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Generalised double-logarithmic equation in QCD

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Abstract

We present a generalisation of the double-logarithmic equation for the anomalous dimension of the non-singlet unpolarized twist-2 operators in QCD. Using known three-loop result this generalisation allows to predict a small x expansion of the four-loop non-singlet splitting functions in QCD for all power of logarithms up to single-logarithm ln x term.

A double-logarithmic equation was formulated initially during the study of the asymptotic behavior of the scattering amplitudes in Quantum Electrodynamics [1–3]. With the help of the method, proposed by Sudakov [4] for the evaluation of the Feynman integrals, the authors of [1–3] derived a Bethe-Saltpeter equation, which sums the leading logarithms terms (in this case $(\alpha_{\rm em} \ln^2 x)$) in all order of perturbative theory.

Using the same methods, the investigations of other regions of applicability of the ressumation procedure were performed by some of the authors of [1–3] and the famous Dokshitzer-Gribov-Lipatov-Altarelli-Parizi (DGLAP) [5–7] and Balitsky-Fadin-Kuraev-Lipatov (BFKL) [8, 9] equations were obtained.

Then, in the papers of Kirschner and Lipatov [10–12] a new approach for the ressummation of the double-logarithmic terms was proposed. This new method allowed to perform a double-logarithmic ressummation for a different amplitudes and a different channels, considerably extend the conventional approach. The main point in new approach consisted in isolating the softest virtual particle with the lowest transverse momentum in the graphs. The authors of ref. [12] proposed a set of equations for the partial waves of the amplitudes for a different channels in the double logarithmic approximation. In general, these equations are ordinary differential equations of Riccatti type and in some cases they are just algebraic ones. The study of the double-logarithmic equation [12] provided an information about behaviour of the structure function in the region of small x [13–16]. Moreover, it was extended to Standard Model [17] and to (super)gravity [18].

In spite of the DGLAP and BFKL equations were studied in the higher-order approximations [19–22], the double-logarithmic equation do not have any general results beyond the leading-logarithm approximation. In this paper we propose such generalisation, which was firstly discovered during the investigation of the analytical properties of the anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory [23].

The double-logarithmic equation for the non-singlet anomalous dimension of the twist-2 operator in QCD near $j = 0 + \omega$, which can be obtained from the corresponding equation for the amplitudes from ref. [12] can be written as:

$$\gamma_{\rm NS}(\omega) = -2\, C_F \frac{a_s}{\omega} + \frac{1}{\omega}\Big(\gamma_{\rm NS}(\omega)\Big)^2\,, \tag{1}$$

where

$$\gamma(j) = \sum_{\ell} a_s^{\ell} \gamma^{(\ell-1)}(j)\,, \qquad\qquad a_s = \frac{\alpha_s}{4\pi} = \frac{g^2}{16\pi^2} \tag{2}$$

and the anomalous dimension $\gamma(j)$ has a poles for all non-positive $j = 0, -1, -2, \ldots$ being the function of the nested harmonic sums, defined as (see [24, 25]):

$$S_a(M) \;=\; \sum_{j=1}^{M} \frac{(\mathrm{sgn}(a))^j}{j^{|a|}}\,, \qquad S_{a_1,\ldots,a_n}(M) \;=\; \sum_{j=1}^{M} \frac{(\mathrm{sgn}(a_1))^j}{j^{|a_1|}}\, S_{a_2,\ldots,a_n}(j)\,. \tag{3}$$

The double-logarithmic equation (1) provides the information about the highest poles in

ω in all order of perturbative theory through a trivial solution:

$$\gamma(0+\omega) = -\frac{\omega}{2} + \frac{\omega}{2}\sqrt{1 - 8\,C_F\frac{a_s}{\omega^2}} = -2\,C_F\frac{a_s}{\omega} - 4\,C_F^2\frac{a_s^2}{\omega^3} - 16\,C_F^3\frac{a_s^3}{\omega^5} - 80\,C_F^4\frac{a_s^4}{\omega^7} + \ldots \quad (4)$$

The double-logarithmic equation (1) sums all terms, which are proportional to $(a_s \ln^2 x)^k$ in all order of perturbative theory if we transform the result (4) into x-space using

$$\mathcal{M}\Big[\ln^k x\Big](N) = (-1)^k \frac{k!}{N^{(k+1)}}\ . \qquad (5)$$

The double-logarithmic equations give important information about the leading behavior of the splitting function at small x. However, this leading order result requires the corrections. Such corrections can be taken into account with the perturbative calculations of the splitting functions and a coefficient functions, which are known at this moment up to next-to-next-to-leading order in QCD [19–21, 26]. But there is no any extension of the double-logarithmic equation (1) itself beyond the leading-logarithm approximation.

Such generalization was discovered in the maximally extended $\mathcal{N}=4$ SYM theory [23], where the anomalous dimension for twist-2 operators are know at this moment up to six loops [27–32]. Perform the analytical continuation for these results, which can be easily done with the help of `HARMPOL` package [33] for `FORM` [34], we can study the changes of this equation with the expansion of the anomalous dimension through the order of perturbative theory g and parameter ω. Such work was started by L.N. Lipatov and A. Onishchenko for a general even $j = 0, -2, -4, -6, \ldots$ at 2006, but was not published, than, some improves of double-logarithmic equation was proposed by L.N. Lipatov in ref. [29]. Surprisingly, that in the most simple case $j = 0+\omega$ the generalisation was found in $\mathcal{N}=4$ SYM theory in a very simply form [23]:

$$\gamma_{\mathcal{N}=4\ \mathrm{SYM}}(2\,\omega + \gamma_{\mathcal{N}=4\ \mathrm{SYM}}) = \sum_{k=1}\sum_{m=0} \mathfrak{C}^k_m\,\omega^m\,g^{2k}\,, \qquad (6)$$

where right-hand side is regular over ω. The solution of the generalised double-logarithmic equation (6) give the corrections to the leading-logarithm approximation (4), that is, if we know ℓ loops anomalous dimension we know the information about all poles up to $(a_s/\omega^2)^k\omega^{2\ell}$ in all order of perturbative theory. As poles in ω correspond to $\ln x$ through eq. (5) we know resummation of the logarithmic terms in all order of perturbative theory up to $({a_s}^k \ln^{2k-2(\ell-1)} x)$ term or in the $\mathrm{N}^{2(\ell-1)}$LLA approximation.

For QCD we know at this moment the non-singlet anomalous dimension up to three loops [19–21, 35, 36]. So, we can expand the results for the anomalous dimension near $j = 0+\omega$, which is looks like

$$\frac{\gamma^{(0)}(\omega)}{C_F} = -\frac{2}{\omega} - 1 + 2(2\zeta_2 - 1)\omega + 2(1-2\zeta_3)\omega^2 + 2(2\zeta_4 - 1)\omega^3 + 2(1-2\zeta_5)\omega^4\,, \quad (7)$$

$$
\begin{aligned}
\frac{\gamma^{(1)}(\omega)}{C_F} =& -\frac{4C_F}{\omega^3} + \frac{1}{\omega^2}\left[\frac{22C_A}{3} - \frac{4n_f}{3} - 4C_F\right] + \frac{1}{\omega}\left[\Big(8\zeta_2+4\Big)C_F - \frac{302C_A}{9} + \frac{44n_f}{9}\right] \\
&+\left(12\zeta_3 + \frac{421}{18}\right)C_A - \frac{29n_f}{9} + \left(-16\zeta_3 - \frac{19}{2}\right)C_F \\
&+\omega\left[C_A\left(\frac{268\zeta_2}{9} - \frac{124\zeta_3}{3} - \zeta_4 - \frac{170}{9}\right) + n_f\left(-\frac{40\zeta_2}{9} + \frac{16\zeta_3}{3} + \frac{20}{9}\right)\right. \\
&\left.+C_F\Big(16 - 8\zeta_2 + 32\zeta_3 - 14\zeta_4\Big)\right] + \omega^2\left[C_A\left(\frac{104}{9} - \frac{160\zeta_3}{9} + 65\zeta_4 - 40\zeta_5\right)\right. \\
&\left.+n_f\left(\frac{40\zeta_3}{9} - 8\zeta_4 - \frac{8}{9}\right) + C_F\Big(-32\zeta_2\zeta_3 + 16\zeta_2 - 54\zeta_4 + 104\zeta_5 - 28\Big)\right], \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\gamma^{(2)}(\omega)}{C_F} =& -\frac{16}{\omega^5}C_F^2 + \frac{1}{\omega^4}\left[C_F\Big(44C_A - 8n_f\Big) - 24C_F^2\right] \\
&+\frac{1}{\omega^3}\left[(208\zeta_2 - 8)C_F^2 + \left(-192\zeta_2 - \frac{944}{9}\right)C_FC_A + \frac{128}{9}C_Fn_f + \frac{88}{9}C_An_f\right. \\
&\left.+\left(60\zeta_2 - \frac{242}{9}\right)C_A^2 - \frac{8}{9}{n_f}^2\right] + \frac{1}{\omega^2}\left[(-192\zeta_2 - 96\zeta_3 - 30)C_F^2\right. \\
&+\left(216\zeta_2 + 48\zeta_3 + \frac{370}{9}\right)C_FC_A - \frac{88}{9}C_Fn_f + \left(8\zeta_2 - \frac{1268}{27}\right)C_An_f \\
&\left.+\left(-92\zeta_2 + \frac{3934}{27}\right)C_A^2 + \frac{88}{27}{n_f}^2\right] + \frac{1}{\omega}\left[(308\zeta_2 + 192\zeta_3 - 324\zeta_4 + 62)C_F^2\right. \\
&+\left(268\zeta_4 - \frac{532}{9} - \zeta_2\frac{1304}{9} - \zeta_3\frac{808}{3}\right)C_AC_F + \frac{325}{9}n_fC_F \\
&+\left(112\zeta_2 + 72\zeta_3 - 81\zeta_4 - \frac{9737}{27}\right)C_A^2 - \frac{32}{9}n_f^2 \\
&\left.+\left(-\zeta_2\frac{160}{9} - \zeta_3\frac{32}{3}\right)n_fC_F + \left(-8\zeta_2 + 16\zeta_3 + \frac{2474}{27}\right)C_An_f\right] \\
&+\left(448\zeta_3\zeta_2 - 560\zeta_2 - 316\zeta_3 + 96\zeta_4 + 304\zeta_5 - \frac{57}{2}\right)C_F^2 \\
&+\left(-480\zeta_2\zeta_3 - 160\zeta_5 - \frac{1951}{36} + \zeta_2\frac{1672}{3} + \zeta_4\frac{86}{3} - \zeta_3\frac{332}{9}\right)C_FC_A \\
&+\left(-\frac{2}{9} - \zeta_2\frac{64}{3} + \zeta_4\frac{28}{3} + \zeta_3\frac{344}{9}\right)n_f \\
&+\left(16\zeta_2 - \frac{80\zeta_3}{3} - \frac{14\zeta_4}{3} - \frac{670}{9}\right)C_An_f + \frac{83}{27}{n_f}^2 \\
&+\left(-176\zeta_2 + 144\zeta_2\zeta_3 + \frac{500\zeta_3}{3} - \frac{133\zeta_4}{3} - 60\zeta_5 + \frac{31087}{108}\right)C_A^2 \qquad (9)
\end{aligned}
$$

and substitute the obtained expressions into the original double-logarithmic equation (1). We have found, that indeed it is modified in a minimal way if we add also QCD β-function in the left hand side* †:

$$\gamma_{\rm NS}(\omega+\gamma_{\rm NS}-\beta) = a_s\Big\{-2-\omega+(4\zeta_2-2)\omega^2+(2-4\zeta_3)\omega^3+(4\zeta_4-2)\omega^4\Big\}C_F$$
$$+a_s^2\Big\{\omega^2\Big[C_AC_F\Big(\frac{268\zeta_2}{9}-56\zeta_3-\zeta_4-\frac{104}{9}\Big)+C_Fn_f\Big(-\frac{40\zeta_2}{9}+8\zeta_3+\frac{8}{9}\Big)$$
$$+C_F^2(-24\zeta_2+40\zeta_3+10\zeta_4+24)\Big]-\frac{335C_AC_F}{9}+\frac{50C_Fn_f}{9}+(13-8\zeta_2)C_F^2$$
$$+\omega\Big[\Big(\frac{44\zeta_2}{3}+12\zeta_3+\frac{289}{18}\Big)C_AC_F-\Big(\frac{8\zeta_2}{3}+\frac{17}{9}\Big)C_Fn_f-\Big(8\zeta_2+\frac{27}{2}\Big)C_F^2\Big]\Big\}$$
$$+a_s^3\Big\{\frac{1}{\omega^2}\Big[144\zeta_2C_F^3-192\zeta_2C_AC_F^2+60\zeta_2C_A^2C_F\Big]$$
$$+\frac{1}{\omega}\Big[8\zeta_2C_AC_Fn_f+304\zeta_2C_AC_F^2-92\zeta_2C_A^2C_F-16\zeta_2C_F^2n_f-240\zeta_2C_F^3\Big]$$
$$+C_AC_Fn_f\Big(-8\zeta_2+8\zeta_3+\frac{608}{9}\Big)+C_A^2C_F\Big(112\zeta_2+116\zeta_3-81\zeta_4-\frac{15455}{54}\Big)$$
$$+C_F^2n_f\Big(\frac{352\zeta_2}{9}-\frac{32\zeta_3}{3}+\frac{154}{9}\Big)+C_F^3(340\zeta_2+128\zeta_3-140\zeta_4+1)$$
$$-\frac{38}{27}C_Fn_f^2+C_AC_F^2\Big(-\frac{4792\zeta_2}{9}-\frac{736\zeta_3}{3}+272\zeta_4+\frac{1771}{18}\Big)\Big\} \tag{10}$$
$$= 4a_s^3\zeta_2C_F\Big(C_A-2C_F\Big)\Big[\frac{3\big(5C_A-6C_F\big)}{\omega^2}-\frac{\big(23C_A-30C_F-2n_f\big)}{\omega}\Big]+\mathcal{O}(\omega^0)\,, \tag{11}$$

where the coefficients for the β-function in QCD

$$\beta(a_s) = -\beta_0a_s^2-\beta_1a_s^3-\beta_2a_s^4 \tag{12}$$

are the following up to three-loop order [37–39]:

$$\beta_0 = \frac{11}{3}C_A-\frac{4}{3}T_Fn_f\,, \tag{13}$$
$$\beta_1 = \frac{34}{3}C_A^2-4C_FT_Fn_f-\frac{20}{3}C_AT_Fn_f\,, \tag{14}$$
$$\beta_2 = \frac{2857}{54}C_A^3+\Big(2C_F^2-\frac{205}{9}C_FC_A-\frac{1415}{27}C_A^2\Big)T_Fn_f+\Big(\frac{44}{9}C_F+\frac{158}{27}C_A\Big)T_F^2n_f^2\,. \tag{15}$$

One can see that the terms in the right hand side of eq. (11), which have a poles in ω, are proportional to ζ_2 and (C_A-2C_F) colour structure, that is they are suppressed by the

*β-function in $\mathcal{N}=4$ SYM theory is equal to zero in all order of perturbative theory

†There is a difference between normalisation of the anomalous dimension in $\mathcal{N}=4$ SYM theory and in QCD, which produce difference in eqs. (10) and (6)

subcolour factor $(C_A-2C_F)=1/N_c$ for $SU(N_c)$. If we assume, that the modification of the original double-logarithmic equation (1) will contains pole terms, which are proportional only ζ_2 or (C_A-2C_F), we will know the resummation of the logarithm in all order of perturbative theory in the form of the following solution:

$$\gamma_{\rm NS}^{\rm N^{2(\ell-1)}LLA}(\omega) = -\frac{\omega-\beta}{2} + \frac{\omega-\beta}{2}\sqrt{1+\frac{4}{\big(\omega-\beta\big)^2}\sum_{\ell=1}\sum_{m=0}\mathfrak{D}^{\ell}_{m}\,\omega^m\,a_s^{\ell}} \tag{16}$$

where coefficients $\mathfrak{D}^{k}_{m}$ can be read directly form eq. (10) and we should drop out all terms proportional ζ_2 or/and suppressed by colour factor (C_A-2C_F) in $\mathfrak{D}^{3}_{m}$.

Let's turn this solution (16) into $\ln x$ with the help of eq. (5) and see its properties. We start with the comparison of the our result eq.(16) with the exact result from the three-loop calculations [21]. Expand the solution (16) only with $\mathfrak{D}^{1}_{m}$ and $\mathfrak{D}^{2}_{m}$ (or using only two-loop results (7) and (8)) we find for the expansion of $\hat{P}^{(2)}_{+,0}(x)$ near $x=0$, which can be written in general as

$$\hat{P}^{(2)}_{+,0}(x) = \hat{D}^{+}_{0}\ln^4 x + \hat{D}^{+}_{1}\ln^3 x + \hat{D}^{+}_{2}\ln^2 x + \hat{D}^{+}_{3}\ln x \tag{17}$$

the following differences with the full results from eq. (4.15) ref. [21]:

$$D^{+}_{2}-\hat{D}^{+}_{2} = -6\zeta_2(C_A-2C_F)(5C_A-6C_F)C_F\,, \tag{18}$$

$$D^{+}_{1}-\hat{D}^{+}_{1} = -4\zeta_2(C_A-2C_F)(23C_A-30C_F-2n_f)C_F\,. \tag{19}$$

We compare the obtained result with fig. (2) from ref. [21], using the same input data. It is clear, that the difference between N^{3}Lx approximation and exact result in fig. (1a) is some constant for small x. Our result (black solid line in fig. (1a)), obtained from the two loops (or using only $\mathfrak{D}^{1}_{m}$ and $\mathfrak{D}^{2}_{m}$), is differ to this constant and the term, which is proportional to $\zeta_2(C_A-2C_F)$. One can see, that we have a very good agrement with the exact result.

Expand the solution (16) with $\mathfrak{D}^{1}_{m}$, $\mathfrak{D}^{2}_{m}$ and $\mathfrak{D}^{3}_{m}$ (or using three-loop results (7)-(9)) we find for the expansion of $\hat{P}^{(3)}_{+,0}(x)$ near $x=0$, which can be written in general as

$$\hat{P}^{(3)}_{+,0}(x) = \hat{D}^{(3),+}_{0}\ln^6 x+\hat{D}^{(3),+}_{1}\ln^5 x+\hat{D}^{(3),+}_{2}\ln^4 x+\hat{D}^{(3),+}_{3}\ln^3 x+\hat{D}^{(3),+}_{4}\ln^2 x+\hat{D}^{(3),+}_{5}\ln x \tag{20}$$

the following predictions:

$$\hat{D}^{(3),+}_{0} = \frac{C_F^4}{9}\,, \tag{21}$$

$$\hat{D}^{(3),+}_{1} = \frac{22}{9}C_AC_F^3 - \frac{4}{9}C_F^3 n_f - \frac{4C_F^4}{3}\,, \tag{22}$$

$$\begin{aligned}\hat{D}^{(3),+}_{2} = &\Big[\frac{16}{3}-\frac{56\zeta_2}{3}\Big]C_F^4 - \frac{44}{9}C_AC_F^2n_f + \frac{170}{9}C_AC_F^3 + \frac{121}{9}C_A^2C_F^2 - \frac{20}{9}C_F^3n_f + \frac{4}{9}C_F^2n_f^2\\ &-2[\boldsymbol{\omega^{-2}}]\zeta_2{C_F}^2\Big(5C_A-6C_F\Big)\Big(C_A-2C_F\Big)\,,\end{aligned} \tag{23}$$

$$
\begin{aligned}
\hat{D}_3^{(3),+} =& -\frac{2092}{27}C_AC_F^2n_f + \frac{44}{27}C_AC_Fn_f^2 - \frac{242}{27}C_A^2C_Fn_f + C_AC_F^3\Big[-176\zeta_2 + 48\zeta_3 + \frac{10}{3}\Big] \\
& + \frac{6530}{27}C_A^2C_F^2 + \frac{1331}{81}C_A^3C_F + \Big[32\zeta_2 - \frac{16}{3}\Big]C_F^3n_f + \frac{152}{27}C_F^2n_f^2 - \frac{8}{81}C_Fn_f^3 \\
& + C_F^4\Big[96\zeta_2 - \frac{320\zeta_3}{3} - \frac{106}{3}\Big] - \frac{8}{3}[\boldsymbol{\omega^{-1}}]\zeta_2{C_F}^2\Big(C_A - 2C_F\Big)\Big(23C_A - 30C_F - 2n_f\Big) \\
& - \frac{2}{3}[\boldsymbol{\omega^{-2}}]\zeta_2C_F\Big(5C_A - 6C_F\Big)\Big(C_A - 2C_F\Big)\Big(11C_A - 6C_F - 2n_f\Big)\,, \qquad (24) \\
\hat{D}_4^{(3),+} =& \, C_AC_F^2n_f\Big[\frac{1024\zeta_2}{9} - 64\zeta_3 - \frac{32968}{81}\Big] + C_AC_F^3\Big[-\frac{1024\zeta_2}{3} + \frac{416\zeta_3}{3} - 532\zeta_4 + 98\Big] \\
& + \frac{644}{27}C_AC_Fn_f^2 - \frac{1390}{9}C_A^2C_Fn_f + C_A^2C_F^2\Big[-\frac{4436\zeta_2}{9} + 32\zeta_3 + 162\zeta_4 + \frac{114740}{81}\Big] \\
& + \frac{25003}{81}C_A^3C_F + C_F^3n_f\Big[\frac{352\zeta_2}{3} + \frac{256\zeta_3}{3} - \frac{170}{3}\Big] + \Big[\frac{2144}{81} - \frac{80\zeta_2}{9}\Big]C_F^2n_f^2 \\
& + C_F^4\Big[-760\zeta_2 - 256\zeta_3 + 1360\zeta_4 - 112\Big] - \frac{88}{81}C_Fn_f^3 \\
& + 24[\boldsymbol{\omega^{-2}}]\zeta_2(2\zeta_2 - 1){C_F}^2\Big(5C_A - 6C_F\Big)\Big(C_A - 2C_F\Big) \\
& - \frac{2}{3}[\boldsymbol{\omega^{-1}}]\zeta_2C_F\Big(C_A - 2C_F\Big)\Big(23C_A - 30C_F - 2n_f\Big)\Big(11C_A - 6C_F - 2n_f\Big)\,, \qquad (25) \\
\hat{D}_5^{(3),+} =& + C_F^4\Big[3072\zeta_2\zeta_3 + 608\zeta_2 + 64\zeta_3 - 1320\zeta_4 + 88\Big] \\
& + C_AC_F^3\Big[-960\zeta_2\zeta_3 - \frac{22604\zeta_2}{9} - \frac{16624\zeta_3}{3} + 2940\zeta_4 + \frac{68}{3}\Big] \\
& + C_A^2C_F^2\Big[-\frac{42008\zeta_2}{27} + \frac{26408\zeta_3}{9} - \frac{3346\zeta_4}{3} + \frac{190748}{81}\Big] \\
& + C_A^3C_F\Big[-\frac{1232\zeta_2}{3} - 264\zeta_3 + 297\zeta_4 + \frac{146482}{81}\Big] \\
& + C_F^3n_f\Big[\frac{3608\zeta_2}{9} + \frac{2368\zeta_3}{3} - 440\zeta_4 + 38\Big] + C_F^2n_f^2\Big[-\frac{1664\zeta_2}{27} + \frac{128\zeta_3}{9} + \frac{7814}{81}\Big] \\
& + C_AC_F^2n_f\Big[\frac{18832\zeta_2}{27} - \frac{4720\zeta_3}{9} + \frac{520\zeta_4}{3} - \frac{77143}{81}\Big] - \frac{64}{27}C_Fn_f^3 \\
& + C_A^2C_Fn_f\Big[104\zeta_2 - \frac{32\zeta_3}{3} - 54\zeta_4 - \frac{68726}{81}\Big] + C_AC_Fn_f^2\Big[-\frac{16\zeta_2}{3} + \frac{32\zeta_3}{3} + \frac{8272}{81}\Big] \\
& + 16[\boldsymbol{\omega^{-1}}]\zeta_2(2\zeta_2 - 1){C_F}^2\Big(C_A - 2C_F\Big)\Big(23C_A - 30C_F - 2n_f\Big) \\
& + 48[\boldsymbol{\omega^{-2}}]\zeta_2(2\zeta_3 - 1){C_F}^2\Big(5C_A - 6C_F\Big)\Big(C_A - 2C_F\Big)\,, \qquad (26)
\end{aligned}
$$

where all terms, which are proportional to $[\boldsymbol{\omega^{-2}}]$ and $[\boldsymbol{\omega^{-1}}]$ come from the first and the second terms in eq. (11). Inserting $C_A = 3$ and $C_F = 4/3$ and the numerical values of

ζ_2, ζ_3 and ζ_5 one can find

$$\hat{D}_0^{(3),+} = 0.351166\,, \tag{27}$$
$$\hat{D}_1^{(3),+} = 13.1687 - 1.05350 n_f\,, \tag{28}$$
$$\hat{D}_2^{(3),+} = 269.244 - 31.3416 n_f + 0.790123 n_f^2 - 13.6469[\boldsymbol{\omega^{-2}}]\,, \tag{29}$$
$$\hat{D}_3^{(3),+} = 2818.6 - 408.66 n_f + 16.5267 n_f^2 - 0.131687 n_f^3 + [\boldsymbol{\omega^{-2}}]\,(6.8234 n_f - 85.293) + [\boldsymbol{\omega^{-1}}]\,(5.1988 n_f - 75.383)\,, \tag{30}$$
$$\hat{D}_4^{(3),+} = 17395. - 2869.9 n_f + 116.470 n_f^2 - 1.44856 n_f^3 + 375.[\boldsymbol{\omega^{-2}}] + [\boldsymbol{\omega^{-1}}]\left(-1.94955 n_f^2 + 52.638 n_f - 353.36\right)\,, \tag{31}$$
$$\hat{D}_5^{(3),+} = 25690. - 7532.6 n_f + 446.36 n_f^2 - 3.16049 n_f^3 + 460.[\boldsymbol{\omega^{-2}}] + [\boldsymbol{\omega^{-1}}]\,(1036. - 71.4 n_f)\,, \tag{32}$$

that the contributions from uncontrolled terms $[\boldsymbol{\omega^{-2}}]$ and $[\boldsymbol{\omega^{-1}}]$ are small. In fig. (1b) we show these results.

In conclusion we want to note, that the generalised double-logarithmic equation (6), obtained in $\mathcal{N}=4$ SYM theory [23], provides us with a new information about ressumation in QCD. The generalised double-logarithmic equation for QCD (10) is violated only with terms, which are proportional to $\zeta_2(C_A - 2C_F)$ in eq. (11). We hope, that one can find the origin of these terms to restore eq. (6). In principle, the result (16) can be improved using our results for the first values of four-loop non-singlet anomalous dimension [40, 41] following the procedure, described in [42].

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References

[1] G. V. Frolov, V. G. Gorshkov and V. N. Gribov, Phys. Lett. **20** (1966) 544.

[2] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov and G. V. Frolov, Sov. J. Nucl. Phys. **6** (1968) 95 [Yad. Fiz. **6** (1967) 129].

[3] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov and G. V. Frolov, Phys. Lett. **22** (1966) 671.

[4] V. V. Sudakov, Sov. Phys. JETP **3** (1956) 65 [Zh. Eksp. Teor. Fiz. **30** (1956) 87].

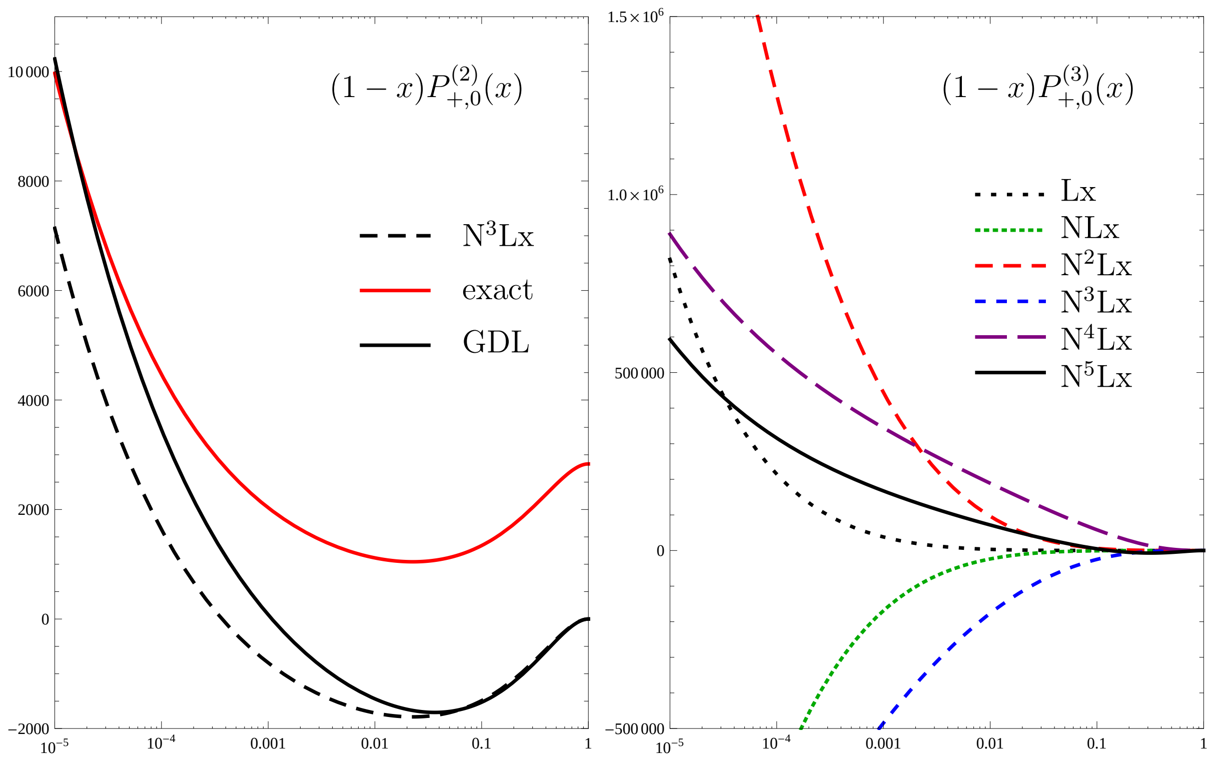

Figure 1: The n_f-independent three-loop contribution $P^{(2)}_{+,0}(x)$ and $P^{(3)}_{+,0}(x)$ to the splitting function $P^{+}_{\rm ns}(x)$, multiplied by $(1-x)$ for display purposes. In the left part exact result from [21] is compared to the small-x approximations and our solution (16). In the right part the predictions for the small-x approximations to four-loop $P^{(3)}_{+,0}(x)$ from our solution (16) are presented.

[5] V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. **15** (1972) 438 [Yad. Fiz. **15** (1972) 781].

[6] G. Altarelli and G. Parisi, Nucl. Phys. B **126** (1977) 298.

[7] Y. L. Dokshitzer, Sov. Phys. JETP **46** (1977) 641 [Zh. Eksp. Teor. Fiz. **73** (1977) 1216].

[8] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **45** (1977) 199 [Zh. Eksp. Teor. Fiz. **72** (1977) 377].

[9] I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822 [Yad. Fiz. **28** (1978) 1597].

[10] R. Kirschner and L. N. Lipatov, Phys. Rev. D **26** (1982) 1202.

[11] R. Kirschner and L. n. Lipatov, Sov. Phys. JETP **56** (1982) 266 [Zh. Eksp. Teor. Fiz. **83** (1982) 488].

[12] R. Kirschner and L. n. Lipatov, Nucl. Phys. B **213** (1983) 122.

[13] B. I. Ermolaev, S. I. Manaenkov and M. G. Ryskin, Z. Phys. C **69** (1996) 259 [arXiv:hep-ph/9502262].

[14] J. Blumlein and A. Vogt, Phys. Lett. B **370** (1996) 149 [arXiv:hep-ph/9510410].

[15] J. Blumlein and A. Vogt, Phys. Rev. D **58** (1998) 014020 [arXiv:hep-ph/9712546].

[16] B. I. Ermolaev, M. Greco and S. I. Troyan, Nucl. Phys. B **594** (2001) 71 [arXiv:hep-ph/0009037].

[17] V. S. Fadin, L. N. Lipatov, A. D. Martin and M. Melles, Phys. Rev. D **61** (2000) 094002 [arXiv:hep-ph/9910338].

[18] J. Bartels, L. N. Lipatov and A. S. Vera, arXiv:1208.3423 [hep-th].

[19] E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545].

[20] A. Gonzalez-Arroyo, C. Lopez and F. J. Yndurain, Nucl. Phys. B **153** (1979) 161.

[21] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B **688** (2004) 101 [arXiv:hep-ph/0403192].

[22] V. S. Fadin and L. N. Lipatov, Phys. Lett. B **429** (1998) 127 [arXiv:hep-ph/9802290].

[23] V. N. Velizhanin, JHEP **1108** (2011) 092 [arXiv:1104.4100 [hep-th]].

[24] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

[25] J. Blumlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241].

[26] J. A. M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005) 3 [arXiv:hep-ph/0504242].

[27] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, Phys. Lett. B **557** (2003) 114 [arXiv:hep-ph/0301021].

[28] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. B **595** (2004) 521 [Erratum-ibid. B **632** (2006) 754] [arXiv:hep-th/0404092].

[29] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, J. Stat. Mech. **0710** (2007) P10003 [arXiv:0704.3586 [hep-th]].

[30] Z. Bajnok, R. A. Janik and T. Lukowski, Nucl. Phys. B **816** (2009) 376 [arXiv:0811.4448 [hep-th]].

[31] T. Lukowski, A. Rej and V. N. Velizhanin, Nucl. Phys. B **831** (2010) 105 [arXiv:0912.1624 [hep-th]].

[32] C. Marboe, V. Velizhanin and D. Volin, arXiv:1412.4762 [hep-th].

[33] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237].

[34] J. A. M. Vermaseren, arXiv:math-ph/0010025.

[35] D. J. Gross and F. Wilczek, Phys. Rev. D **8** (1973) 3633.

[36] H. Georgi and H. D. Politzer, Phys. Rev. D **9** (1974) 416.

[37] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343.

[38] D. R. T. Jones, Nucl. Phys. B **75** (1974) 531.

[39] O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B **93** (1980) 429.

[40] V. N. Velizhanin, Nucl. Phys. B **860** (2012) 288 [arXiv:1112.3954 [hep-ph]].

[41] V. N. Velizhanin, arXiv:1411.1331 [hep-ph].

[42] W. L. van Neerven and A. Vogt, Nucl. Phys. B **568** (2000) 263 [arXiv:hep-ph/9907472].