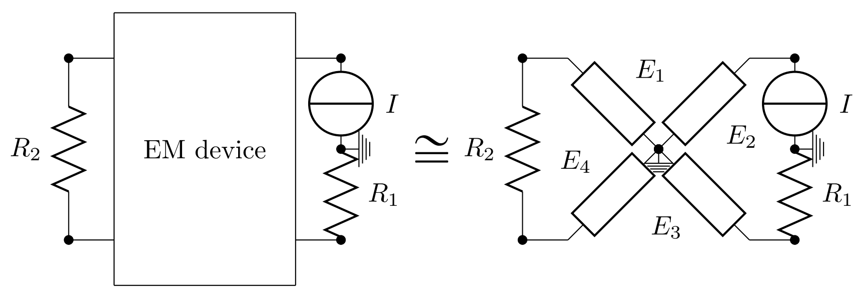


Figure 4.1: Two interlocking copper half loops connected to an electrical circuit.

half loop while a current source and a resistor are connected to the contact areas of the other one. As in the example each electromagnetic device has a number of contact areas. To include the electromagnetic devices into the network framework we add one edge to the circuit graph for every contact area of every device. Each of these edges is connected to one node and the mass node. The nodes, except the mass node, which are connected to the new edges are the ones which are connected to the contact areas of the device.



If a current flows into or out of an electromagnetic device this current leaves or enters the circuit. To reconcil this fact with the conservation of energy the other end of the added edges is connected to the mass node. By choice all new edges are directed to the mass node. Due to this modeling strategy the incidence matrix A can now be split into

$$A = \begin{pmatrix} A_C & A_R & A_L & A_E & A_V & A_I \end{pmatrix}.$$

This allows us to formulate the coupled circuit/field problem with an extended version of the MNA:

$$\begin{aligned} A_C q_C'(A_C^T e, t) + A_R g_R(A_R^T e, t) + A_L j_L + A_E i_E + A_V j_V + A_I i_s(t) &= 0 \\ \phi_L'(j_L, t) - A_L^T e &= 0 \end{aligned}$$

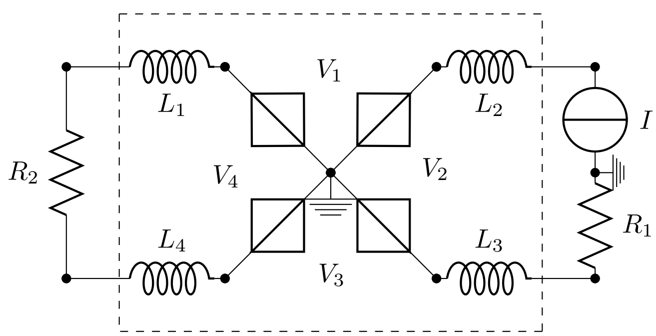
$$
\begin{aligned}
L_E j_E' - A_E^T e + V_E E &= 0 \\
A_V^T e - v_s(t) &= 0 \\
M_\varepsilon E' + M_\sigma E - C^T M_\nu(B) B &= 0 \\
B' + CE - C \Lambda A_E^T e &= 0
\end{aligned}
$$

This set of equations is a combination of (2.1) and (3.11).

Remark 4.1. It is possible to understand the electromagnetic device as a combination of linear constant inductors and controlled voltage sources. To realize that we reorder the current coupling equation of the electromagnetic inductor equations

$$
L_E j_E' - u_E + V_E E = 0 \Leftrightarrow \begin{cases} L_E j_E' - u_E^1 = 0 \\ u_E^2 = V_E E \\ u_E^1 + u_E^2 = u_E \end{cases} .
$$

Now we got as many linear inductors with inductance L_E and controlled voltage sources as we got contact areas. The voltage sources are controlled by the electric field which means they are indirectly controlled by the potential at the contact node of the device since these potentials serve as input functions for the Maxwell equations. With this perception we can redraw the circuit of the example and obtain:



Again the similarity of a classic nonlinear inductor and the electromagnetic inductor attracts attention. In the remark we even saw that the electromagnetic inductor can be assembled with the help of constant inductors. With

$$
j_{\mathcal{L}} = \begin{pmatrix} j_L \\ j_E \end{pmatrix}, \quad A_{\mathcal{L}} = \begin{pmatrix} A_L & A_E \end{pmatrix}, \quad \phi_{\mathcal{L}}(j_{\mathcal{L}}, t) = \begin{pmatrix} \phi_L(j_L, t) \\ L_E j_E \end{pmatrix}
$$

and

$$
\Lambda_{\mathcal{L}} = \begin{pmatrix} 0 & \Lambda \end{pmatrix}, \quad V_{\mathcal{L}} = \begin{pmatrix} 0 \\ V_E \end{pmatrix}
$$

we join these two classes of inductors into a general inductor class.

With this general inductor class the extended MNA reads:

$$\begin{aligned} A_C q_C'(A_C^T e,t) + A_R g_R(A_R^T e,t) + A_{\mathcal{L}} j_{\mathcal{L}} + A_V j_V + A_I i_s(t) &= 0 \\ \phi_{\mathcal{L}}'(i_{\mathcal{L}},t) - A_{\mathcal{L}}^T e + V_{\mathcal{L}} E &= 0 \\ A_V^T e - v_s(t) &= 0 \\ M_\varepsilon E' + M_\sigma E - C^T M_\nu(B) B &= 0 \\ B' + CE - C\Lambda_{\mathcal{L}} A_{\mathcal{L}}^T e &= 0 \end{aligned}$$

4.2. Extended Topological Index Conditions

In this last part we generalize Theorem 2.3 to the extended MNA. Therefore we need to calculate the matrix chain again. We start by defining

$$B(x,t) := \begin{pmatrix} A_R G(A_R^T e,t) A_R^T & A_{\mathcal{L}} & A_V & 0 & 0 \\ -A_{\mathcal{L}}^T & 0 & 0 & V_{\mathcal{L}} & 0 \\ -A_V^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_\sigma & K_\nu(B) \\ -C\Lambda_{\mathcal{L}} A_{\mathcal{L}}^T & 0 & 0 & C & 0 \end{pmatrix}$$

and

$$G_0(x,t) = \begin{pmatrix} A_C C(A_C^T e,t) A_C^T & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}(i_{\mathcal{L}},t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_\varepsilon & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

with $K_\nu(B) := -\frac{\partial}{\partial B}(C^T M_\nu(B) B)$. Similar to the the matrix chain of the classical MNA we get

$$\mathrm{p} = \mathrm{v} := \begin{pmatrix} \mathrm{p}_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad \mathrm{q} = \mathrm{w} := \begin{pmatrix} \mathrm{q}_C & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore

$$G_1(x,t) = \mathrm{v}^T G(x,t) \mathrm{p} = \begin{pmatrix} \mathrm{p}_C^T A_C C(A_C^T e,t) A_C^T \mathrm{p}_C & 0 & 0 & 0 \\ 0 & \mathcal{L}(i_{\mathcal{L}},t) & 0 & 0 \\ 0 & 0 & M_\varepsilon & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and

$$B_x^{\mathrm{v}}(x,t) = \mathrm{v}^T B(x,t) \mathrm{p} = \begin{pmatrix} \mathrm{p}_C^T A_R G(A_R^T e,t) A_R^T \mathrm{p}_C & \mathrm{p}_C^T A_{\mathcal{L}} & 0 & 0 \\ -A_{\mathcal{L}}^T \mathrm{p}_C & 0 & V_{\mathcal{L}} & 0 \\ 0 & 0 & M_\sigma & K_\nu(B) \\ -C\Lambda_{\mathcal{L}} A_{\mathcal{L}}^T \mathrm{p}_C & 0 & C & 0 \end{pmatrix},$$

$$B_y^{\mathrm{v}}(x,t) = \mathrm{v}^T B(x,t)\mathrm{q} = \begin{pmatrix} \mathrm{p}_C^T A_R G(A_R^T e,t) A_R^T \mathrm{q}_C & \mathrm{p}_C^T A_V \\ -A_{\bar{C}\mathcal{L}}^T & 0 \\ 0 & 0 \\ -C\Lambda_{\mathcal{L}} A_{\bar{C}\mathcal{L}}^T & 0 \end{pmatrix},$$

$$B_x^{\mathrm{w}}(x,t) = \mathrm{w}^T B(x,t)\mathrm{p} = \begin{pmatrix} A_{\bar{C}R} G(A_R^T e,t) A_R^T \mathrm{p}_C & A_{\bar{C}\mathcal{L}} & 0 & 0 \\ -A_V^T \mathrm{p}_C & 0 & 0 & 0 \end{pmatrix},$$

$$B_y^{\mathrm{w}}(x,t) = \mathrm{w}^T B(x,t)\mathrm{q} = \begin{pmatrix} A_{\bar{C}R} G(A_R^T e,t) A_{\bar{C}R}^T & A_{\bar{C}V} \\ -A_{\bar{C}V}^T & 0 \end{pmatrix}.$$

Again we compute

$$\mathrm{w}_y^T B_x^{\mathrm{w}} = \begin{pmatrix} 0 & A_{\bar{C}\bar{V}\bar{R}\mathcal{L}} & 0 & 0 \\ -\mathrm{w}_{\bar{C}V}^T A_V^T \mathrm{p}_C & 0 & 0 & 0 \end{pmatrix},$$

$$B_y^{\mathrm{v}} \mathrm{q}_y = \begin{pmatrix} 0 & \mathrm{p}_C^T A_V \mathrm{w}_{\bar{C}V} \\ -A_{\bar{C}\bar{V}\bar{R}\mathcal{L}}^T & 0 \\ 0 & 0 \\ -C\Lambda_{\mathcal{L}} A_{\bar{C}\bar{V}\bar{R}\mathcal{L}}^T & 0 \end{pmatrix}$$

to obtain the basis functions

$$\mathrm{q}_x := \begin{pmatrix} \mathrm{q}_{CV} & 0 & 0 & 0 \\ 0 & \mathrm{q}_{\mathcal{L}I} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \text{ and } \mathrm{w}_x := \begin{pmatrix} \mathrm{q}_{CV} & 0 & 0 & 0 \\ 0 & \mathrm{q}_{\mathcal{L}I} & 0 & -\Lambda_{\mathcal{L}}^T C^T \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

The chain then ends with

$$\mathrm{w}_x^T G_1(x,t)\mathrm{q}_x = \begin{pmatrix} \mathrm{q}_{CV}^T \mathrm{p}_C^T A_C C(A_C^T e,t) A_C^T \mathrm{p}_C \mathrm{q}_{CV} & 0 & 0 & 0 \\ 0 & \mathrm{q}_{\mathcal{L}I}^T \mathcal{L}(i_{\mathcal{L}},t)\mathrm{q}_{\mathcal{L}I} & 0 & 0 \\ 0 & 0 & M_\varepsilon & 0 \\ 0 & -C\Lambda_{\mathcal{L}}\mathcal{L}(i_{\mathcal{L}},t)\mathrm{q}_{\mathcal{L}I} & 0 & I \end{pmatrix}.$$

Analogously to the previous theorem there are topological index conditions for the extended version of the MNA.

Theorem 4.2.
The extended MNA has tractability-strangeness index

(i) 0, if and only if there is a spanning tree in the circuit consisting only of capacitors and there are no voltage sources in the circuit.

(ii) 1, or lower if there are no loop consisting of capacitors and voltage sources with at least one voltage source and no cutsets consisting of general inductors and current sources.

(iii) 2, else.

Proof. The proof follows the lines of Theorem 2.3 except for the addition that M_ε and I are non-singular. □

Due to the classification of the electromagnetic device as an inductor-like and the framework of the tractability-strangeness index we are able to canonically generalize the index conditions.

5. Conclusion and Outlook

We saw that the known topological index conditions for electrical circuits also hold for the tractability-strangeness index. In the framework of this mixed index concept it was possible to extend these topological index conditions to circuits including electromagnetic devices in a canonically way.
The next step will be to classify semiconductor devices as capacitor-like elements and extend the topological index conditions once more.

Acknowledgement

I would like to thank Sascha Baumanns for the inspiring discussions about coupled systems which enabled me to write this paper.

References

[Bau12] S. Baumanns. *Coupled Electromagnetic Field/Circuit Simulation: Modeling and Numerical Analysis.* PhD thesis, Universität zu Köln, 2012.

[Ben06] G. Benderskaya. *Numerical Methods for Transient Field-Circuit Coupled Simulations Based on the Finite Integration Technique and a Mixed Circuit Formulation.* Dissertation, Technische Universität Darmstadt, 2006.

[Cam87] S.L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM journal on Mathematical Analysis*, 18(4):1101–1115, 1987.

[CDK87] L.O. Chua, C.A. Desoer, and E.S. Kuh. *Linear and nonlinear circuits.* McGraw-Hill, Singapore, 1987.

[CL75] L.O. Chua and P.M. Lin. *Computer-aided analysis of electronic circuits: algorithms and computational techniques.* Prentice-Hall series in electrical and computer engineering. Prentice-Hall, 1975.

[CW01] M. Clemens and T. Weiland. Discrete Electromagnetism with the Finite Integration Technique. *Progress In Electromagnetics Research*, 32:65–87, 2001.

[DK84] C.A. Desoer and E.S. Kuh. *Basic Circuit Theory.* International student edition. McGraw-Hill, 1984.

[GM86] E. Griepentrog and R. März. *Differential-Algebraic Equations and Their Numerical Treatment.* Teubner, Leipzig, 1986.

[Gün01] M. Günther. *Partielle differential-algebraische Systeme in der numerischen Zeitbereichsanalyse elektrischer Schaltungen.* Number 343 in Fortschritt-Berichte VDI, Reihe 20. VDI Verlag, Düsseldorf, 2001.

[HLR89] E. Hairer, C. Lubich, and M. Roche. *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods.* Springer, New York, 1989.

[KM06] P. Kunkel and V. Mehrmann. *Differential-algebraic equations: analysis and numerical solution.* EMS textbooks in mathematics. European Mathematical Society, 2006.

[LMT13] R. Lamour, R. März, and C. Tischendorf. *Differential Algebraic Equations: A Projector Based Analysis.* to appear in Springer, 2013.

[Mär02] R. März. The index of linear differential algebraic equations with properly stated leading terms. *Results in Mathematics*, 42(3-4):308–338, 2002.

[RR90] P.J. Rabier and W.C. Rheinboldt. *A General Existence and Uniqueness Theorem for Implicit Differential-Algebraic Equations.* Defense Technical Information Center, 1990.

[RR02] P.J. Rabier and W.C. Rheinboldt. *Techniques of scientific computing (part 4) - theoretical and numerical analysis of differential-algebraic equations.*, volume 8 of *Handbook of numerical analysis.* Elsevier, Amsterdam, 2002.

[Sch11] S. Schöps. *Multiscale Modeling and Multirate Time-Integration of Field/Circuit Coupled Problems.* Dissertation, Bergischen Universität Wuppertal, 2011.

[Tis99] C. Tischendorf. Topological index calculation of DAEs in circuit simulation. *Surveys on Mathematics for Industry*, 8(3-4):187–199, 1999.

[TW96] P. Thoma and T. Weiland. A consistent subgridding scheme for the finite difference time domain method. *International journal of Numerical Modelling: Electronic Networks, Devices and Fields*, 9(5):359–374, 1996.

[Wei77] T. Weiland. A discretization model for the solution of Maxwell's equations for six-component fields. *Archiv Elektronik und Übertragungstechnik*, 31(3):116–120, 1977.