

# Twisted elliptic multiple zeta values and non-planar one-loop open-string amplitudes

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## Abstract

We consider a generalization of elliptic multiple zeta values, which we call twisted elliptic multiple zeta values. These arise as iterated integrals on an elliptic curve from which a rational lattice has been removed. At the cusp, twisted elliptic multiple zeta values are shown to degenerate to cyclotomic multiple zeta values in the same way as elliptic multiple zeta values degenerate to classical multiple zeta values. We investigate properties of twisted elliptic multiple zeta values and consider them in the context of the non-planar part of the four-point one-loop open-string amplitude.

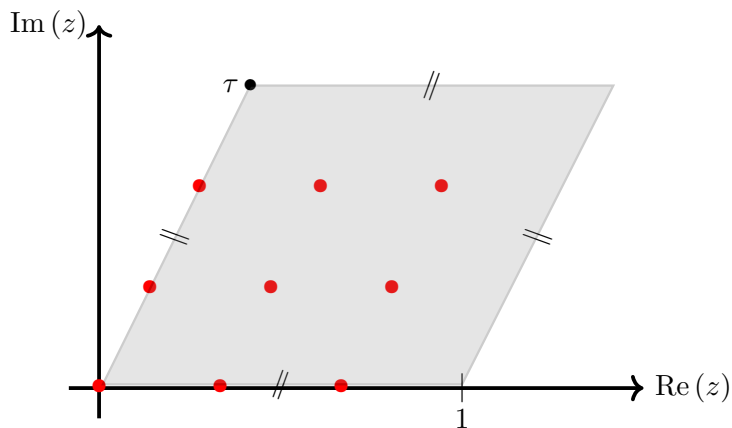
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# 1 Introduction

Within the program of studying iterated integrals on Riemann surfaces of various genera, the genus-zero case, which leads to multiple zeta values (MZVs) [1–3], is the starting point and takes the most prominent rôle. During the last years, however, the genus-one situation has received more attention: various iterated integrals on the elliptic curve as well as associated periods and elliptic associators have been investigated [4–7].

The simplest genus-one generalizations of MZVs are elliptic multiple zeta values (eMZVs), which arise from iterated integrals on the once-punctured elliptic curve, that is the elliptic curve where the origin is removed [8]. In this article, the notion of eMZVs is extended to *twisted* elliptic multiple zeta values (teMZVs), which are iterated integrals on a multiply-punctured elliptic curve. While for eMZVs it is sufficient to remove the origin, teMZVs arise when a lattice with rational coordinates as visualized in figure 1 is removed from the elliptic curve. The iterated integrals to be considered are performed over a path parallel to the real axis and are therefore a generalization of Enriquez’  $A$ -cycle eMZVs.<sup>1</sup> A slight technical difficulty, which was absent for eMZVs, is that the integrands giving rise to teMZVs might have additional poles along the path of integration. We address the problem by suggesting a rather natural regularization scheme, which essentially amounts to integrating over an infinitesimal deformation of the real axis.



**Figure 1:** The shaded region represents the elliptic curve  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ , where edges marked by  $//$  and  $\cong$  are identified. In the setup of teMZVs, points from the lattice  $\mathbb{Q} + \mathbb{Q}\tau$  are removed from the elliptic curve. Here we show the example  $\{0, \frac{1}{3}, \frac{2}{3}\} + \{0, \frac{1}{3}, \frac{2}{3}\}\tau$ .

A crucial tool in the study of eMZVs was the existence of a certain first-order linear differential equation, expressing eMZVs as special linear combinations of iterated integrals of Eisenstein series and MZVs [7, 8, 10, 11]. One of the main results of this article is the generalization of this differential equation to teMZVs. In particular, one can again identify a procedure delivering the boundary data for teMZVs at the cusp  $i\infty$  of the modular parameter  $\tau$ , and relating them to integrals over genus-zero Riemann surfaces in a natural way. While for eMZVs this procedure leads to MZVs, in the case of teMZVs we obtain *cyclotomic* MZVs [3, 12–14]. A further parallel to eMZVs is the existence of shuffle and Fay relations.

Scattering amplitudes in open-superstring theories have been recently noticed as a rewarding setup where iterated integrals on Riemann surfaces appear naturally. Generalizing the ubiquity

<sup>1</sup>It is worth noting that there is no structural problem in defining twisted  $B$ -cycle eMZVs by considering integration paths parallel to the direction of the modular parameter  $\tau$  [9].

of MZVs in tree-level amplitudes<sup>2</sup>, one-loop scattering amplitudes (corresponding to genus-one surfaces) provide a natural testing ground for eMZVs [28]. However, the analysis in ref. [28] was focused on the *planar* sector of the one-loop amplitude where the integrations are performed over a single boundary of a genus-one surface with cylinder topology.

In this article, teMZVs will be identified as a convenient language tailored to the *non-planar* part of the open-string one-loop amplitude: the extension of the iterated integrals to both boundaries of the cylinder leads to a class of teMZVs with twist  $\tau/2$ . We will employ these teMZVs to calculate non-planar contributions to the low-energy expansion<sup>3</sup> of the four-point one-loop open-string amplitude. Explicit results will be given up to the third subleading low-energy order which are checked to match the expressions available in the literature at the first subleading low-energy order [29] and at the cusp [30].

Interestingly, our results for the string scattering amplitudes in the cases considered can be expressed without using teMZVs: they appear in combinations which ultimately boil down to eMZVs. We will provide arguments bolstering the conjecture that this feature will persist to all orders in the low-energy expansion.

Finally, we expect the teMZVs defined here to be closely related to the monodromy of the universal twisted elliptic KZB equation to be studied in upcoming work of Calaque and Gonzalez [9]. A particularly important aspect of their work is the definition of a twisted version of the derivation algebra, the untwisted version of which [31, 5, 32, 33] already appeared in the study of eMZVs [10, 34, 11]. Similar to the situation for eMZVs, this twisted derivation algebra might be capable of encoding the number of indecomposable teMZVs of a given weight and length.

In section 2 we introduce teMZVs, and discuss the expansion of their constituents with respect to the modular parameter of the elliptic curve. Thereby we set the stage for section 3, where a differential equation for teMZVs w.r.t.  $\tau$  as well as a procedure to extract their  $\tau \rightarrow i\infty$  limit is presented. In section 4, the formalism is applied to the calculation of the non-planar contribution to the open-string one-loop scattering amplitude and the rôle of teMZVs therein is discussed. After concluding and pointing out a couple of open problems in section 5 we provide various appendices containing collections of definitions for the numerous objects appearing as well as several detailed calculations omitted in the main text.

## 2 From elliptic to twisted elliptic multiple zeta values

Elliptic multiple zeta values can be represented as iterated integrals on the multiply punctured elliptic curve  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \setminus \{b_1, \dots, b_\ell\}$  with parameter  $\tau$  in the upper half plane  $\mathbb{H}$ , where we denote  $q = \exp(2\pi i\tau)$ . Starting from  $\Gamma(; z) = 1$ , elliptic iterated integrals are defined recursively via

$$\Gamma\left(\begin{matrix} n_1 & n_2 & \dots & n_\ell \\ b_1 & b_2 & \dots & b_\ell \end{matrix}; z\right) = \int_0^z dt f^{(n_1)}(t - b_1) \Gamma\left(\begin{matrix} n_2 & \dots & n_\ell \\ b_2 & \dots & b_\ell \end{matrix}; t\right), \quad z \in [0, 1], \quad (2.1)$$

where the interval of integration is  $[0, z]$ . As will be discussed in subsection 2.1, regularization prescriptions have to be specified, if  $n_i = 1$  and  $b_i \in [0, z]$ . In particular, regularization of the divergences has been discussed in section 2 of ref. [10].

<sup>2</sup>See [15–17] for a discussion of the contributing iterated integrals on a genus-zero surface in the mathematics literature and [18, 19] for a treatment via polylogarithms in a physics context. Moreover, the expansion of  $n$ -point disk integrals has been addressed via motivic MZVs [20] and the Drinfeld associator [21] (also see [22]). As a complementary approach, the relation of disk integrals to hypergeometric functions has been used to obtain ( $n \leq 5$ )-point expansions [23, 24] and certain ranges of low-energy orders at  $n \leq 7$  points, see e.g. [25–27].

<sup>3</sup>The low-energy expansion of string amplitudes refers to an expansion in the inverse string tension  $\alpha'$ .

The weighting functions  $f^{(n)}(z, \tau)$ , arise as expansion coefficients of the doubly-periodic completion of the Eisenstein–Kronecker series, starting with

$$f^{(0)}(z, \tau) = 1, \quad f^{(1)}(z, \tau) = \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)} + 2\pi i \frac{\text{Im}(z)}{\text{Im}(\tau)}, \quad (2.2)$$

see appendix B for details and conventions. They are doubly-periodic functions of alternating parity

$$f^{(n)}(z+1, \tau) = f^{(n)}(z+\tau, \tau) = f^{(n)}(z, \tau), \quad f^{(n)}(-z, \tau) = (-1)^n f^{(n)}(z, \tau), \quad (2.3)$$

and the function  $f^{(1)}(z-b_i, \tau)$  in eq. (2.2) acquires a pole at  $z=b_i$  which requires regularization of eq. (2.1). Throughout the article, we will frequently omit noting the  $\tau$ -dependence of both weighting functions  $f^{(n)}$  and elliptic iterated integrals eq. (2.1).

In refs. [28, 10], the main focus was on elliptic multiple zeta values, whose shifting parameters  $b_i$  – referred to as *twists* – have been limited to  $b_i=0$ . Correspondingly, the elliptic curve in question has a single puncture only:  $E_\tau^\times = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \setminus \{0\}$ . Evaluating this subclass of elliptic iterated integrals at  $z=1$  leads to the definition of Enriquez’  $A$ -cycle elliptic multiple zeta values or eMZVs for short:

$$\begin{aligned} \omega(n_1, n_2, \dots, n_\ell) &= \int_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1) dz_1 f^{(n_2)}(z_2) dz_2 \dots f^{(n_\ell)}(z_\ell) dz_\ell \\ &= \Gamma \left( \begin{matrix} n_\ell & \dots & n_2 & n_1 \\ 0 & \dots & 0 & 0 \end{matrix}; 1 \right) = \Gamma(n_\ell, \dots, n_2, n_1; 1). \end{aligned} \quad (2.4)$$

The quantities  $w = \sum_{i=1}^\ell n_i$ , and the number  $\ell$  of integrations in eqs. (2.1) and (2.4) are referred to as *weight* and *length* of the elliptic iterated integral and the corresponding eMZV, respectively.

Allowing for rational values  $s_i$  and  $r_i$  in  $b_i = s_i + r_i\tau$ , leads to *twisted* elliptic multiple zeta values or tEMZVs:

$$\begin{aligned} \omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) &= \int_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1 - b_1) dz_1 f^{(n_2)}(z_2 - b_2) dz_2 \dots f^{(n_\ell)}(z_\ell - b_\ell) dz_\ell \\ &= \Gamma \left( \begin{matrix} n_\ell & n_{\ell-1} & \dots & n_1 \\ b_\ell & b_{\ell-1} & \dots & b_1 \end{matrix}; 1 \right), \end{aligned} \quad (2.5)$$

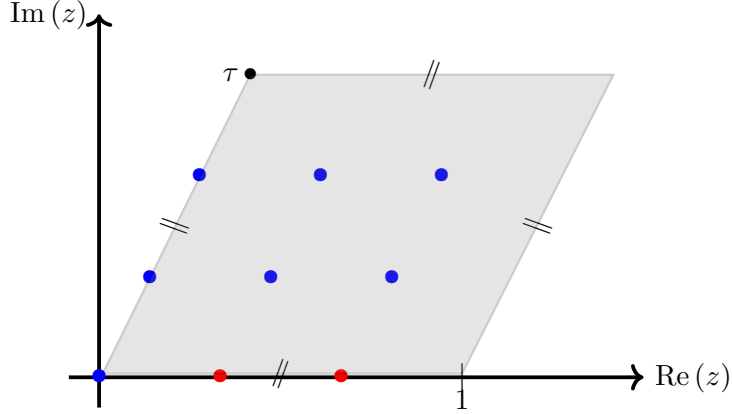
where the notion of *weight* and *length* carry over from eq. (2.4) directly. Taking the double-periodicity (2.3) of the weighting functions  $f^{(n)}$  into account, one can limit the attention to  $r_i, s_i \in [0, 1)$ , which corresponds to the shaded region in figure 2.

In this article we are going to limit our attention to twists  $\mathbb{Q} + \mathbb{Q}\tau$ , that is  $r_i, s_i \in \mathbb{Q}$ . In order to classify those, let us introduce

$$\Lambda_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}, \quad \Lambda_n^\times = \Lambda_n \setminus \{0\}. \quad (2.6)$$

If  $b_i \in \Lambda_n^\times$ , the twist is referred to as *proper rational*. Correspondingly, all other twists – that is those with  $b_i \in (\Lambda_n + \Lambda_n\tau) \setminus \Lambda_n^\times$  as visualized in figure 2 – are called *generic* twists. While in the latter situation divergences occur at endpoints only and can be addressed using the methods in ref. [8], the presences of a proper rational twist requires more work as discussed in subsection 2.1.

Twisted eMZVs based on proper rational twists do not make an appearance in the open-string one-loop amplitude. However, they are interesting from a number-theoretic point of view because their constant terms give rise to cyclotomic generalizations of MZVs or “cyclotomic

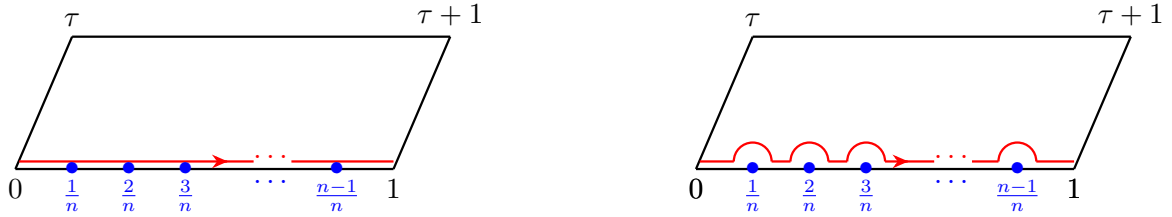


**Figure 2:** Example of the lattice  $\Lambda_n + \Lambda_n \tau$  at  $n = 3$ : Proper rational twists and generic twists are marked in red and blue, respectively. Edges marked by  $//$  and  $\cong$  are identified in  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ , respectively.

MZVs” for short [3, 12–14]. The set of (generic) twists  $b_i \in \{0, \tau/2\}$  turns out to lead to teMZVs relevant for the non-planar open-string amplitude, which we are going to discuss in section 4.

## 2.1 Regularization

In order to regularize the divergences in eq. (2.5) caused by twists  $b_1, \dots, b_\ell \in \Lambda_n^\times$ , we propose to replace the straight line  $[0, 1]$  by the domain of integration  $[0, 1]_\varepsilon$  in the right panel of figure 3.



**Figure 3:** Deformation of the straight-line path  $[0, 1]$  to the path  $[0, 1]_\varepsilon$ , avoiding the possible singularities of  $f^{(1)}$ .

Here,  $\varepsilon > 0$  is an additional real parameter, which determines the radii of the semicircles around proper rational twists in figure 3. One then defines regularized values of teMZVs

$$\omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) = \lim_{\varepsilon \rightarrow 0} \int_{[0, 1]_\varepsilon} f^{(n_1)}(z_1 - b_1) dz_1 f^{(n_2)}(z_2 - b_2) dz_2 \dots f^{(n_\ell)}(z_\ell - b_\ell) dz_\ell, \quad (2.7)$$

which agree with eq. (2.5) if all twists are generic. The existence of the limit in eq. (2.7) requires some explanation because  $f^{(1)}(z - b_i)$  has a pole at  $z = b_i$ . For a single proper rational twist  $b \in \Lambda_n^\times$ , the path  $[0, 1]_\varepsilon$  can be written as the composition of a straight line from 0 to  $b - \varepsilon$ , followed by a semicircle from  $b - \varepsilon$  to  $b + \varepsilon$  above  $b$ , and then followed by a straight line from  $b + \varepsilon$  to 1:

$$\omega \left( \begin{matrix} 1 \\ b \end{matrix} \right) = \lim_{\varepsilon \rightarrow 0} \int_{[0, b-\varepsilon]} dz f^{(1)}(z - b) + \int_{[b-\varepsilon, b+\varepsilon]} dz f^{(1)}(z - b) + \int_{[b+\varepsilon, 1]} dz f^{(1)}(z - b). \quad (2.8)$$

Clearly, the contribution to eq. (2.8) coming from the non-holomorphic part  $2\pi i \frac{\text{Im}(z)}{\text{Im}(\tau)}$  of  $f^{(1)}$

in eq. (2.2) vanishes in the limit  $\varepsilon \rightarrow 0$ . The contribution coming from the closed one-form  $\frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)} dz$  in turn is independent of  $\varepsilon$  by Stokes theorem, since the paths  $[0, 1]_\varepsilon$  belong to the same homotopy class. Computing eq. (2.8), we find that

$$\omega\left(\frac{1}{b}\right) = \log\left(\frac{\theta_1(1-b, \tau)}{\theta_1(b, \tau)}\right) - i\pi = -i\pi, \quad (2.9)$$

where the first term comes from the first and third integral in eq. (2.8) and vanishes by reflection and periodicity of the  $\theta_1$  function. The contribution of  $-i\pi$  is due to the second integral in eq. (2.8).

The higher-length case is handled similarly. First we note that on the semicircles we have additional contributions from the non-meromorphic parts of the weighting functions  $f^{(n_i)}(z_i - b_i)$  (cf. eq. (2.17) below), given by powers of  $2\pi i \frac{\text{Im}(z_i - b_i)}{\text{Im}(\tau)}$ . These additional contributions on the semicircle are bounded by  $\text{Im}(z_i) \leq \varepsilon$ , and the accompanying meromorphic functions have at most a simple pole at  $b_i$ . Hence, the overall integrand on the semicircle is finite as  $\varepsilon \rightarrow 0$ .

Subsequently, we may use the composition of paths formula for iterated integrals (cf. eq. (C.3)) to check that the contributions from the non-meromorphic parts on the semicircles are in fact of  $\mathcal{O}(\varepsilon)$  and therefore do not contribute in the limit  $\varepsilon \rightarrow 0$ . Thus we are left with integrals over meromorphic functions of the  $z_i$ , which do not depend on  $\varepsilon$  by homotopy of all paths  $[0, 1]_\varepsilon$ .

The upshot is then that, up to terms which vanish in the limit  $\varepsilon \rightarrow 0$ , the right hand side of eq. (2.7) is independent on  $\varepsilon$ , thus convergent. An example at length two can be found in appendix C.

## 2.2 General properties of elliptic iterated integrals and (t)eMZVs

In general, iterated integrals of the form (2.1) satisfy shuffle relations. In terms of combined letters  $B_i = \frac{n_i}{b_i}$ , the shuffle relation for elliptic iterated integrals reads

$$\Gamma(B_1, B_2, \dots, B_{\ell_B}; z) \Gamma(C_1, \dots, C_{\ell_C}; z) = \Gamma((B_1, B_2, \dots, B_{\ell_B}) \sqcup (C_1, \dots, C_{\ell_C}); z), \quad (2.10)$$

where  $\sqcup$  denotes the shuffle product [35]. Naturally, the shuffle relation eq. (2.10) straightforwardly carries over to eMZVs,

$$\omega(n_1, n_2, \dots, n_{\ell_n}) \omega(m_1, m_2, \dots, m_{\ell_m}) = \omega((n_1, n_2, \dots, n_{\ell_n}) \sqcup (m_1, m_2, \dots, m_{\ell_m})), \quad (2.11)$$

and tMZVs

$$\omega(B_1, B_2, \dots, B_{\ell_B}) \omega(C_1, \dots, C_{\ell_C}) = \omega((B_1, B_2, \dots, B_{\ell_B}) \sqcup (C_1, \dots, C_{\ell_C})), \quad (2.12)$$

where the  $B_i, C_i$  are combined letters as defined above.

Taking into account the parity property (2.3) of the weighting functions  $f^{(n_i)}$  and the definition of elliptic iterated integrals, one finds the reflection identity

$$\Gamma\left(\frac{n_1}{b_1} \frac{n_2}{b_2} \dots \frac{n_\ell}{b_\ell}; z\right) = (-1)^{n_1+n_2+\dots+n_\ell} \Gamma\left(\frac{n_\ell}{z-b_\ell} \dots \frac{n_2}{z-b_2} \frac{n_1}{z-b_1}; z\right), \quad (2.13)$$

which is, however, valid only if the combined letter  $B_i = \frac{1}{b_i}$  with  $b_i$  a proper rational twist does not occur. The need to exclude such instances of  $B_i$  stems from the regularization of section 2.1 which does not preserve the reflection property.

Again, as in the case of the shuffle relation, there is an echo of the reflection identity for

eMZVs and teMZVs:

$$\begin{aligned}\omega(n_1, n_2, \dots, n_{\ell-1}, n_\ell) &= (-1)^{n_1+n_2+\dots+n_\ell} \omega(n_\ell, n_{\ell-1}, \dots, n_2, n_1) \\ \omega\left(\begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix}\right) &= (-1)^{n_1+n_2+\dots+n_\ell} \omega\left(\begin{matrix} n_\ell, n_{\ell-1}, \dots, n_1 \\ \tilde{b}_\ell, \tilde{b}_{\ell-1}, \dots, \tilde{b}_1 \end{matrix}\right),\end{aligned}\quad (2.14)$$

where  $\tilde{b}_i = 1 + \tau - b_i$  and letters  $B_i = \frac{1}{b_i}$  with  $b_i$  a proper rational twist are again excluded.

### 2.3 $q$ -expansion of teMZVs

In contrast to usual MZVs, which are just numbers, eMZVs and teMZVs are functions of the modular parameter  $\tau$  and can be expanded in its exponentiated cousin  $q = e^{2\pi i\tau}$ . The  $q$ -expansions of eMZVs and teMZVs rely on the available  $q$ -expansions of the weighting functions  $f^{(n)}$ . The discussion below will simplify considerably, if we consider in addition a class of meromorphic weighting functions  $g^{(n)}$ .

While the weighting functions  $f^{(n)}$  appear as expansion coefficients of the doubly-periodic completion  $\Omega(z, \alpha, \tau)$  of the Eisenstein–Kronecker series  $F(z, \alpha, \tau)$  (cf. eq. (B.1)) [6]

$$\Omega(z, \alpha, \tau) = \exp\left(2\pi i\alpha \frac{\text{Im}(z)}{\text{Im}(\tau)}\right) F(z, \alpha, \tau) = \sum_{n=0}^{\infty} f^{(n)}(z, \tau) \alpha^{n-1}, \quad (2.15)$$

the functions  $g^{(n)}$  are the expansion coefficients of the Eisenstein–Kronecker series [36, 37]

$$F(z, \alpha, \tau) = \sum_{n=0}^{\infty} g^{(n)}(z, \tau) \alpha^{n-1}. \quad (2.16)$$

The set of meromorphic functions  $g^{(n)}(z, \tau)$  starts with  $g^{(0)} = 1$  and  $g^{(1)}(z, \tau) = \frac{\theta'_1(z, \tau)}{\theta_1(z, \tau)}$  and can be related to their doubly-periodic but non-meromorphic<sup>4</sup> completions via eq. (2.15):

$$f^{(n)}(z, \tau) = \sum_{j=0}^n \frac{1}{(n-j)!} \left(2\pi i \frac{\text{Im}(z)}{\text{Im}(\tau)}\right)^{n-j} g^{(j)}(z, \tau). \quad (2.17)$$

Quasi-periodicity and the reflection property of  $F(z, \alpha, \tau)$  (see eqs. (B.5) and (B.6)) imply the following properties of the  $g^{(n)}(z, \tau)$ ,

$$g^{(n)}(z) = g^{(n)}(z+1), \quad g^{(n)}(z+\tau) = \sum_{j=0}^n \frac{(-2\pi i)^j}{j!} g^{(n-j)}(z), \quad g^{(n)}(-z) = (-1)^n g^{(n)}(z), \quad (2.18)$$

and their Fourier expansions are given by [37, 6, 28]

$$\begin{aligned}g^{(1)}(z, \tau) &= \pi \cot(\pi z) - 2i(2\pi i) \sum_{n,m=1}^{\infty} \sin(2\pi m z) q^{mn} \\ g^{(2k)}(z, \tau) &= -2\zeta_{2k} - 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n,m=1}^{\infty} \cos(2\pi m z) n^{2k-1} q^{mn}, \quad k > 0 \\ g^{(2k+1)}(z, \tau) &= -2i \frac{(2\pi i)^{2k+1}}{(2k)!} \sum_{n,m=1}^{\infty} \sin(2\pi m z) n^{2k} q^{mn}, \quad k > 0.\end{aligned}\quad (2.19)$$

<sup>4</sup>Note that by Liouville's theorem, every meromorphic, doubly-periodic function, which has at most a simple pole at zero must be constant. Therefore, one either has to include non-holomorphic factors (as we do here, following [6]) or allow poles of order  $\geq 2$  (as in [4], §5.1.2).



For real values of  $z$  one finds from eqs. (2.15) and (2.17) that  $f^{(n)}(z) = g^{(n)}(z)$  and their  $q$ -expansions agree. In particular, they can be employed to find  $q$ -expansions for eMZVs

$$\omega(n_1, n_2, \dots, n_\ell) = \omega_0(n_1, n_2, \dots, n_\ell) + \sum_{k=1}^{\infty} c_k(n_1, n_2, \dots, n_\ell) q^k. \quad (2.20)$$

The  $q$ -independent quantity  $\omega_0$  in eq. (2.20) is called the *constant term* of the eMZV  $\omega$  and is known to be a  $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combination of MZVs (see refs. [28, 10, 11]).

In order to describe the  $q$ -dependence of teMZVs in a similar manner, we consider the twist  $b = s + r\tau$  in the weighting function  $f^{(n)}(z - b)$  eq. (2.17) for real values of  $z$ :

$$f^{(n)}(z - s - r\tau, \tau) = \sum_{j=0}^n \frac{(-2\pi i r)^{n-j}}{(n-j)!} g^{(j)}(z - s - r\tau, \tau), \quad z \in \mathbb{R}. \quad (2.21)$$

Employing eqns. (2.19), the functions  $g^{(j)}(z - b, \tau)$ , can be expanded in non-negative rational powers of  $q$ ,

$$\begin{aligned} g^{(2k+1)}(z - s - r\tau, \tau) &= \delta_{k,0} \pi \cot(\pi(z - s - r\tau)) + \frac{(2\pi i)^{2k}}{(2k)!} \sum_{m,n=1}^{\infty} n^{2k} q^{mn} \\ &\times \left\{ \cos(2\pi m(z - s))(q^{mr} - q^{-mr}) - i \sin(2\pi m(z - s))(q^{mr} + q^{-mr}) \right\}, \quad k \geq 0 \\ g^{(2k)}(z - s - r\tau, \tau) &= -2\zeta_{2k} - \frac{(2\pi i)^{2k-1}}{(2k-1)!} \sum_{m,n=1}^{\infty} n^{2k-1} q^{mn} \\ &\times \left\{ \cos(2\pi m(z - s))(q^{mr} + q^{-mr}) - i \sin(2\pi m(z - s))(q^{mr} - q^{-mr}) \right\}, \quad k > 0. \end{aligned} \quad (2.22)$$

The cotangent term in  $g^{(1)}$  may be rewritten as

$$\pi \cot(\pi(z - s - r\tau)) = i\pi(1 + q^r e^{2\pi i(s-z)}) \sum_{n=0}^{\infty} (q^r e^{2\pi i(s-z)})^n. \quad (2.23)$$

On these grounds,  $f^{(n)}(z - s - r\tau)$  can be expanded in powers of  $q^r$  and  $q^{1-r}$  such that every teMZV admits an expansion in  $q^p$ ,

$$\omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) = \omega_0 \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) + \sum_{k=1}^{\infty} c_k \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) (q^p)^k, \quad (2.24)$$

where  $1/p \in \mathbb{Q}$  is the least common denominator of all occurring  $r_i$ . The  $q$ -independent quantity  $\omega_0$  in eq. (2.24) is called the *constant term* of the teMZV, which we are going to study in section 3. Depending on the set of twists  $b_i$ , different classes of objects appear as constant terms: while MZVs cover constant terms for generic twists, proper rational twists lead to cyclotomic MZVs [3, 12–14]. We will refer to teMZVs for which  $c_k \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) = 0$  for all  $k \in \mathbb{N}^+$  as *constant*.

### 3 $q$ -expansion for twisted elliptic multiple zeta values

The goal of this section is to set up an initial value problem for teMZVs eq. (2.5) and to obtain their  $q$ -expansion without performing any integral over their trigonometric constituents in eq. (2.22). Following the strategy for computing the usual eMZV's  $q$ -expansion in [8, 10], in

a first step we derive a first-order differential equation in  $\tau$  for teMZVs. In the second step, a boundary value at the cusp  $\tau \rightarrow i\infty$  will be determined to identify a unique solution to the differential equation. Since the action of  $\partial_\tau$  reduces the length of teMZVs, one can derive the  $q$ -expansion for teMZVs recursively.

For eMZVs, Eisenstein series and MZVs are the building blocks for the  $\tau$ -derivative and constant term respectively [8, 10, 11]. Similarly, we will show that the weighting functions  $f^{(k)}(b, \tau)$  evaluated at lattice points  $b \in \Lambda_n + \Lambda_n\tau$  and cyclotomic MZVs are suitable generalizations thereof for teMZVs.

After deriving the differential equation in subsection 3.1, the constant term will be elaborated on in subsection 3.2 for generic twists and modifications when including proper rational twists are discussed in subsection 3.3.

### 3.1 Differential equation

We begin by defining a generating series for teMZVs of length  $\ell$ ,

$$\begin{aligned} \mathbb{T} \left[ \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right] &= \int_{0 \leq z_i \leq z_{i+1} \leq 1} \Omega(z_1 - b_1, \alpha_1) dz_1 \Omega(z_2 - b_2, \alpha_2) dz_2 \dots \Omega(z_\ell - b_\ell, \alpha_\ell) dz_\ell \\ &= \sum_{n_1, n_2, \dots, n_\ell = 0}^{\infty} \alpha_1^{n_1-1} \alpha_2^{n_2-1} \dots \alpha_\ell^{n_\ell-1} \omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right), \end{aligned} \quad (3.1)$$

generalizing a construction of [8]. As a consequence of the mixed heat equation (B.4),  $\Omega(z-b, \alpha)$  satisfies the differential equation

$$\begin{aligned} \partial_\tau \Omega(z - s - r\tau, \alpha) &= \exp(-2\pi i r \alpha) \partial_\tau F(z - s - r\tau, \alpha) \\ &= \exp(-2\pi i r \alpha) \left( -r \partial_z + \frac{1}{2\pi i} \partial_z \partial_\alpha \right) F(z - s - r\tau, \alpha) \\ &= \frac{1}{2\pi i} \partial_z \partial_\alpha \Omega(z - s - r\tau, \alpha), \quad z \in \mathbb{R}. \end{aligned} \quad (3.2)$$

The  $\tau$ -derivative of the generating function (3.1) reads

$$\begin{aligned} 2\pi i \frac{\partial}{\partial \tau} \mathbb{T} \left[ \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right] &= \int_{0 \leq z_i \leq z_{i+1} \leq 1} dz_1 dz_2 \dots dz_\ell \sum_{i=1}^{\ell} \partial_{z_i} \partial_{\alpha_i} \Omega(z_i - b_i, \alpha_i) \prod_{j \neq i}^{\ell} \Omega(z_j - b_j, \alpha_j) \\ &= \partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] - \partial_{\alpha_1} \Omega(-b_1, \alpha_1) \mathbb{T} \left[ \begin{matrix} \alpha_2, \dots, \alpha_\ell \\ b_2, \dots, b_\ell \end{matrix} \right] \\ &\quad + \sum_{i=2}^{\ell} \left( \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_{i-1}} \Omega(b_i - b_{i-1}, \alpha_{i-1}) \right. \\ &\quad \left. - \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_i} \Omega(b_{i-1} - b_i, \alpha_i) \right), \end{aligned} \quad (3.3)$$

where we used eq. (3.2) in the first line. In the second equality, the number of integrations is reduced by evaluating  $\int dz_i \partial_{z_i} \partial_{\alpha_i} \Omega(z_i - b_i, \alpha_i)$  via boundary terms  $\partial_{\alpha_i} \Omega(z_i - b_i, \alpha_i)|_{z_i=0}^{z_i=1}$  with  $z_0 = 0$  and  $z_{\ell+1} = 1$ . The resulting products of the form  $\Omega(z_i - b_{i-1}, \alpha_{i-1}) \Omega(z_i - b_i, \alpha_i)$  are rewritten using the Fay identity eq. (B.3) such that each integration variable  $z_i$  appears in at most one factor of  $\Omega$ . The details of the computation can be found in appendix E.

Upon expanding  $\Omega$  and  $T$  in eq. (3.3) in  $\alpha_i$ , one can compare the coefficients of the monomials  $\alpha_1^{m_1} \dots \alpha_\ell^{m_\ell}$ . The coefficient of each monomial is a linear combination of some  $f^{(n)}$  multiplied

by teMZV of length  $\ell - 1$ . Working out the details yields the following differential equation for teMZVs ( $\ell \geq 2$ ),

$$\begin{aligned}
2\pi i \partial_\tau \omega \left( \begin{matrix} n_1, \dots, n_\ell \\ b_1, \dots, b_\ell \end{matrix} \right) &= h^{(n_\ell+1)}(-b_\ell) \omega \left( \begin{matrix} n_1, \dots, n_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right) - h^{(n_1+1)}(-b_1) \omega \left( \begin{matrix} n_2, \dots, n_\ell \\ b_2, \dots, b_\ell \end{matrix} \right) \\
&+ \sum_{i=2}^{\ell} \left[ \theta_{n_i \geq 1} \sum_{k=0}^{n_{i-1}+1} \binom{n_i+k-1}{k} h^{(n_{i-1}-k+1)}(b_i - b_{i-1}) \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, n_i+k, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) \right. \\
&\quad - \theta_{n_{i-1} \geq 1} \sum_{k=0}^{n_i+1} \binom{n_{i-1}+k-1}{k} h^{(n_i-k+1)}(b_{i-1} - b_i) \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, n_{i-1}+k, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right) \\
&\quad \left. + (-1)^{n_i+1} \theta_{n_{i-1} \geq 1} \theta_{n_i \geq 1} h^{(n_{i-1}+n_i+1)}(b_i - b_{i-1}) \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) \right], \quad (3.4)
\end{aligned}$$

where we have introduced  $\theta_{n \geq 1} = 1 - \delta_{n,0}$  for non-negative  $n$  to indicate that some of the contributions in the last three lines vanish for  $n_i = 0$ . Furthermore we used the shorthand

$$h^{(n)}(z, \tau) = (n-1) f^{(n)}(z, \tau). \quad (3.5)$$

For vanishing twists  $b_i = 0$ , eq. (3.4) reduces to the differential equation for eMZVs stated in eq. (2.47) of ref. [10] since the weighting functions  $f^{(n)}$  are related to holomorphic Eisenstein series (with  $G_0(\tau) = -1$ ) via

$$-f^{(k)}(0, \tau) = G_k(\tau) = \begin{cases} 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : k \text{ even} \\ 0 & : k \text{ odd} \end{cases}. \quad (3.6)$$

In other words, the functions  $h^{(n)}(b_i, \tau)$  occurring in the differential equation (3.4) take the rôle of Eisenstein series in the differential equation for eMZVs.

The differential equation (3.4) is shown to be compatible with the regularization of teMZVs with proper rational twists  $b_i \in \Lambda_n^\times$  (described in subsection 2.1) in appendix F.

### 3.2 Constant terms for generic twists

In this subsection we are going to extend the constant-term procedure for eMZVs studied in [10,11] to a procedure delivering the constant terms for teMZVs. Calculating the constant term for teMZVs amounts to the computation of the limit  $\tau \rightarrow i\infty$  of eq. (2.5). This limit will figure as the initial value for the differential equation (3.4) discussed in the previous subsection.

In order to make the bookkeeping more efficient, it is convenient to consider the following generating series, which is a generalization of the  $A$ -part of Enriquez' elliptic KZB associator [7] to the realm of teMZVs:

$$\begin{aligned}
A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau) &= \sum_{\ell \geq 0} (-1)^\ell \sum_{\substack{n_1, n_2, \dots, n_\ell \geq 0 \\ b_1, b_2, \dots, b_\ell \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}} \omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) \text{ad}_{x_{b_\ell}}^{n_\ell}(y) \dots \text{ad}_{x_{b_2}}^{n_2}(y) \text{ad}_{x_{b_1}}^{n_1}(y) \\
&= \tilde{\mathcal{P}} \exp \left( - \int_0^1 dz \sum_{k=0}^{\infty} \sum_{b \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times} f^{(k)}(z-b, \tau) \text{ad}_{x_b}^k(y) \right), \quad (3.7)
\end{aligned}$$

where  $\Lambda_n$  and  $\Lambda_n^\times$  were defined in eq. (2.6). There is no loss of generality in studying the lattice  $\Lambda_n + \Lambda_n \tau$  rather than  $\Lambda_n + \Lambda_m \tau$  with  $m \neq n$ : the latter can be embedded into the lattice

$\Lambda_N + \Lambda_N \tau$  with  $N$  the smallest common multiple of  $m$  and  $n$ .

Note that proper rational twists  $b \in \Lambda_n^\times$  have been excluded from the summation range for  $b$  in eq. (3.7) in order to relegate a discussion of the additional ingredients required in these cases to section 3.3. Moreover,  $\tilde{\mathcal{P}} \exp(\dots)$  denotes the path-ordered exponential with reverted order of multiplication for the non-commutative variables

$$\text{ad}_x(y) = [x, y], \quad \text{ad}_x^n(y) = \underbrace{[x, \dots [x, [x, y]] \dots]}_{n \text{ times}}, \quad (3.8)$$

in comparison to the order of the integration variables  $z$ . Furthermore – although composed of several non-commutative variables  $x_{b_i}$  and  $y$  – each nested commutator  $\text{ad}_{x_{b_i}}^{n_i}(y)$  is treated as a single letter when reversing the order of multiplication.

The series (3.7) combines different instances of the generating series  $\mathbb{T} \left[ \begin{smallmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_\ell \\ b_1 & b_2 & \dots & b_\ell \end{smallmatrix} \right]$  in eq. (3.1),

$$A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau) \longleftrightarrow \sum_{\ell \geq 0} (-1)^\ell \sum_{b_1, \dots, b_\ell \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times} \mathbb{T} \left[ \begin{smallmatrix} \alpha_\ell & \alpha_{\ell-1} & \dots & \alpha_1 \\ b_\ell & b_{\ell-1} & \dots & b_1 \end{smallmatrix} \right], \quad (3.9)$$

summing over all values of the length  $\ell \geq 0$  and the generic twists  $b_i \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times$ . The non-commutative product of  $\text{ad}_{x_{b_i}}^{k_i}(y)$  corresponds to commutative variables  $\alpha_i^{k_i-1}$  in eq. (3.1), which accompany individual teMZVs in the respective generating series. While the organization via  $\alpha_i^{k_i-1}$  is better suited for the study of the differential equation of teMZVs, the non-commutative variables  $\text{ad}_{x_{b_i}}^{k_i}(y)$  in eq. (3.8) are well adapted to the subsequent analysis of their constant terms<sup>5</sup>.

### 3.2.1 Degeneration of weighting functions

In order to compute  $\lim_{\tau \rightarrow i\infty} A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau)$ , we need to study the degeneration of the weighting functions  $f^{(k)}(z - b, \tau)$  as  $\tau \rightarrow i\infty$  or  $q \rightarrow 0$ . Conveniently, the limit is expressed in the variables

$$w = e^{2\pi iz}, \quad dz = \frac{1}{2\pi i} \frac{dw}{w}. \quad (3.10)$$

Using the  $q$ -expansions eqs. (2.22) and (2.23) together with eq. (2.21) we obtain, for generic twists and  $k > 1$ ,

$$\begin{aligned} \lim_{\tau \rightarrow i\infty} f^{(k)}(z - s - r\tau) dz &= \left( \frac{\pi i (-2\pi i r)^{k-1}}{(k-1)!} - 2 \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-2\pi i r)^{k-2m}}{(k-2m)!} \zeta_{2m} \right) \frac{1}{2\pi i} \frac{dw}{w} \\ &= -\frac{dw}{w} \sum_{m=0}^k \frac{B_m (-2\pi i)^{m-1} (-2\pi i r)^{k-m}}{m! (k-m)!}, \quad k > 1. \end{aligned} \quad (3.11)$$

Here, we have used  $\zeta_{2m} = -\frac{B_{2m} (2\pi i)^{2m}}{2(2m)!}$ , where  $B_k$  denote the Bernoulli numbers (such that  $B_1 = -\frac{1}{2}$ ). While  $f^{(0)} = 1$ , the case of  $f^{(1)}(z - b)$  is special and we find

$$\lim_{\tau \rightarrow i\infty} f^{(1)}(z - s - r\tau) dz = \begin{cases} \left( \frac{1}{2} - r \right) \frac{dw}{w} & : r \neq 0 \\ -\frac{1}{2} \frac{dw}{w} + \frac{dw}{w-1} & : r = 0 \end{cases}. \quad (3.12)$$

<sup>5</sup>The use of two, essentially equivalent, generating series of teMZVs goes back to Enriquez' original work on eMZVs [8].

Combining eqs. (3.11) and (3.12) allows to rewrite the exponent of eq. (3.7) as follows:

$$\lim_{\tau \rightarrow i\infty} -dz \sum_{k=0}^{\infty} \left( \sum_{b \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times} f^{(k)}(z-b) \text{ad}_{x_b}^k(y) \right) = \tilde{y}_n \frac{dw}{w} + t \frac{dw}{w-1}, \quad (3.13)$$

where

$$\tilde{y}_n = -\frac{\text{ad}_{x_0}}{e^{2\pi i \text{ad}_{x_0}} - 1}(y) + \sum_{b \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n} \frac{\text{ad}_{x_b} e^{-2\pi i r \text{ad}_{x_b}}}{e^{-2\pi i \text{ad}_{x_b}} - 1}(y), \quad t = [y, x_0]. \quad (3.14)$$

These definitions of  $\tilde{y}_n$  and  $t$  are tailored to track the appearance of the forms  $\frac{dw}{w}$  and  $\frac{dw}{w-1}$  in the degeneration limits eqs. (3.11) and (3.12) of  $f^{(k)}(z-b, \tau)$ . In absence of twists, for instance, the first contribution to  $\tilde{y}_n$  in eq. (3.14) stems from setting  $r = 0$  in eq. (3.11) and identifying the generating series of  $\frac{B_k(-2\pi i)^{k-1}}{k!} \text{ad}_{x_0}^k(y)$ :

$$\begin{aligned} \lim_{\tau \rightarrow i\infty} -dz \sum_{k=0}^{\infty} \left( f^{(k)}(z) \text{ad}_{x_0}^k(y) \right) &= -\frac{dw}{w} \sum_{k=0}^{\infty} \frac{B_k(2\pi i)^{k-1}}{k!} \text{ad}_{x_0}^k(y) - \frac{dw}{w-1} \text{ad}_{x_0}(y) \\ &= -\frac{\text{ad}_{x_0}}{e^{2\pi i \text{ad}_{x_0}} - 1}(y) \frac{dw}{w} - \frac{dw}{w-1} \text{ad}_{x_0}(y). \end{aligned} \quad (3.15)$$

In the generalization to non-zero twists  $r \neq 0$ , the second contribution to  $\tilde{y}_n$  in eq. (3.14) arises as the generating series of  $\sum_{m=0}^k \frac{B_m(-2\pi i)^{m-1}}{m!} \frac{(-2\pi i r)^{k-m}}{(k-m)!} \text{ad}_{x_b}^k(y)$ . Note in particular that the dependence on  $r$  gives rise to the factor of  $e^{-2\pi i r \text{ad}_{x_b}}$  in the numerator.

By comparing the degeneration behaviour in eq. (3.13) with eq. (3.7), we deduce that

$$\lim_{\tau \rightarrow i\infty} A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau) = \tilde{\mathcal{P}} \exp \left( \int_{C_0^{2\pi}(1)} \left[ \frac{\tilde{y}_n}{w} + \frac{t}{w-1} \right] dw \right), \quad (3.16)$$

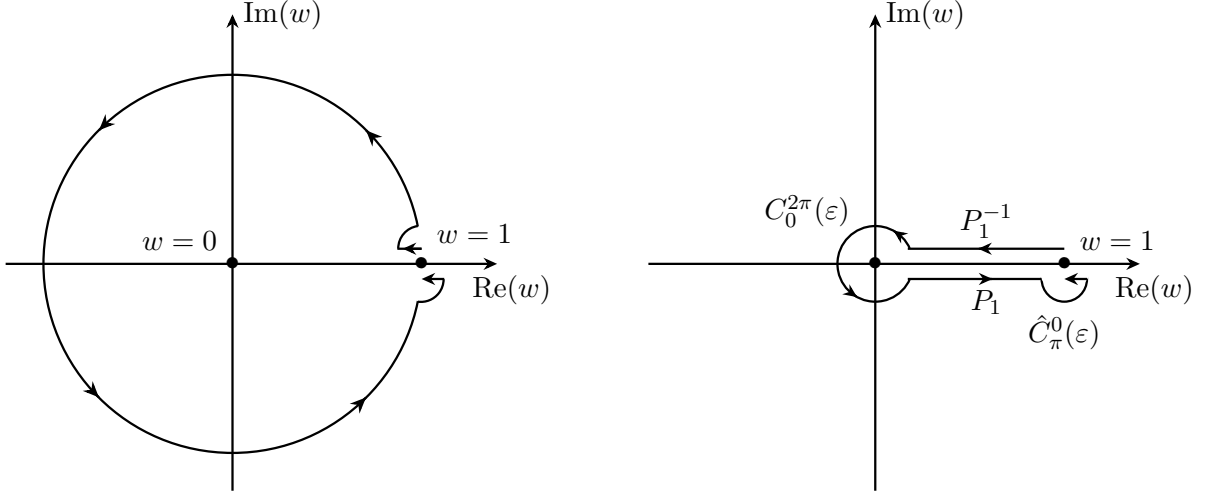
where the unit circle  $w \in C_0^{2\pi}(1)$  arises from the path of integration  $[0, 1] \subset \mathbb{C}$  under the change of variables eq. (3.10). Strictly speaking, eq. (3.16) requires regularization, due to divergences at  $w = 1$ . They are treated in analogy to eMZVs as described in refs. [8, 11, 28] and cause modifications to be pointed out in the subsequent discussion.

### 3.2.2 Deforming the integration contour

We have expressed  $\lim_{\tau \rightarrow i\infty} A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau)$  as a generating series of iterated integrals of explicit differential forms along the unit circle  $C_0^{2\pi}(1)$ , see the left panel of figure 4 below. Since the integrand in eq. (3.13) is holomorphic on  $\mathbb{C} \setminus \{0, 1\}$ , the resulting path-ordered exponential eq. (3.16) is homotopy-invariant. Therefore one can replace the unit circle by a contour homotopic to it, visualized in the right panel of figure 4.

This deformed contour can in turn be viewed as the composition of straight paths  $P_1, P_1^{-1}$  connecting the points  $w = 0, 1$  along with a circle  $C_0^{2\pi}(\varepsilon)$  of infinitesimal radius around the origin, as shown in figure 4. Moreover, the regularization alluded to above manifests itself in figure 4: both paths  $C_0^{2\pi}(1)$  as well as the composition  $P_1^{-1} C_0^{2\pi}(\varepsilon) P_1 \hat{C}_\pi^0(\varepsilon)$  have to leave  $w = 1$  with velocity  $-1$  and arrive back at  $w = 1$  with the same velocity. More precisely, both paths, which really are smooth functions  $[0, 1] \rightarrow \mathbb{C}^\times$  must have a derivative equal to  $-\frac{\partial}{\partial w} \in T_1(\mathbb{C}^\times)$ , where  $T_1$  denotes the tangent space at 1. This is also the reason for the semicircle  $\hat{C}_\pi^0(\varepsilon)$ .

The virtue of deforming the path of integration is that eq. (3.16) can now be computed



**Figure 4:** Deformation of the unit circle  $C_0^{2\pi}(1)$  to the path composition  $P_1^{-1}C_0^{2\pi}(\epsilon)P_1\hat{C}_\pi^0(\epsilon)$ .

rather explicitly. First, in view of the reversal operation contained in the definition of  $\tilde{\mathcal{P}}$ , the composition of paths  $\alpha$  and  $\beta$  translates into a concatenation of the non-commutative series with reversed order,

$$\tilde{\mathcal{P}} \exp \left( \int_{\alpha\beta} \omega \right) = \tilde{\mathcal{P}} \exp \left( \int_{\beta} \omega \right) \tilde{\mathcal{P}} \exp \left( \int_{\alpha} \omega \right) \quad (3.17)$$

regardless of the differential form  $\omega$ . Hence, the equality of homotopy classes of paths (relative to the tangent vector  $-\frac{\partial}{\partial w}$  at 1)  $[C_0^{2\pi}(1)] = [P_1^{-1}C_0^{2\pi}(\epsilon)P_1\hat{C}_\pi^0(\epsilon)]$  allows to rewrite (3.16) as

$$\lim_{\tau \rightarrow i\infty} A_{(\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times}^{\text{twist}}(\tau) = e^{i\pi t} \Phi(\tilde{y}_n, t) e^{2\pi i \tilde{y}_n} \Phi(\tilde{y}_n, t)^{-1}, \quad (3.18)$$

where  $\Phi(\tilde{y}_n, t)$  denotes the Drinfeld associator [38]. In deducing (3.18), we have used the identities

$$\Phi(\tilde{y}_n, t) = \tilde{\mathcal{P}} \exp \left( \int_{P_1} \left[ \frac{\tilde{y}_n}{w} + \frac{t}{w-1} \right] dw \right), \quad (3.19)$$

$$e^{2\pi i \tilde{y}_n} = \tilde{\mathcal{P}} \exp \left( \int_{C_0^{2\pi}(\epsilon)} \left[ \frac{\tilde{y}_n}{w} + \frac{t}{w-1} \right] dw \right), \quad (3.20)$$

$$e^{i\pi t} = \tilde{\mathcal{P}} \exp \left( \int_{\hat{C}_\pi^0(\epsilon)} \left[ \frac{\tilde{y}_n}{w} + \frac{t}{w-1} \right] dw \right). \quad (3.21)$$

Since the coefficients of  $\Phi$  are  $\mathbb{Q}$ -linear combinations of MZVs [39], an implementation of eq. (3.18) using a standard computer algebra system can be used to explicitly write the constant terms of teMZVs for generic twists as  $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combinations of MZVs. Equation (3.18) is a generalization of a similar formalism for eMZVs, which has been established in refs. [7, 10]. Examples for constant terms of teMZVs computed via eq. (3.18) are gathered in appendix G.1.

### 3.3 Constant terms for all twists

So far, we have only considered the constant terms of teMZVs with generic twists. The presence of proper rational twists  $b \in \Lambda_n^\times$  requires a separate discussion due to the additional features of

the corresponding weighting function  $f^{(1)}(z - b)$ :

- its simple pole in the *interior* of the domain of integration requires the regularization procedure of subsection 2.1
- its  $\tau \rightarrow i\infty$  limit introduces singularities  $\frac{dw}{w-\zeta}$  with  $\zeta$  denoting a root of unity

In order to facilitate the computation of the constant terms of teMZVs including proper rational twists, we again introduce a generating series for bookkeeping purposes

$$\begin{aligned} A_{\Lambda_n + \Lambda_n \tau}^{\text{twist}}(\tau) &= \sum_{\ell \geq 0} (-1)^\ell \sum_{\substack{n_1, n_2, \dots, n_\ell \geq 0 \\ b_1, b_2, \dots, b_\ell \in \Lambda_n + \Lambda_n \tau}} \omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) \text{ad}_{x_{b_\ell}}^{n_\ell}(y) \dots \text{ad}_{x_{b_2}}^{n_2}(y) \text{ad}_{x_{b_1}}^{n_1}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{P}} \exp \left( - \int_{[0,1]^\varepsilon} dz \sum_{k=0}^{\infty} \sum_{b \in \Lambda_n + \Lambda_n \tau} f^{(k)}(z - b, \tau) \text{ad}_{x_b}^k(y) \right), \end{aligned} \quad (3.22)$$

which generalizes eq. (3.7) to the case of arbitrary twists.

### 3.3.1 Degeneration of weighting functions

As in the above situation we need to determine the degeneration limit of the weighting functions  $f^{(k)}(z - b, \tau)$  as  $\tau \rightarrow i\infty$ . The only case, where this degeneration limit differs from the results of the previous subsection (cf. eqs. (3.11) and (3.12)) is  $k = 1$  and  $r = 0$ :

$$\lim_{\tau \rightarrow i\infty} f^{(1)}(z - s, \tau) dz = -\frac{1}{2} \frac{dw}{w} + \frac{dw}{w - e^{2\pi i s}}. \quad (3.23)$$

Note the occurrence of the root of unity  $e^{2\pi i s}$ . Denoting the set of  $n^{\text{th}}$  roots of unity by

$$\mu_n = \{e^{2\pi i s} \mid s \in \Lambda_n\}, \quad (3.24)$$

the generalization of eq. (3.13) to twists  $b \in \Lambda_n + \Lambda_n \tau$  reads

$$\lim_{\tau \rightarrow i\infty} -dz \sum_{k=0}^{\infty} \sum_{b \in \Lambda_n + \Lambda_n \tau} f^{(k)}(z - b) \text{ad}_{x_b}^k(y) = \tilde{y}_n \frac{dw}{w} + \sum_{\zeta \in \mu_n} t_\zeta \frac{dw}{w - \zeta}, \quad (3.25)$$

where<sup>6</sup>

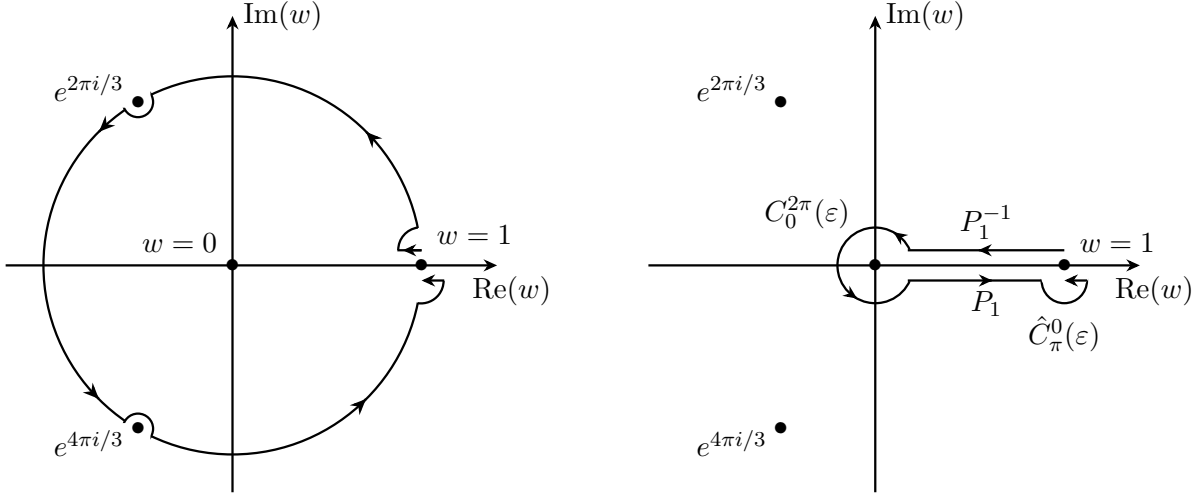
$$\begin{aligned} \tilde{y}_n &= - \sum_{b \in \Lambda_n} \frac{\text{ad}_{x_b}}{e^{2\pi i \text{ad}_{x_b}} - 1}(y) + \sum_{b \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n} \frac{\text{ad}_{x_b} e^{-2\pi i r \text{ad}_{x_b}}}{e^{-2\pi i \text{ad}_{x_b}} - 1}(y) \\ t_\zeta &= [y, x_s], \quad \text{for } \zeta = e^{2\pi i s} \in \mu_n. \end{aligned} \quad (3.26)$$

Equation (3.26) is the generalization of eq. (3.14) to arbitrary twists in the lattice  $\Lambda_n + \Lambda_n \tau$ , and can be proved along the lines of the previous subsection. In particular, the expression for  $\tilde{y}_n$  follows by repeating the steps which have been detailed around eq. (3.15).

<sup>6</sup>Note that the definition of  $\tilde{y}_n$  for all twists is different from eq. (3.14) in the previous subsection which is valid for generic twists only.

### 3.3.2 Deforming the integration contour

Now the image of the integration contour  $[0, 1]_\varepsilon$  under the transformation  $z \mapsto w = e^{2\pi iz}$  is the unit circle around 0, which is dented at the roots of unity  $e^{2\pi is} \in \mu_n$  as pictured in figure 5 below. However, the point  $w = 1$  is special and will be taken care of by the regularization of subsection 3.2. Similar to the situation above, the dented unit circle is homotopic to  $[P_1^{-1}C_0^{2\pi}(\varepsilon)P_1\hat{C}_\pi^0(\varepsilon)]$  as depicted in figure 5 for twists in  $\Lambda_3$ . Hence, eq. (3.22) can be generalized to



**Figure 5:** Deformation of the dented unit circle to the path composition  $P_1^{-1}C_0^{2\pi}(\varepsilon)P_1\hat{C}_\pi^0(\varepsilon)$ . Proper rational twists  $s = 0, \frac{1}{n}, \dots, \frac{n-1}{n}$  are mapped to unit roots  $\zeta = e^{2\pi is} \in \mu_n$ .

$$\lim_{\tau \rightarrow i\infty} A_{\Lambda_n + \Lambda_n \tau}^{\text{twist}}(\tau) = e^{i\pi t_1} \Phi_n(\tilde{y}_n, (t_\zeta)_{\zeta \in \mu_n}) e^{2\pi i \tilde{y}_n} \Phi_n(\tilde{y}_n, (t_\zeta)_{\zeta \in \mu_n})^{-1}, \quad (3.27)$$

where  $\Phi_n(e_0, (e_\zeta)_{\zeta \in \mu_n})$  is the cyclotomic version of the Drinfeld associator [40], defined by

$$\Phi_n(e_0, (e_\zeta)_{\zeta \in \mu_n}) = \tilde{\mathcal{P}} \exp \left( \int_0^1 \left[ \frac{e_0}{w} + \sum_{\zeta \in \mu_n} \frac{e_\zeta}{w - \zeta} \right] dw \right). \quad (3.28)$$

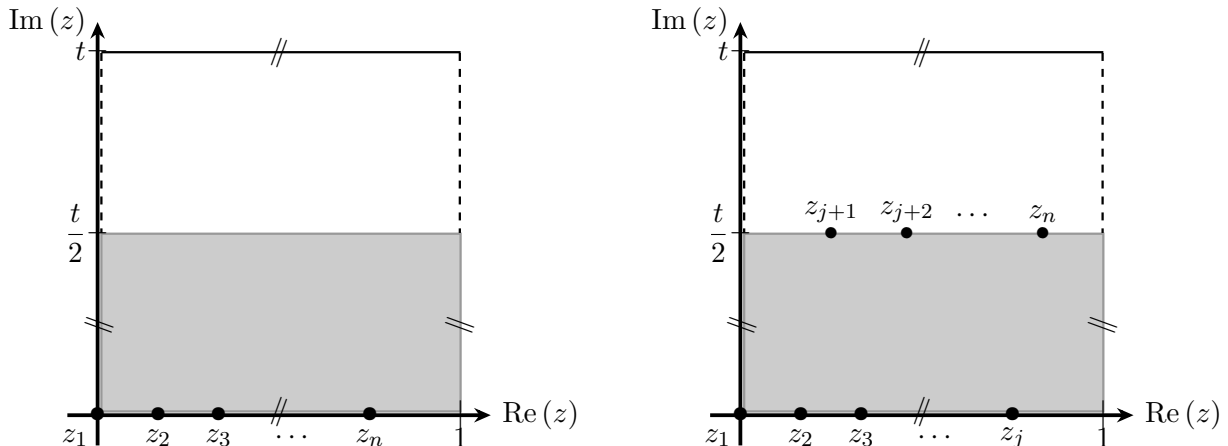
Since  $\Phi_n$  is the generating series of  $n$ -cyclotomic MZVs [3, 12–14], the constant terms of teMZVs for arbitrary twists in the lattice  $\Lambda_n + \Lambda_n \tau$  are  $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combinations of cyclotomic MZVs. Definitions and properties of cyclotomic MZVs are collected in appendix D, and examples for constant terms of teMZVs with proper rational twists can be found in appendix G.2.

As exemplified by  $\omega_0 \left( \frac{1}{1/2} \right) = -i\pi$ , it is the regularization of divergences occurring for proper rational twists, which spoils the validity of the reflection property eq. (2.14) for letters  $B = \frac{1}{b}$  with  $b \in \Lambda_n^\times$ . It would be interesting to identify an alternative regularization scheme where eq. (2.14) is preserved.

## 4 One-loop open-string amplitude

This section is devoted to the discussion of the appearance of teMZVs in a physics context – in the low-energy expansion of scattering amplitudes in string theory. In general,  $g$ -loop string amplitudes can be represented by integrals over the moduli space of punctured Riemann surfaces of





**Figure 6:** Worldsheets of cylinder topology are mapped to the shaded region, see the left and the right panel for the planar and the non-planar case, respectively. The punctures on the boundaries are taken to have coordinates with  $\text{Re}(z_j) \in [0, 1]$  and either  $\text{Im}(z_j) = 0$  or  $\text{Im}(z_j) = \frac{t}{2}$ . The identification of edges is marked by  $\cong$  and  $//$ , respectively, and inherited from a torus with modular parameter  $\tau = it$ . The Möbius topology is not drawn here, because all its contributions to the amplitude can be inferred from the planar cylinder topology. This can be seen by the change of variables described in ref. [41].

genus  $g$ . For one-loop scattering of open strings, the Riemann surfaces or *worldsheets* of interest are the cylinder and the Möbius strip. The punctures – the insertion points of vertex operators for external states – are then integrated over the boundary components of the worldsheets. These boundary integrals are weighted by traces over Lie-algebra generators  $t^a$  associated with the gauge degrees of freedom of the open-string states: Each boundary component contributes a separate trace factor, in each of which the order of multiplication matches the ordering of the associated punctures.

A convenient parametrization of the one-loop open-string topologies – the cylinder and the Möbius strip – can be obtained starting from the torus by restricting the modular parameter to  $\tau_C = it$  and to  $\tau_M = it + \frac{1}{2}$ , respectively, where  $t \in \mathbb{R}_+$ .

In both cases, the boundary is parametrized via  $z \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$  with  $\text{Re}(z) \in [0, 1]$  and  $\text{Im}(z) = 0$  or  $\text{Im}(z) = \frac{t}{2}$  which is sometimes referred to as the closed-string channel.

In this setup, elliptic iterated integrals eq. (2.1) appear naturally when integrating over moduli spaces of cylinder- and Möbius-strip punctures. This is yet another example, where the iterated integrals on the boundary of open-string worldsheets yield special values of polylogarithms tailored to the corresponding Riemann surfaces, generalizing the ubiquity of MZVs at genus zero.

Cylindrical worldsheets with all insertions on the same boundary are referred to as *planar* cylinders. For these contributions to the amplitude, all integrals over the punctures were shown to boil down to eMZVs in [28]. Moreover, the only difference between integrals over punctures on the Möbius strip and those on the planar cylinder is the value of the modular parameter  $\tau$  [41]: therefore one can straightforwardly convert the contributions from the planar cylinder to those of the Möbius strip by

- replacing  $q_C = e^{2\pi i\tau_C} = e^{-2\pi t}$  in the Fourier expansion of the eMZVs in the planar-cylinder contribution by  $q_M = e^{2\pi i\tau_M} = e^{-2\pi t + i\pi} = -q_C$ ; this results in alternating relative signs between the coefficients in the  $q_C$ -expansion of the Möbius-strip contributions and the cylinder contributions.

- inserting a factor of  $\pm \frac{32}{N}$  for the Möbius strip to account for its single boundary of doubled length compared to individual boundary components of the cylinder [41]. The “+” sign is for gauge groups  $USp(N)$  and “−” for  $SO(N)$ .

Hence, for a gauge group  $SO(32)$ , the constant term in the  $q$ -expansion w.r.t.  $q_C$  which would give rise to a UV divergence upon integration over  $t \in \mathbb{R}_+$  cancels between the cylinder and the Möbius strip [41].

The double-trace contributions, on the other hand, stem entirely from the cylinder topology with punctures on two different boundaries – *non-planar* cylinder diagrams. We will see that the integrals over the punctures boil down to teMZVs with purely imaginary modular parameter and twists  $b \in \{0, \frac{\tau_C}{2}\} = \{0, \frac{it}{2}\}$ .

In the planar case mentioned above the link between eMZVs and the worldsheet integral over the cylinder boundary was established as follows: Punctures on the same boundary enter through the genus-one Green function at real arguments which can be written as an integral over the weighting function  $f^{(1)}(x)$  with real argument  $x \in (0, 1)$  [28]. Consequently, we will show that the new ingredient in the case of the non-planar cylinder, i.e. the genus-one Green function for two insertions on different boundaries, is related to integrals over  $f^{(1)}(x - \tau/2)$ . Hence, the non-planar contributions may be expressed as iterated integrals on  $E_\tau^\times \setminus \{\tau/2\}$  described above.

#### 4.1 The four-point integrals

We will illustrate the emergence of teMZVs through the non-planar contribution to the four-point one-loop amplitude of the open superstring. Its dependence on the external polarizations enters through a prefactor  $K$  universal to all worldsheet topologies [42] and is irrelevant to the subsequent discussion. Then, setting  $q_C = q$  and  $q_M = -q$  as discussed above, the complete expression for the one-loop open-string four-point amplitude reads [41]

$$\mathcal{A}_4^{1\text{-loop}} = K \int_0^1 \frac{dq}{q} \left\{ \text{Tr}(t^1 t^2 t^3 t^4) [N I_{1234}(q) - 32 I_{1234}(-q)] \right. \\ \left. + \text{Tr}(t^1 t^2) \text{Tr}(t^3 t^4) I_{12|34}(q) + \text{cyc}(2, 3, 4) \right\}, \quad (4.1)$$

where  $t^a$  are traceless  $SO(N)$  generators and the traces are taken in the fundamental representation of the gauge group. The accompanying integrals are given by<sup>7</sup>

$$I_{1234}(q) = \int_0^1 dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \delta(x_1) \prod_{i<j}^4 \exp \left[ \frac{1}{2} s_{ij} G(x_{ij}, \tau) \right], \quad (4.2)$$

$$I_{12|34}(q) = \int_0^1 dx_4 \int_0^1 dx_3 \int_0^1 dx_2 \int_0^1 dx_1 \delta(x_1) \\ \times \exp \left[ \frac{1}{2} s_{12} G(x_{12}, \tau) + \frac{1}{2} s_{34} G(x_{34}, \tau) + \frac{1}{2} \sum_{\substack{i=1,2 \\ j=3,4}} s_{ij} G(x_{ij} - \tau/2, \tau) \right], \quad (4.3)$$

where the insertion points  $z_{1,2} = x_{1,2}$  and  $z_{3,4} = x_{3,4} + \tau/2$  of the vertex operators are parametrized by real integration variables  $x_i$  with  $x_{ij} = x_i - x_j$ , see figure 6. Translation invariance on a genus-one surface has been used to fix  $x_1 = 0$  through the above delta function. The genus-one Green

<sup>7</sup>Note that  $I_{1234}(q)$  was denoted by  $I_{4\text{pt}}(1, 2, 3, 4)$  in ref. [28] and that  $I_{12|34}(q)$  is defined with a factor of two in comparison to the integral  $h(s, u)$  in ref. [29] because we do not impose  $x_3 < x_4$  as done in the latter reference.

functions [30]

$$G(z, \tau) = \log \left| \frac{\theta_1(z, \tau)}{\theta_1(0, \tau)} \right|^2 - \frac{2\pi}{\text{Im}(\tau)} \text{Im}(z)^2 \quad (4.4)$$

depend on the differences of punctures  $z_i$  and their second argument  $\tau$  will often be suppressed. Considering the parametrization of the cylinder visualized in figure 6, vertex insertions on different boundaries give rise to arguments  $x_{ij} - \tau/2$  as for instance seen in the contribution  $G(x_{13} - \tau/2, \tau)$  to the exponent of eq. (4.3).

Given the definition of and relations between the dimensionless Mandelstam invariants,

$$s_{ij} = \alpha'(k_i + k_j)^2, \quad s_{34} = s_{12}, \quad s_{14} = s_{23}, \quad s_{13} = s_{24} = -s_{12} - s_{23}, \quad (4.5)$$

the integrands of eqs. (4.2) and (4.3) are unchanged if the Green function eq. (4.4) is shifted by a  $z$ -independent function. This feature will be made use of in the following subsections.

Configurations with three punctures on the same boundary lead to color factors such as  $\text{Tr}(t^1 t^2 t^3) \text{Tr}(t^4)$  which vanish for traceless  $SO(N)$  generators considered in eq. (4.1). Nevertheless, the accompanying integral

$$I_{123|4}(q) = \int_0^1 dx_4 \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^1 dx_1 \delta(x_1) \quad (4.6)$$

$$\times \exp \left[ \frac{1}{2} \sum_{1 \leq i < j}^3 s_{ij} G(x_{ij}, \tau) + \frac{1}{2} \sum_{j=1}^3 s_{j4} G(x_{j4} - \tau/2, \tau) \right]$$

plays an important role for one-loop monodromy relations [43, 29]. It will be demonstrated in appendix H that eq. (4.6) may be expanded in terms of teMZVs using the same techniques as will be applied to the integral  $I_{12|34}(q)$  in eq. (4.3) along with two punctures on each boundary. For both non-planar integrals  $I_{12|34}(q)$  and  $I_{123|4}(q)$ , our results up to and including the order of  $s_{ij}^3$  can be simplified to ultimately yield combinations of eMZVs, i.e. all of their twisted counterparts are found to drop out at the orders considered.

#### 4.1.1 Analytic versus non-analytic momentum dependence

The one-loop four-point amplitude is a non-analytic function of the Mandelstam invariants eq. (4.5): From the integration over  $q$  in eq. (4.1), the region with  $t \rightarrow \infty$  or  $q = e^{-2\pi t} \rightarrow 0$  leads to branch cuts well-known from the Feynman integrals in the field-theory limit [42]. Moreover, the non-planar contribution  $I_{12|34}(q)$  additionally exhibits kinematic poles in  $s_{12}$  reflecting the exchange of massless closed-string states between the cylinder boundaries [44]. Since both the poles and the branch cuts stem from the integration over  $q$ , the integrals eqs. (4.2) and (4.3) over the punctures by themselves do not reflect the singularity structure of the one-loop amplitude.

At fixed values of  $q$  it is possible to separate the overall amplitude into analytic and non-analytic parts. For the four-point closed-string one-loop amplitude, a careful procedure to isolate the logarithmic dependence on  $s_{ij}$  has been developed in ref. [45]: this method allows for a focused study of the analytic sector where modular graph functions take the role of eMZVs [46, 47].

While the integrals in the expansion of eq. (4.2) have been performed at fixed value of  $q$ , integrating for instance eq. (4.7) below and its counterpart from the Mœbius strip over  $q$  introduces divergences, also for the gauge group  $SO(32)$ . The choice of regularization scheme for these divergences (see [29] for an example at the first subleading order in  $\alpha'$ ) reflects a particular way of splitting the analytic and non-analytic parts of the final expression for the

amplitude after integrating over  $q$ .

### 4.1.2 The low-energy expansion in the single-trace sector

In the following, we will study the analytic part of the non-planar one-loop four-point amplitude by Taylor-expansion of eq. (4.3) in  $s_{ij}$  and thereby in  $\alpha'$ , probing the low-energy behaviour. The analogous low-energy expansion for the single-trace integral eq. (4.2) has been performed in [28],

$$I_{1234}(q) = \omega(0, 0, 0) - 2\omega(0, 1, 0, 0)(s_{12} + s_{23}) + 2\omega(0, 1, 1, 0, 0)(s_{12}^2 + s_{23}^2) \quad (4.7)$$

$$- 2\omega(0, 1, 0, 1, 0)s_{12}s_{23} + \beta_5(s_{12}^3 + 2s_{12}^2s_{23} + 2s_{12}s_{23}^2 + s_{23}^3) + \beta_{2,3}s_{12}s_{23}(s_{12} + s_{23}) + \mathcal{O}(\alpha'^4)$$

with the following combinations of eMZVs at order  $\alpha'^3$ :

$$\beta_5 = \frac{4}{3} [\omega(0, 0, 1, 0, 0, 2) + \omega(0, 1, 1, 0, 1, 0) - \omega(2, 0, 1, 0, 0, 0) - \zeta_2 \omega(0, 1, 0, 0)] \quad (4.8)$$

$$\beta_{2,3} = \frac{\zeta_3}{12} + \frac{8\zeta_2}{3} \omega(0, 1, 0, 0) - \frac{5}{18} \omega(0, 3, 0, 0) \quad (4.9)$$

It was explained in the reference that the dependence of the single-trace integral eq. (4.7) on  $q$  is captured by eMZVs at any order in  $\alpha'$ . Note that the contributions of the planar cylinder and the Möbius strip to eq. (4.1) are obtained by integrating eq. (4.7) over arguments  $q \rightarrow e^{-2\pi t}$  and  $q \rightarrow -e^{-2\pi t}$ , respectively, with  $t \in \mathbb{R}_+$ .

In analogy with eq. (4.7), we will determine the  $\alpha'$ -expansion of the non-planar integral eq. (4.3) in the framework of teMZVs. Since the main emphasis of this article is to exemplify the use of teMZVs in the calculation of non-planar one-loop amplitudes, a detailed analysis of the singularity structure after integration over  $q$  is left for the future.

## 4.2 The genus-one Green function as elliptic iterated integral

The link between open-string amplitudes and the framework of elliptic iterated integrals is the (holomorphic) derivative

$$\partial_z G(z, \tau) = f^{(1)}(z, \tau), \quad z \in \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau), \quad (4.10)$$

of the genus-one Green function eq. (4.4) in the integrals eqs. (4.2) and (4.3). It allows to rewrite the exponent in the non-planar integral eq. (4.3) into the form

$$\begin{aligned} & \frac{1}{2}s_{12}G(x_{12}, \tau) + \frac{1}{2}s_{34}G(x_{34}, \tau) + \sum_{\substack{i=1,2 \\ j=3,4}} \frac{1}{2}s_{ij}G(x_{ij} - \tau/2, \tau) \\ & = s_{12}P(x_{12}) + s_{34}P(x_{34}) + \sum_{\substack{i=1,2 \\ j=3,4}} s_{ij}Q(x_{ij}), \end{aligned} \quad (4.11)$$

where the entire dependence on the real parts<sup>8</sup>  $x_i \in \mathbb{R}$  of the punctures  $z_{1,2} = x_{1,2}$  and  $z_{3,4} = x_{3,4} + \tau/2$  is captured via elliptic iterated integrals eq. (2.1):

$$P(x) = \int_0^x dy f^{(1)}(y) = \Gamma\left(\frac{1}{0}; x\right) \quad (4.12)$$

---

<sup>8</sup>Restricting the first argument  $G(x, \tau)$  to be real (as appropriate for our parametrization of the cylinder) leads to a relative factor of two between  $\partial_x G(x, \tau) = 2f^{(1)}(x, \tau)$ ,  $x \in \mathbb{R}$  and eq. (4.10). This factor of two has been neglected in early versions of [28], and it is not altered by a complex shift  $\partial_x G(x - \tau/2, \tau) = 2f^{(1)}(x - \tau/2, \tau)$ ,  $x \in \mathbb{R}$ .

$$Q(x) = c(q) + \int_0^x dy f^{(1)}(y - \tau/2) = c(q) + \Gamma\left(\frac{1}{\tau/2}; x\right). \quad (4.13)$$

The appearance of the  $x_i$ -independent quantity

$$c(q) = -\frac{i\pi}{2} - \frac{1}{8} \log(q) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{q^m}{1-q^m} - \frac{q^{m/2}}{1-q^m} \right] \quad (4.14)$$

is special to the non-planar cylinder and caused by the different properties of the Green functions  $G(x_{ij})$  and  $G(x_{ij} - \tau/2)$  connecting punctures on the same or different boundaries of the cylinder.

In passing to the right hand side of eq. (4.11) we have used momentum conservation  $\sum_{i < j}^4 s_{ij} = 0$  to discard

$$\frac{1}{2}G(x) - P(x) = \frac{i\pi}{2} - \log(2\pi) = \frac{1}{2}G(x - \tau/2) - Q(x). \quad (4.15)$$

A detailed explanation of the relations above and the underlying regularization will be given in the following two subsections.

#### 4.2.1 $G(x)$ versus $P(x)$

In the case of punctures on the same boundary of the cylinder, the Green functions with arguments  $x \in [0, 1]$  and  $\tau \in i\mathbb{R}_+$  reduces to  $\frac{1}{2}G(x, \tau) = \log \frac{\theta_1(x, \tau)}{\theta_1'(0, \tau)}$ . Up to an additive constant, this expression can be recovered by the following integration with a regulator  $\varepsilon > 0$  in the lower limit:

$$\begin{aligned} \int_{\varepsilon}^x dy f^{(1)}(y) &= \log(\theta_1(x)) - \log(\theta_1(\varepsilon)) \\ &= \log(\theta_1(x)) - \log(\theta_1'(0)) - \log(\varepsilon) + \mathcal{O}(\varepsilon) \\ &= \frac{1}{2}G(x) - \log(-2\pi i \varepsilon) + \log(-2\pi i) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.16)$$

The regularization scheme for the limit  $\varepsilon \rightarrow 0$  has to be chosen consistently with the treatment of divergent eMZVs: Following the conventions of [28], the regularized value of an eMZV is defined to be the constant term in an expansion<sup>9</sup> w.r.t.  $\log(-2\pi i \varepsilon)$ , and we choose the principal branch of the logarithm, such that  $\log(-i) = -\frac{i\pi}{2}$ . Hence,  $\log(-2\pi i \varepsilon)$  is formally set to zero in eq. (4.16), and we obtain

$$P(x) = \lim_{\varepsilon \rightarrow 0} \text{Reg} \int_{\varepsilon}^x dy f^{(1)}(y) = \frac{1}{2}G(x) + \log(-2\pi i) = \frac{1}{2}G(x) - \frac{i\pi}{2} + \log(2\pi), \quad (4.17)$$

reproducing the first equality in eq. (4.15).

#### 4.2.2 $G(x - \tau/2)$ versus $Q(x)$

Pairs of punctures on different boundaries of the cylinder lead to arguments  $x - \tau/2$  of the Green function eq. (4.4), where  $x \in [0, 1]$ . In these cases, the Green function may be related to the integral

$$\int_0^x dy f^{(1)}(y - \tau/2) = \log(\theta_1(x - \tau/2)) - i\pi x - \log(\theta_1(-\tau/2))$$

<sup>9</sup>The expansion in  $\log(-2\pi i \varepsilon)$  will ensure that the constant terms of eMZVs are  $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combinations of eMZVs, as opposed to  $\mathbb{Q}[(2\pi i)^{-1}, \log(2\pi)]$ -linear combinations of eMZVs (as in [8], Proposition 2.8).

$$\begin{aligned}
&= \log |\theta_1(x - \tau/2)| - \frac{i\pi}{2} - \log(-\theta_1(\tau/2)) \tag{4.18} \\
&= \frac{1}{2}G(x - \tau/2) + \log \theta'_1(0) - \frac{1}{8} \log(q) - \frac{i\pi}{2} - \log(-iq^{-1/8}\theta_4(0)) \\
&= \frac{1}{2}G(x - \tau/2) + \log\left(\frac{\theta'_1(0)}{\theta_4(0)}\right) = \frac{1}{2}G(x - \tau/2) - \frac{1}{2}G(\tau/2),
\end{aligned}$$

without any need for regularization. In passing to the second line, we have chosen the principal branch of the logarithm to relate  $\log(\theta_1(x - \tau/2)) - i\pi x = \log |\theta_1(x - \tau/2)| - \frac{i\pi}{2}$ , see appendix A for our conventions and several identities for the theta functions. In the next step, we have introduced the Green function via  $\log |\theta_1(x - \tau/2)| = \frac{1}{2}G(x - \tau/2) + \log \theta'_1(0) - \frac{1}{8} \log(q)$  and used the theta-function identity  $\theta_1(\tau/2) = iq^{-1/8}\theta_4(0)$ , cf. eq. (A.3).

Using the infinite-product representations eq. (A.1) of the theta functions, the result of eq. (4.18) can be rewritten as [48]

$$\begin{aligned}
&\int_0^x dy f^{(1)}(y - \tau/2) - \frac{1}{2}G(x - \tau/2) = \log\left(\frac{\theta'_1(0)}{\theta_4(0)}\right) \\
&= \log(2\pi q^{1/8}) + 2 \sum_{n=1}^{\infty} \left[ \log(1 - q^n) - \log(1 - q^{n-1/2}) \right] \\
&= \log(2\pi q^{1/8}) + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{q^{m/2}}{1 - q^m} - \frac{q^m}{1 - q^m} \right] \tag{4.19} \\
&= -\frac{i\pi}{2} + \log(2\pi) - c(q).
\end{aligned}$$

In passing to the third line, we have rearranged the infinite sums<sup>10</sup> to identify the quantity  $c(q)$  in eq. (4.14). In this way, one arrives at

$$Q(x) = c(q) + \int_0^x dy f^{(1)}(y - \tau/2) = \frac{1}{2}G(x - \tau/2) - \frac{i\pi}{2} + \log(2\pi), \tag{4.20}$$

and the second equality in eq. (4.15) is confirmed.

### 4.3 Non-planar contribution to the four-point amplitude

Using the identities discussed in the previous subsections, the integral eq. (4.3) relevant to the non-planar cylinder can be written as

$$I_{12|34}(q) = \int_{12}^{34} \exp \left[ s_{12}P(x_{12}) + s_{34}P(x_{34}) + \sum_{\substack{i=1,2 \\ j=3,4}} s_{ij}Q(x_{ij}) \right], \tag{4.21}$$

where  $P(x)$  and  $Q(x)$  are given by the elliptic iterated integrals eqs. (4.12) and (4.13) and we have introduced the following shorthand for the integration measure:

$$\int_{12}^{34} = \int_0^1 dx_4 \int_0^1 dx_3 \int_0^1 dx_2 \int_0^1 dx_1 \delta(x_1). \tag{4.22}$$

<sup>10</sup>Note the following useful intermediate expressions whose sum is denoted by  $-\frac{1}{2}Q_3$  in ref. [29]

$$\sum_{n=1}^{\infty} \log(1 - q^n) = - \sum_{m,n=1}^{\infty} \frac{q^{mn}}{m} = - \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1 - q^m}, \quad - \sum_{n=1}^{\infty} \log(1 - q^{n-1/2}) = \sum_{m,n=1}^{\infty} \frac{q^{mn-m/2}}{m} = \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^{m/2}}{1 - q^m}.$$

We will now investigate the Taylor expansion of eq. (4.21) in the dimensionless Mandelstam invariants eq. (4.5) and thus in  $\alpha'$  by expanding the exponentials  $e^{s_{ij}P(x_{ij})}$  and  $e^{s_{ij}Q(x_{ij})}$  in the integrand:

$$I_{12|34}(q) = e^{-2s_{12}c(q)} \int_{12}^{34} \sum_{n_{ij}=0}^{\infty} \frac{(s_{12}\Gamma(\frac{1}{0}; x_{12}))^{n_{12}} (s_{34}\Gamma(\frac{1}{0}; x_{34}))^{n_{34}}}{n_{12}! n_{34}!} \prod_{\substack{i=1,2 \\ j=3,4}} \frac{(s_{ij}\Gamma(\frac{1}{\tau/2}; x_{ij}))^{n_{ij}}}{n_{ij}!} . \quad (4.23)$$

The punctures  $x_i$  only enter via elliptic iterated integrals, and Fay relations among the weighting functions  $f^{(n)}$  guarantee that individual integration over  $x_i$  can always be performed in terms of further elliptic iterated integrals, see appendix I. Hence, each order in  $\alpha'$  can be expressed in terms of teMZVs and the quantity  $c(q)$  in eq. (4.14), where the latter will also be related to teMZVs in eq. (4.49). Moreover, the explicit results up to the third order can in fact be expressed in terms of eMZVs only, without the need to involve their twisted counterparts. Whether this behaviour persists at any order in  $\alpha'$  will be discussed in subsection 4.3.5.

#### 4.3.1 Structure of the leading orders $\alpha'^{\leq 3}$

As a first step towards an expansion in terms of teMZVs, we classify the inequivalent integrals w.r.t. the cycle structure of  $\text{Tr}(t^1 t^2) \text{Tr}(t^3 t^4)$  which occur at the orders  $\alpha'^{\leq 3}$  of eq. (4.23): We will use the shorthand  $P_{ij} = P(x_{ij})$  and  $Q_{ij} = Q(x_{ij})$  for the two integrals at order  $\alpha'^1$ ,

$$d_1^1 = \int_{12}^{34} P_{12} , \quad d_2^1 = \int_{12}^{34} Q_{13} , \quad (4.24)$$

the six integrals at order  $\alpha'^2$ ,

$$\begin{aligned} d_1^2 &= \frac{1}{2} \int_{12}^{34} P_{12}^2 , & d_2^2 &= \int_{12}^{34} P_{12} Q_{13} , & d_3^2 &= \int_{12}^{34} Q_{13} Q_{14} \\ d_4^2 &= \frac{1}{2} \int_{12}^{34} Q_{13}^2 , & d_5^2 &= \int_{12}^{34} P_{12} P_{34} , & d_6^2 &= \int_{12}^{34} Q_{13} Q_{24} , \end{aligned} \quad (4.25)$$

and the twelve integrals at order  $\alpha'^3$ :

$$\begin{aligned} d_1^3 &= \frac{1}{6} \int_{12}^{34} P_{12}^3 , & d_2^3 &= \frac{1}{2} \int_{12}^{34} P_{12}^2 P_{34} , & d_3^3 &= \int_{12}^{34} P_{12} Q_{13} Q_{24} \\ d_4^3 &= \frac{1}{6} \int_{12}^{34} Q_{13}^3 , & d_5^3 &= \frac{1}{2} \int_{12}^{34} Q_{13}^2 Q_{14} , & d_6^3 &= \int_{12}^{34} P_{12} Q_{13} Q_{14} \\ d_7^3 &= \frac{1}{2} \int_{12}^{34} P_{12}^2 Q_{13} , & d_8^3 &= \frac{1}{2} \int_{12}^{34} Q_{13}^2 Q_{24} , & d_9^3 &= \int_{12}^{34} P_{34} Q_{13} Q_{14} \\ d_{10}^3 &= \frac{1}{2} \int_{12}^{34} P_{12} Q_{13}^2 , & d_{11}^3 &= \int_{12}^{34} P_{12} P_{34} Q_{13} , & d_{12}^3 &= \int_{12}^{34} Q_{13} Q_{14} Q_{23} . \end{aligned} \quad (4.26)$$

In fact, some of the above  $d_i^j$  can be related via cyclicity and reflection properties of five-point generalization

$$\int_{123}^{45} = \int_0^1 dx_5 \int_0^1 dx_4 \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \delta(x_1) \quad (4.27)$$

of the integration measure eq. (4.22): Using  $\partial_i Q_{ij} = -\partial_j Q_{ij}$  and the vanishing of  $\int_0^1 dx_j \partial_j Q_{ij}$  by double periodicity of the Green function, we find one relation

$$0 = \int_{123}^{45} Q_{25} \partial Q_{14} = \int_{123}^{45} Q_{25} (Q_{24} - Q_{34}) \Rightarrow d_5^2 = d_6^2 \quad (4.28)$$

among the integrals eq. (4.25) at order  $\alpha'^2$ . The same methods yield the two relations

$$\left. \begin{aligned} 0 &= \int_{123}^{45} Q_{25}^2 \partial Q_{14} \\ 0 &= \int_{123}^{45} P_{23} Q_{25} \partial Q_{14} \end{aligned} \right\} \Rightarrow \begin{cases} d_7^3 = d_6^3 \\ d_9^3 = d_{10}^3 \end{cases} \quad (4.29)$$

among the integrals eq. (4.26) at order  $\alpha'^3$ .

### 4.3.2 teMZVs at orders $\alpha'^{\leq 3}$

As a next step, we exploit the representations eqs. (4.12) and (4.13) of  $P_{ij}$  and  $Q_{ij}$  to express the above  $d_j^i$  in terms of teMZVs: The two instances at order  $\alpha'$  yield

$$d_1^1 = \int_0^1 dx_2 \int_0^{x_2} dy f^{(1)}(y) = \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right) \quad (4.30)$$

$$\begin{aligned} d_2^1 &= c(q) + \int_0^1 dx_3 \int_0^{x_3} dy f^{(1)}(y - \tau/2) \\ &= c(q) + \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right), \end{aligned} \quad (4.31)$$

and we can similarly convert the five independent integrals at order  $\alpha'^2$  to

$$d_1^2 = \omega \left( \begin{matrix} 1, 1, 0 \\ 0, 0, 0 \end{matrix} \right) \quad (4.32)$$

$$d_2^2 = \frac{1}{2} c(q)^2 + c(q) \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) + \omega \left( \begin{matrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{matrix} \right) \quad (4.33)$$

$$d_3^2 = d_1^1 d_2^1 = \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right) \left( c(q) + \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) \right) \quad (4.34)$$

$$d_4^2 = (d_1^1)^2 = \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right)^2 \quad (4.35)$$

$$d_5^2 = (d_2^1)^2 = \left( c(q) + \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) \right)^2 \quad (4.36)$$

and the ten independent integrals at order  $\alpha'^3$  to

$$d_1^3 = \omega \left( \begin{matrix} 1, 1, 1, 0 \\ 0, 0, 0, 0 \end{matrix} \right) \quad (4.37)$$

$$d_2^3 = \frac{1}{6} c(q)^3 + \frac{1}{2} c(q)^2 \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) + c(q) \omega \left( \begin{matrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{matrix} \right) + \omega \left( \begin{matrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{matrix} \right) \quad (4.38)$$

$$d_3^3 = d_1^2 d_2^1 = \omega \left( \begin{matrix} 1, 1, 0 \\ 0, 0, 0 \end{matrix} \right) \left( c(q) + \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) \right) \quad (4.39)$$

$$d_4^3 = d_1^1 d_2^2 = \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right) \left( \frac{1}{2} c(q)^2 + c(q) \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) + \omega \left( \begin{matrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{matrix} \right) \right) \quad (4.40)$$

$$d_5^3 = d_1^1 d_1^2 = \omega \left( \begin{matrix} 1, 1, 0 \\ 0, 0, 0 \end{matrix} \right) \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right) \quad (4.41)$$



$$d_7^3 = d_2^1 d_2^2 = \left( c(q) + \omega \left( \frac{1}{\tau/2}, 0 \right) \right) \left( \frac{1}{2} c(q)^2 + c(q) \omega \left( \frac{1}{\tau/2}, 0 \right) + \omega \left( \frac{1}{\tau/2}, \frac{1}{\tau/2}, 0 \right) \right) \quad (4.42)$$

$$d_8^3 = d_1^1 d_3^2 = (d_1^1)^2 d_2^1 = \omega \left( \frac{1}{0}, 0 \right)^2 \left( c(q) + \omega \left( \frac{1}{\tau/2}, 0 \right) \right) \quad (4.43)$$

$$d_9^3 = d_2^1 d_3^2 = d_1^1 (d_2^1)^2 = \omega \left( \frac{1}{0}, 0 \right) \left( c(q) + \omega \left( \frac{1}{\tau/2}, 0 \right) \right)^2 \quad (4.44)$$

$$d_{11}^3 = \omega \left( \frac{1}{0}, 0 \right) \left( c(q) + \omega \left( \frac{1}{\tau/2}, 0 \right) \right)^2 + \frac{1}{3} \omega(0, 3, 0, 0) \quad (4.45)$$

$$d_{12}^3 = d_2^1 d_5^2 = (d_2^1)^3 = \left( c(q) + \omega \left( \frac{1}{\tau/2}, 0 \right) \right)^3 . \quad (4.46)$$

Note that  $d_6^2, d_6^3$  and  $d_{10}^3$  are determined by eqs. (4.28) and (4.29), and the derivation of the more involved integral  $d_{11}^3$  is detailed in appendix I.

### 4.3.3 Assembling the orders $\alpha'^{\leq 3}$

Once we apply momentum conservation eq. (4.5) to the integral eq. (4.23), its leading orders in the  $\alpha'$ -expansion simplify to

$$\begin{aligned} I_{12|34}(q) = & 1 + 2s_{12} (d_1^1 - d_2^1) + s_{12}^2 (2d_1^2 + 2d_2^2 - 4d_3^2 + d_4^2 + d_5^2) + s_{13}s_{23} (2d_5^2 - 4d_2^2) \\ & + 2s_{12}^3 (d_1^3 - d_2^3 - 2d_3^3 + 2d_4^3 + d_5^3 - d_7^3 - d_8^3 + d_9^3) \\ & - 2s_{12}s_{13}s_{23} (d_{12}^3 - 3d_2^3 + 4d_4^3 - d_7^3 - 2d_{11}^3) + \mathcal{O}(\alpha'^4) . \end{aligned} \quad (4.47)$$

The  $q$ -dependence of the relevant teMZVs can be determined by solving the initial-value problem set up in section 3, yielding for instance

$$\omega \left( \frac{1}{0}, 0 \right) = -\frac{i\pi}{2} + 2 \sum_{n,m=1}^{\infty} \frac{q^{mn}}{m} , \quad \omega \left( \frac{1}{\tau/2}, 0 \right) = 2 \sum_{n,m=1}^{\infty} \frac{q^{m(n-1/2)}}{m} , \quad (4.48)$$

and further examples can be found in appendix G.4. By comparing the  $q$ -expansion with the expression eq. (4.14) for  $c(q)$ , we infer that

$$c(q) = \omega \left( \frac{1}{0}, 0 \right) - \omega \left( \frac{1}{\tau/2}, 0 \right) - \frac{1}{8} \log(q) , \quad (4.49)$$

which identifies the prefactor  $e^{-2s_{12}c(q)}$  in eq. (4.23) as  $q^{s_{12}/4}$  multiplied by a series in teMZVs  $\omega \left( \frac{1}{0}, 0 \right)$  and  $\omega \left( \frac{1}{\tau/2}, 0 \right)$ . Moreover, eq. (4.49) simplifies the order  $\alpha'^1$  of eq. (4.47) to

$$I_{12|34}(q) \Big|_{s_{12}} = 2(d_1^1 - d_2^1) = 2 \left\{ \omega \left( \frac{1}{0}, 0 \right) - \omega \left( \frac{1}{\tau/2}, 0 \right) - c(q) \right\} = \frac{1}{4} \log(q) , \quad (4.50)$$

in agreement with [29]. Also at higher orders of eq. (4.47), we convert any appearance of  $c(q)$  into  $-\frac{1}{8} \log(q)$  via eq. (4.49) and obtain

$$\begin{aligned} I_{12|34}(q) \Big|_{s_{12}^2} &= \frac{(\log(q)^2)}{32} + 2\omega \left( \frac{1}{0}, \frac{1}{0}, 0 \right) - \omega \left( \frac{1}{0}, 0 \right)^2 + 2\omega \left( \frac{1}{\tau/2}, \frac{1}{\tau/2}, 0 \right) - \omega \left( \frac{1}{\tau/2}, 0 \right)^2 \\ &= \frac{(\log(q)^2)}{32} + \frac{7\zeta_2}{6} + 2\omega(0, 0, 2) \end{aligned} \quad (4.51)$$

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = 2\omega \left( \frac{1}{\tau/2}, 0 \right)^2 - 4\omega \left( \frac{1}{\tau/2}, \frac{1}{\tau/2}, 0 \right) = -2\omega(0, 0, 2) - \frac{2\zeta_2}{3} \quad (4.52)$$

via teMZV relations eqs. (G.11) and (G.12) as well as

$$\begin{aligned}
I_{12|34}(q) \Big|_{s_{12}^3} &= \frac{1}{3!} \left( \frac{\log(q)}{4} \right)^3 + \frac{1}{4} \log(q) \left( \frac{7\zeta_2}{6} + 2\omega(0, 0, 2) \right) \\
&\quad + \frac{2}{3} \omega \left( \begin{smallmatrix} 1, 0 \\ 0, 0 \end{smallmatrix} \right)^3 - 2\omega \left( \begin{smallmatrix} 1, 0 \\ 0, 0 \end{smallmatrix} \right) \omega \left( \begin{smallmatrix} 1, 1, 0 \\ 0, 0, 0 \end{smallmatrix} \right) + 2\omega \left( \begin{smallmatrix} 1, 1, 1, 0 \\ 0, 0, 0, 0 \end{smallmatrix} \right) \\
&\quad - \frac{2}{3} \omega \left( \begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix} \right)^3 + 2\omega \left( \begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix} \right) \omega \left( \begin{smallmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{smallmatrix} \right) - 2\omega \left( \begin{smallmatrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{smallmatrix} \right) \\
&= \frac{1}{3!} \left( \frac{\log(q)}{4} \right)^3 + \frac{1}{4} \log(q) \left( \frac{7\zeta_2}{6} + 2\omega(0, 0, 2) \right) - 4\zeta_2 \omega(0, 1, 0, 0)
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
I_{12|34}(q) \Big|_{s_{12}s_{23}s_{13}} &= \frac{1}{4} \log(q) \left( -2\omega(0, 0, 2) - \frac{2\zeta_2}{3} \right) + \frac{4}{3} \omega(0, 3, 0, 0) \\
&\quad + 2\omega \left( \begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix} \right)^3 - 6\omega \left( \begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix} \right) \omega \left( \begin{smallmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{smallmatrix} \right) + 6\omega \left( \begin{smallmatrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{smallmatrix} \right) \\
&= \frac{1}{4} \log(q) \left( -2\omega(0, 0, 2) - \frac{2\zeta_2}{3} \right) + \frac{5}{3} \omega(0, 3, 0, 0) + 4\zeta_2 \omega(0, 1, 0, 0) - \frac{1}{2} \zeta_3
\end{aligned} \tag{4.54}$$

via teMZV relations eqs. (G.13) and (G.14). As will be detailed shortly, the cancellation of teMZVs with nonzero twist manifests the absence of unphysical poles in the string amplitude after integration over  $q$ .

#### 4.3.4 Summary of the orders $\alpha'^{\leq 3}$

As is clear by comparing eq. (4.23) with eq. (4.49), any appearance of  $\log(q)$  can be traced back to the expansion of  $q^{s_{12}/4}$ . Hence, the above orders  $\alpha'^{\leq 3}$  can be summarized as

$$\begin{aligned}
I_{12|34}(q) &= q^{s_{12}/4} \left\{ 1 + s_{12}^2 \left( \frac{7\zeta_2}{6} + 2\omega(0, 0, 2) \right) - 2s_{13}s_{23} \left( \frac{\zeta_2}{3} + \omega(0, 0, 2) \right) \right. \\
&\quad \left. - 4\zeta_2 \omega(0, 1, 0, 0) s_{12}^3 + s_{12}s_{13}s_{23} \left( \frac{5}{3} \omega(0, 3, 0, 0) + 4\zeta_2 \omega(0, 1, 0, 0) - \frac{1}{2} \zeta_3 \right) + \mathcal{O}(\alpha'^4) \right\},
\end{aligned} \tag{4.55}$$

and the integral  $I_{123|4}(q)$  in eq. (4.6) admits a similar low-energy expansion in terms of eMZVs only, see eq. (H.26). While the  $q$ -expansions of the above eMZVs are listed in appendix G.4, their constant terms yield

$$I_{12|34}(q) = q^{s_{12}/4} \left\{ 1 + \frac{1}{2} \zeta_2 s_{12}^2 - \frac{1}{2} \zeta_3 s_{12}^3 + \mathcal{O}(q, \alpha'^4) \right\}, \tag{4.56}$$

in agreement with the all-order  $\alpha'$  expression given in [29]

$$I_{12|34}(q) = \frac{2^{2s_{12}} q^{s_{12}/4}}{\pi} \left( \frac{\Gamma(\frac{1}{2} + \frac{s_{12}}{2})}{\Gamma(1 + \frac{s_{12}}{2})} \right)^2 + \mathcal{O}(q). \tag{4.57}$$

With the above strategy and the techniques exemplified in appendix I, there is no limitation in obtaining higher orders of the  $\alpha'$ -expansion eq. (4.55). Similarly, non-planar open-string amplitudes with five and more external legs can be expanded along the same lines<sup>11</sup> because their integrands depend on the punctures and  $\tau$  through products of  $f^{(n)}$  and Eisenstein series [49, 50, 28].

<sup>11</sup>See section 5.1 of [28] for the analogous expansion of the planar five-point one-loop amplitude in terms of eMZVs.

### 4.3.5 Higher orders in $\alpha'$ and eMZVs

The exclusive appearance of eMZVs at the leading orders of the  $\alpha'$ -expansion eq. (4.55) illustrates a general property of the non-planar integral  $I_{12|34}(q)$ : apart from the prefactor  $q^{s_{12}/4}$ , its  $q$ -expansion comprises integer powers only. This property is shared by eMZVs but not by typical teMZVs involving twists  $\tau/2$ .

The absence of half-odd integer powers  $q^{n+1/2}$  with  $n \in \mathbb{N}_0$  can be explained from physical constraints on the pole structure of the open-string amplitude: Performing the  $q$ -integration in the amplitude prescription eq. (4.1) over expansions of the schematic form

$$I_{12|34}(q) = q^{s_{12}/4} \sum_{n=0}^{\infty} (a_n q^n + c_n q^{n+1/2}) \quad (4.58)$$

yields kinematic poles at  $s_{12} = -4n$  in case of integer powers of  $q$  and at  $s_{12} = -4n - 2$  in case of half-odd integer powers, respectively, with  $n \in \mathbb{N}_0$ :

$$\int_0^1 \frac{dq}{q} I_{12|34}(q) = \sum_{n=0}^{\infty} \left\{ \frac{4a_n}{s_{12} + 4n} + \frac{4c_n}{s_{12} + 4n + 2} \right\}. \quad (4.59)$$

The expansion coefficients  $a_n$  and  $c_n$  are understood to be polynomials in  $s_{ij}$  accompanied by  $\mathbb{Q}[(2\pi i)^{-1}]$ -linear combinations of MZVs.

The singular values of Mandelstam variables  $s_{12} = \alpha'(k_1 + k_2)^2$  in scattering amplitudes correspond to internal masses  $-\alpha' m^2$ . In particular, the poles in eq. (4.59) with residues proportional to  $a_n$  and  $c_n$  signal the exchange of closed-string states with masses  $m^2 = \frac{4n}{\alpha'}$  and  $m^2 = \frac{4n+2}{\alpha'}$ , respectively, with  $n \in \mathbb{N}_0$ . However, the closed-superstring spectrum only comprises masses  $m^2 = \frac{4n}{\alpha'}$ , whereas states with  $m^2 = \frac{4n+2}{\alpha'}$  cannot be found in GSO projected string theories [51, 44, 52]. Hence, the pole structure of  $\frac{4c_n}{s_{12}+4n+2}$  due to half-odd integer powers of  $q$  would signal the propagation of unphysical states and violate unitarity if some of the  $c_n$  were nonzero.

However, it is not immediately clear if an integer-power  $q$ -expansion eq. (4.58) with  $c_n = 0$  is necessarily expressible in terms of eMZVs. Given that  $q^{-s_{12}/4} I_{12|34}(q)$  was argued to comprise teMZVs with twists  $\in \{0, \tau/2\}$ , this leads to the following, purely mathematical question: If a linear combination of teMZVs with twists  $\in \{0, \tau/2\}$  is such that its  $q$ -expansion has integer exponents only, can it be written as a linear combination of eMZVs (i.e. teMZVs with twists  $\in \{0\}$ ) only? We expect that an answer to this question will necessitate a closer study of the decomposition of teMZVs into iterated  $\tau$ -integrals of the functions  $h^{(n)}(b)$  in eq. (3.5), with  $b \in \{0, \tau/2\}$ , which generalizes the decomposition of eMZVs into linear combinations of iterated Eisenstein integrals [10, 34]. In particular, a natural first step would be to prove linear independence of iterated  $\tau$ -integrals comprised of the integrands  $h^{(n)}(b)$  with  $b \in \{0, \tau/2\}$ , which generalizes the main result of [53], and can presumably be proved along similar lines.

In any case, based on the arguments presented in this subsection, we conjecture that all orders in the  $\alpha'$ -expansion eq. (4.55) are furnished by eMZVs.

## 5 Conclusions

In this article, teMZVs have been introduced as iterated integrals on an elliptic curve with multiple punctures on a lattice  $\mathbb{Q} + \mathbb{Q}\tau$ . Our main result is the identification of an initial-value problem satisfied by teMZVs, which expresses them in terms of linear combinations of iterated

$\tau$ -integrals of the weighting functions  $f^{(n)}$  with coefficients given by cyclotomic MZVs.

As an application of teMZVs in physics we have studied one-loop scattering amplitudes of open-string states. In the non-planar sector of the four-point amplitude, the low-energy expansion can be computed by integrals over the two boundaries of a cylinder. A systematic procedure is established, which allows to evaluate these integrals in terms of teMZVs. There is no conceptual bottleneck in extending the procedure to one-loop amplitudes with an arbitrary number of external states. Having calculated the non-planar amplitude up to the third subleading low-energy order, we find that the results can ultimately be simplified to eMZVs.

The results of this article trigger a variety of questions: From a mathematical perspective, the differential equation of teMZVs could serve as a starting point to classify their relations and to understand the underlying algebraic principles. In the untwisted case, a crucial role was played by a certain derivation algebra. We expect that a suitable twisted analogue of the derivation algebra [9] will likewise control the algebraic structure of teMZVs.

In a physics context, the methods of this article allow to compute higher orders in the low-energy expansion of non-planar one-loop open-string amplitudes and to investigate its structure. More interestingly, higher-loop open-string amplitudes should require an extension of elliptic iterated integrals to Riemann surfaces of higher genus and a suitable generalization of teMZVs to accommodate multiple boundaries.

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## Appendix

### A Jacobi theta functions

For later use we note as well the expansion of Jacobi theta functions (and derivatives thereof) as an expansion in the parameter  $q$  [48]

$$\begin{aligned}
\theta_1(z, \tau) &= 2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n)(1 - 2q^n \cos(2\pi z) + q^{2n}) \\
\theta_1'(0, \tau) &= 2\pi q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)^3 \\
\theta_4(z, \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - 2q^{n-1/2} \cos(2\pi z) + q^{2n-1}),
\end{aligned} \tag{A.1}$$

which all turn out to be positive given  $z \in [0, 1]$  and  $q \in [0, 1]$ . Furthermore, a periodicity property useful for calculating eq. (4.18) as well as a relation between  $\theta_1$  and  $\theta_4$  read

$$\theta_1(z + m + n\tau, \tau) = (-1)^{n+m} q^{-n^2/2} e^{-2\pi i n z} \theta_1(z, \tau) \quad (\text{A.2})$$

$$\theta_4(z, \tau) = -i e^{i\pi z} q^{1/8} \theta_1(z + \tau/2, \tau). \quad (\text{A.3})$$

## B Weighting functions

The weighting functions  $f^{(n)}(z, \tau)$  arise as expansion coefficients of the doubly-periodic completion  $\Omega(z, \alpha, \tau)$  of the Eisenstein–Kronecker series  $F(z, \alpha, \tau)$ ,

$$\Omega(z, \alpha, \tau) = \exp\left(2\pi i \alpha \frac{\text{Im}(z)}{\text{Im}(\tau)}\right) F(z, \alpha, \tau) = \sum_{n=0}^{\infty} f^{(n)}(z, \tau) \alpha^{n-1}, \quad (\text{B.1})$$

which, in turn is given by [54, 6]

$$F(z, \alpha, \tau) = \frac{\theta_1'(0, \tau) \theta_1(z + \alpha, \tau)}{\theta_1(z, \tau) \theta_1(\alpha, \tau)}. \quad (\text{B.2})$$

The odd Jacobi theta function  $\theta_1$  is defined in appendix A, and the derivative with respect to the first argument is denoted by a tick. For real  $z$ , expressions eqs. (B.1) and (B.2) agree, and the lowest-order examples of  $f^{(n)}$  are spelt out in eq. (2.2). In fact,  $f^{(1)}$  is the only weighting function with a simple pole on the lattice  $\mathbb{Z} + \mathbb{Z}\tau$ , while all  $f^{(n)}$  with  $n \neq 1$  are non-singular on the entire elliptic curve.

Both the Eisenstein–Kronecker series  $F(z, \alpha, \tau)$  and its doubly periodic completion  $\Omega(z, \alpha, \tau)$  satisfy the Fay identity

$$\begin{aligned} \Omega(z_1, \alpha_1, \tau) \Omega(z_2, \alpha_2, \tau) &= \Omega(z_1, \alpha_1 + \alpha_2, \tau) \Omega(z_2 - z_1, \alpha_2, \tau) \\ &+ \Omega(z_2, \alpha_1 + \alpha_2, \tau) \Omega(z_1 - z_2, \alpha_1, \tau). \end{aligned} \quad (\text{B.3})$$

Furthermore, the Eisenstein–Kronecker series satisfies the mixed heat equation

$$2\pi i \partial_\tau F(z, \alpha, \tau) = \partial_\alpha \partial_z F(z, \alpha, \tau). \quad (\text{B.4})$$

Both the Fay identity eq. (B.3) and the mixed heat equation (B.4) are relevant in the calculations of section 3. Starting from the quasi-periodicity of the Eisenstein–Kronecker series,

$$F(z + 1, \alpha, \tau) = F(z, \alpha, \tau), \quad F(z + \tau, \alpha, \tau) = \exp(-2\pi i \alpha) F(z, \alpha, \tau), \quad (\text{B.5})$$

and its reflection property

$$F(-z, -\alpha, \tau) = -F(z, \alpha, \tau), \quad (\text{B.6})$$

it is straightforward to derive properties eq. (2.3) of the weighting functions  $f^{(n)}$ .

## C A teMZV with proper real twist $b = 1/2$

We illustrate the definition of teMZVs in the case of proper rational twists  $b \in \Lambda_n^\times = \{\frac{1}{n}, \dots, \frac{n-1}{n}\}$ , via an explicit computation of  $\omega\left(\begin{smallmatrix} 0, & 1 \\ 0, & 1/2 \end{smallmatrix}\right)$ . Our starting point is the definition eq. (2.7) of teMZVs

with twists  $b_i \in \{0, 1/2\}$  through the integral

$$\omega \left( \begin{matrix} n_1, n_2, \dots, n_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right) = \lim_{\varepsilon \rightarrow 0} \int_{\alpha_1 \delta_\varepsilon \alpha_2} f^{(n_1)}(z - b_1) dz_1 f^{(n_2)}(z - b_2) dz_2 \dots f^{(n_\ell)}(z - b_\ell) dz_\ell. \quad (\text{C.1})$$

We choose the parametrization of the individual path segments depicted in figure 3 as

$$\begin{aligned} \alpha_1(t) &= \left( \frac{1}{2} - \varepsilon \right) t \\ \delta_\varepsilon(t) &= \frac{1}{2} - \varepsilon \exp(-i\pi t) \\ \alpha_2(t) &= \frac{1}{2} + \varepsilon + \left( \frac{1}{2} - \varepsilon \right) t. \end{aligned} \quad (\text{C.2})$$

Then we may compute the iterated integral using the composition of paths formula, valid for smooth one-forms  $\omega_i = f^{(n_i)}(z - b_i) dz$  (cf. [55], Proposition 2.9)

$$\int_{\alpha\beta} \omega_1 \omega_2 \dots \omega_\ell = \sum_{k=0}^{\ell} \int_{\alpha} \omega_1 \omega_2 \dots \omega_k \int_{\beta} \omega_{k+1} \dots \omega_\ell, \quad (\text{C.3})$$

where the paths  $\alpha, \beta$  are such that  $\alpha(1) = \beta(0)$  and the empty integral is defined to be one.

As the forms  $\omega_i = f^{(n_i)}(z - b_i) dz$  admit an expansion in  $q$  we may treat the  $q^0$  term separately from the rest, assuming the  $q$ -expansion can be exchanged with the integration. Then, as the coefficients of the  $j^{\text{th}}$  power  $q^j$  for  $j \neq 0$  are well defined on the real line, we may exchange the limit  $\varepsilon \rightarrow 0$  with the integration and compute this part of the integral over the much more mundane path  $\gamma(t) = t$ . Specifically, the  $q$  dependent part is given by

$$I_q = -2i(2\pi i) \int_{0 < t_1 < t_2 < 1} dt_1 dt_2 \sum_{n,m=1}^{\infty} q^{mn} \sin \left( 2\pi m \left( t_2 - \frac{1}{2} \right) \right) = -2 \sum_{n,m=1}^{\infty} \frac{(-1)^m q^{mn}}{m}. \quad (\text{C.4})$$

We note that this can be reproduced from the differential equation (3.4) for real twists

$$\begin{aligned} \omega \left( \begin{matrix} 0, 1 \\ 0, 1/2 \end{matrix} \right) &= \omega_0 \left( \begin{matrix} 0, 1 \\ 0, 1/2 \end{matrix} \right) + \int_0^q \frac{d \log(q_1)}{-4\pi^2} \left[ \omega \left( \begin{matrix} 2 \\ 1/2 \end{matrix} \right) - f^{(2)} \left( \frac{1}{2}, q_1 \right) \right] \\ &= \omega_0 \left( \begin{matrix} 0, 1 \\ 0, 1/2 \end{matrix} \right) + \int_0^q \frac{d \log(q_1)}{-4\pi^2} \left[ 8\pi^2 \sum_{m,n=1}^{\infty} (-1)^m q_1^{mn} \right], \end{aligned} \quad (\text{C.5})$$

where the integration constant  $\omega_0 \left( \begin{matrix} 0, 1 \\ 0, 1/2 \end{matrix} \right)$  remains to be determined.

Application of eq. (C.3) for the constant term  $f_0^{(n)}(z - b)$  of the  $q$  expansion of  $f^{(n)}(z - b)$  yields, bearing in mind that  $f^{(0)}(z) = 1$ ,

$$\begin{aligned} I_0 &= \int_{\alpha_1 \delta_\varepsilon \alpha_2} dz_1 f_0^{(1)} \left( z_2 - \frac{1}{2} \right) dz_2 \\ &= \int_{\alpha_1} dz_1 f_0^{(1)} \left( z_2 - \frac{1}{2} \right) dz_2 + \int_{\delta_\varepsilon} dz_1 f_0^{(1)} \left( z_2 - \frac{1}{2} \right) dz_2 + \int_{\alpha_2} dz_1 f_0^{(1)} \left( z_2 - \frac{1}{2} \right) dz_2 \\ &\quad + \int_{\alpha_1} dw \int_{\delta_\varepsilon} f_0^{(1)} \left( z - \frac{1}{2} \right) dz + \int_{\alpha_1} dw \int_{\alpha_2} f_0^{(1)} \left( z - \frac{1}{2} \right) dz + \int_{\delta_\varepsilon} dw \int_{\alpha_2} f_0^{(1)} \left( z - \frac{1}{2} \right) dz. \end{aligned} \quad (\text{C.6})$$

The individual integrals are given by

$$\begin{aligned} \int_{\alpha_1} dz_1 f_0^{(1)}\left(z_2 - \frac{1}{2}\right) dz_2 &= \int_{0 < t_1 < t_2 < 1} dt_1 dt_2 \left(\frac{1}{2} - \varepsilon\right)^2 \pi \cot\left(\pi \left[\left(\frac{1}{2} - \varepsilon\right) t_2 - \frac{1}{2}\right]\right) \\ &= \frac{\log(2)}{2} + \frac{\log(\pi\varepsilon)}{2} + \mathcal{O}(\varepsilon) \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} \int_{\delta_\varepsilon} dz_1 f_0^{(1)}\left(z_2 - \frac{1}{2}\right) dz_2 &= \int_{0 < t_1 < t_2 < 1} dt_1 dt_2 (i\pi)^2 \varepsilon^2 e^{-i\pi(t_1+t_2)} \pi \cot(-\pi\varepsilon e^{-i\pi t_2}) \\ &= \mathcal{O}(\varepsilon) \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned} \int_{\alpha_2} dz_1 f_0^{(1)}\left(z_2 - \frac{1}{2}\right) dz_2 &= \int_{0 < t_1 < t_2 < 1} dt_1 dt_2 \left(\frac{1}{2} - \varepsilon\right)^2 \pi \cot\left(\pi \left[\left(\frac{1}{2} - \varepsilon\right) t_2 + \varepsilon\right]\right) \\ &= \frac{\log(2)}{2} + \varepsilon \log(\pi\varepsilon) + \mathcal{O}(\varepsilon) \end{aligned} \quad (\text{C.9})$$

$$\int_{\alpha_1} dw \int_{\delta_\varepsilon} f_0^{(1)}\left(z - \frac{1}{2}\right) dz = i\pi \left(\varepsilon - \frac{1}{2}\right) \quad (\text{C.10})$$

$$\int_{\alpha_1} dw \int_{\alpha_2} f_0^{(1)}\left(z - \frac{1}{2}\right) dz = \left(\varepsilon - \frac{1}{2}\right) \log(\sin(\pi\varepsilon)) \quad (\text{C.11})$$

$$\int_{\delta_\varepsilon} dw \int_{\alpha_2} f_0^{(1)}\left(z - \frac{1}{2}\right) dz = -2\varepsilon \log(\sin(\pi\varepsilon)). \quad (\text{C.12})$$

Note that due to  $\lim_{\varepsilon \rightarrow 0} \varepsilon \log(\sin(\pi\varepsilon)) = 0$  the only singular contributions stem from the integrals in eqs. (C.7) and (C.11), which cancel in their sum. Then, we arrive at

$$\lim_{\varepsilon \rightarrow 0} I_0 = -\frac{i\pi}{2} + \log(2), \quad (\text{C.13})$$

as predicted by the constant-term procedure eq. (3.27), see eq. (G.7). Finally, upon combination with the  $q$ -series in eq. (C.4), the desired teMZV is given by

$$\omega\left(\begin{smallmatrix} 0, & 1 \\ 0, & 1/2 \end{smallmatrix}\right) = \lim_{\varepsilon \rightarrow 0} (I_0 + I_q) = -\frac{i\pi}{2} + \log(2) - 2 \sum_{n,m=1}^{\infty} \frac{(-1)^m q^{mn}}{m}. \quad (\text{C.14})$$

## D MZVs and cyclotomic MZVs

Cyclotomic MZVs (also called ‘‘multiple polylogarithms at roots of unity’’) are generalizations of MZVs. While MZVs are represented by nested sums of the form

$$\zeta_{n_1, n_2, \dots, n_r} = \zeta(n_1, n_2, \dots, n_r) = \sum_{0 < k_1 < k_2 < \dots < k_r}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}, \quad n_1, \dots, n_{r-1} \geq 1, n_r \geq 2, \quad (\text{D.1})$$

cyclotomic MZVs are represented by nested sums with additional ‘‘coloring’’ given by  $n$ -th roots of unity  $\sigma_1, \dots, \sigma_r \in \mu_n$ :

$$\zeta\left(\begin{smallmatrix} n_1, n_2, \dots, n_r \\ \sigma_1, \sigma_2, \dots, \sigma_r \end{smallmatrix}\right) = \sum_{0 < k_1 < k_2 < \dots < k_r} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \dots \sigma_r^{k_r}}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}, \quad n_1, \dots, n_r \geq 1, n_r \geq 2 \text{ if } \sigma_r = 1. \quad (\text{D.2})$$

Likewise, the integral representation of MZVs

$$\zeta(n_1, \dots, n_r) = \int_{0 \leq t_i \leq t_{i+1} \leq 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \cdots \frac{dt_{n_1}}{t_{n_1}} \frac{dt_{n_1+1}}{1-t_{n_1+1}} \cdots \frac{dt_w}{t_w}, \quad w = n_1 + \dots + n_r, \quad (\text{D.3})$$

generalizes to an integral representation for cyclotomic MZVs

$$\zeta \left( \begin{matrix} n_1, n_2, \dots, n_r \\ \sigma_1, \sigma_2, \dots, \sigma_r \end{matrix} \right) = \int_{0 \leq t_i \leq t_{i+1} \leq 1} \frac{dt_1}{\eta_1 - t_1} \frac{dt_2}{t_2} \cdots \frac{dt_{n_1}}{t_{n_1}} \frac{dt_{n_1+1}}{\eta_2 - t_{n_1+1}} \cdots \frac{dt_w}{t_w}, \quad (\text{D.4})$$

where  $\eta_i = (\sigma_i \sigma_{i+1} \cdots \sigma_r)^{-1}$ . The positive integer  $n$ , implicit in the definition of cyclotomic MZVs, is sometimes considered an additional datum, and one speaks of  $n$ -cyclotomic MZVs to emphasize the choice of  $n$ .

Cyclotomic MZVs have first been considered by Goncharov [3]. Suitable references for cyclotomic MZVs include [12–14]. For a detailed study of  $n$ -cyclotomic MZVs where  $n = 2, 3, 4, 6, 8$ , see [56]. More recently, the case  $n = 6$  has generated further interest [57, 58].

## E Details on the differential equation of teMZVs

In this appendix, we give a detailed derivation of the differential equation (3.4) in the case  $b_1, b_\ell \neq 0$ . The case  $b_1 = 0$  or  $b_\ell = 0$  is technically more complicated, since the iterated integrals involved need to be regularized according to ref. [8]. However, using Proposition 3.1 of [8], the arguments of this section go through for  $b_1 = 0$  or  $b_\ell = 0$  as well.

Using the mixed heat equation (3.2) for  $\Omega(z - b, \alpha)$ , we may rewrite the  $\tau$ -derivative of the generating function eq. (3.1) of length- $\ell$  teMZVs as

$$\begin{aligned} 2\pi i \frac{\partial}{\partial \tau} \text{T} \left[ \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right] &= \int_{0 \leq z_p \leq z_{p+1} \leq 1} dz_1 dz_2 \dots dz_\ell \sum_{i=1}^{\ell} \partial_{\alpha_i} \partial_{z_i} \Omega(z_i - b_i, \alpha_i) \prod_{j \neq i}^{\ell} \Omega(z_j - b_j, \alpha_j) \\ &= \int_{0 \leq z_p \leq z_{p+1} \leq 1} dz_2 \dots dz_\ell \partial_{\alpha_1} \Omega(z_2 - b_1, \alpha_1) \prod_{j=2}^{\ell} \Omega(z_j - b_j, \alpha_j) - \partial_{\alpha_1} \Omega(-b_1, \alpha_1) \text{T} \left[ \begin{matrix} \alpha_2, \dots, \alpha_\ell \\ b_2, \dots, b_\ell \end{matrix} \right] \\ &\quad + \partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \text{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] - \int_{0 \leq z_p \leq z_{p+1} \leq 1} dz_1 \dots dz_{\ell-1} \partial_{\alpha_\ell} \Omega(z_{\ell-1} - b_\ell, \alpha_\ell) \prod_{j=1}^{\ell-1} \Omega(z_j - b_j, \alpha_j) \\ &\quad + \sum_{i=2}^{\ell-1} \int_{0 \leq z_p \leq z_{p+1} \leq 1} dz_1 \dots dz_{i-1} dz_{i+1} \dots dz_r \partial_{\alpha_i} \Omega(z_i - b_i, \alpha_i) \Big|_{z_i=z_{i-1}}^{z_i=z_{i+1}} \prod_{j \neq i}^{\ell} \Omega(z_j - b_j, \alpha_j) \\ &= \partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \text{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] - \partial_{\alpha_1} \Omega(-b_1, \alpha_1) \text{T} \left[ \begin{matrix} \alpha_2 \dots \alpha_\ell \\ b_2 \dots b_\ell \end{matrix} \right] \\ &\quad + \int_{0 \leq z_p \leq z_{p+1} \leq 1} dz_2 dz_3 \dots dz_\ell \sum_{i=2}^{\ell} \prod_{j \neq i, 1}^{\ell} \Omega(z_j - b_j, \alpha_j) (\partial_{\alpha_{i-1}} - \partial_{\alpha_i}) \Omega(z_i - b_{i-1}, \alpha_{i-1}) \Omega(z_i - b_i, \alpha_i) \\ &= \partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \text{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] - \partial_{\alpha_1} \Omega(-b_1, \alpha_1) \text{T} \left[ \begin{matrix} \alpha_2, \dots, \alpha_\ell \\ b_2, \dots, b_\ell \end{matrix} \right] \\ &\quad + \sum_{i=2}^{\ell} \left( \text{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_{i-1}} \Omega(b_i - b_{i-1}, \alpha_{i-1}) \right. \\ &\quad \left. - \text{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_i} \Omega(b_{i-1} - b_i, \alpha_i) \right), \end{aligned} \quad (\text{E.1})$$



where we mapped the integrals over  $\partial_{z_i}(\dots)$  to boundary terms in the second equality and used the Fay identity (B.3) in the last equality to simplify

$$\begin{aligned}
& (\partial_{\alpha_{i-1}} - \partial_{\alpha_i})\Omega(z_i - b_{i-1}, \alpha_{i-1})\Omega(z_i - b_i, \alpha_i) \\
&= (\partial_{\alpha_{i-1}} - \partial_{\alpha_i})\left[\Omega(z_i - b_i, \alpha_{i-1} + \alpha_i)\Omega(b_i - b_{i-1}, \alpha_{i-1}) + \Omega(z_i - b_{i-1}, \alpha_{i-1} + \alpha_i)\partial_{\alpha_i}\Omega(b_{i-1} - b_i, \alpha_i)\right] \\
&= \Omega(z_i - b_i, \alpha_{i-1} + \alpha_i)\partial_{\alpha_{i-1}}\Omega(b_i - b_{i-1}, \alpha_{i-1}) - \Omega(z_i - b_{i-1}, \alpha_{i-1} + \alpha_i)\partial_{\alpha_i}\Omega(b_{i-1} - b_i, \alpha_i).
\end{aligned} \tag{E.2}$$

From the above differential equation for the generating function one may deduce a differential equation for the teMZV

$$\begin{aligned}
2\pi i \partial_\tau \mathbb{T} \left[ \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_\ell \\ b_1, b_2, \dots, b_\ell \end{matrix} \right] &= 2\pi i \sum_{n_1, n_2, \dots, n_\ell=0}^{\infty} \alpha_1^{n_1-1} \dots \alpha_\ell^{n_\ell-1} \partial_\tau \omega \left( \begin{matrix} n_1, \dots, n_\ell \\ b_1, \dots, b_\ell \end{matrix} \right) \\
&= \partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] - \partial_{\alpha_1} \Omega(-b_1, \alpha_1) \mathbb{T} \left[ \begin{matrix} \alpha_2, \dots, \alpha_\ell \\ b_2, \dots, b_\ell \end{matrix} \right] \\
&\quad + \sum_{i=2}^{\ell} \left( \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_{i-1}} \Omega(b_i - b_{i-1}, \alpha_{i-1}) \right. \\
&\quad \left. - \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_i} \Omega(b_{i-1} - b_i, \alpha_i) \right).
\end{aligned} \tag{E.3}$$

Eventually we want to equate coefficients to extract the  $\tau$ -derivative of a particular teMZV. For this purpose let us consider the terms in eq. (E.3) separately. Recalling the definition eq. (3.5) of  $h^{(n)}$  we may rewrite the first term on the right hand side of eq. (E.3) as

$$\begin{aligned}
\partial_{\alpha_\ell} \Omega(-b_\ell, \alpha_\ell) \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right] &= \sum_{n_\ell=0}^{\infty} h^{(n_\ell)}(-b_\ell) \alpha_\ell^{n_\ell-2} \sum_{n_1, n_2, \dots, n_{\ell-1}=0}^{\infty} \alpha_1^{n_1-1} \dots \alpha_{\ell-1}^{n_{\ell-1}-1} \omega \left( \begin{matrix} n_1, \dots, n_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right) \\
&= \sum_{n_1, n_2, \dots, n_\ell=0}^{\infty} \alpha_1^{n_1-1} \dots \alpha_{\ell-1}^{n_{\ell-1}-1} \alpha_\ell^{n_\ell-1} h^{(n_\ell+1)}(-b_\ell) \omega \left( \begin{matrix} n_1, \dots, n_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right) \\
&\quad + h^{(0)}(-b_\ell) \alpha_\ell^{-2} \sum_{n_1, n_2, \dots, n_{\ell-1}=0}^{\infty} \alpha_1^{n_1-1} \dots \alpha_{\ell-1}^{n_{\ell-1}-1} \omega \left( \begin{matrix} n_1, \dots, n_{\ell-1} \\ b_1, \dots, b_{\ell-1} \end{matrix} \right),
\end{aligned} \tag{E.4}$$

and similarly for the second term. The sum  $\sim \alpha_\ell^{-2}$  is canceled by contributions from the last two lines of eq. (E.3), which we will now turn to

$$\begin{aligned}
& \mathbb{T} \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \dots, \alpha_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right] \partial_{\alpha_{i-1}} \Omega(b_i - b_{i-1}, \alpha_{i-1}) \\
&= \sum_{n_1, \dots, n_{i-2}, n_{i+1}, \dots, n_\ell=0}^{\infty} \alpha_1^{n_1-1} \dots \alpha_{i-2}^{n_{i-2}-1} \alpha_{i+1}^{n_{i+1}-1} \dots \alpha_\ell^{n_\ell-1} \\
&\quad \times \underbrace{\left[ (\alpha_{i-1} + \alpha_i)^{-1} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) \sum_{j=0}^{\infty} h^{(j)}(b_i - b_{i-1}) \alpha_{i-1}^{j-2} \right]}_{= B_{i,+}} \\
&\quad + \underbrace{\sum_{j,k=0}^{\infty} \sum_{p=0}^k \binom{k}{p} \alpha_{i-1}^{k-p+j-2} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j)}(b_i - b_{i-1})}_{= C_{i,+}} \tag{E.5}
\end{aligned}$$

with an analogous definition for  $B_{i,-}$  and  $C_{i,-}$  relevant to the last term of eq. (E.3)

$$B_{i,-} = (\alpha_{i-1} + \alpha_i)^{-1} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right) \sum_{j=0}^{\infty} h^{(j)}(b_{i-1} - b_i) \alpha_i^{j-2} \quad (\text{E.6})$$

$$C_{i,-} = \sum_{j,k=0}^{\infty} \sum_{p=0}^k \binom{k}{p} \alpha_i^{k-p+j-2} \alpha_{i-1}^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j)}(b_{i-1} - b_i) . \quad (\text{E.7})$$

In the following the manipulations only affect pairs  $(\alpha_{i-1}, \alpha_i)$ , hence we will suppress the summation over the other  $\alpha$ 's. Since  $h^{(0)}(b) = -1$  does not depend on  $b$ , we can set it to zero for the iterated integral

$$\omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_{i-1}, b_{i+1}, \dots, b_\ell \end{matrix} \right) = \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) . \quad (\text{E.8})$$

Hence, for the terms not expressible by the binomial law  $B_{i,\pm}$ , we can use  $h^{(0)} = -1$ ,  $h^{(1)} = 0$ ,  $h^{(2)}(b) = h^{(2)}(-b)$  as well as

$$(\alpha_{i-1}^j - (-1)^j \alpha_i^j) = (\alpha_{i-1} + \alpha_i) \sum_{a=0}^{j-1} (-1)^{j-1-a} \alpha_{i-1}^a \alpha_i^{j-1-a} , \quad j > 0 \quad (\text{E.9})$$

to obtain

$$\begin{aligned} & B_{i,+} - B_{i,-} \\ &= (\alpha_{i-1} + \alpha_i)^{-1} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) \sum_{j=0}^{\infty} \left( h^{(j)}(b_i - b_{i-1}) \alpha_{i-1}^{j-2} - h^{(j)}(b_{i-1} - b_i) \alpha_i^{j-2} \right) \\ &= \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) \left( \frac{\alpha_i^{-2} - \alpha_{i-1}^{-2}}{\alpha_{i-1} + \alpha_i} + \sum_{j=0}^{\infty} h^{(j+3)}(b_i - b_{i-1}) \sum_{a=0}^j (-1)^{j-a} \alpha_{i-1}^a \alpha_i^{j-a} \right) . \end{aligned} \quad (\text{E.10})$$

The singular term  $\frac{\alpha_i^{-2} - \alpha_{i-1}^{-2}}{\alpha_{i-1} + \alpha_i} = \frac{1}{\alpha_{i-1} \alpha_i} \left( \frac{1}{\alpha_i} - \frac{1}{\alpha_{i-1}} \right)$  will eventually cancel among different  $B_{i,+} - B_{i,-}$ . The remaining contribution may be rewritten into a form where we can easily read off the coefficient of a given monomial in the  $\alpha_i$ ,

$$\begin{aligned} & \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) \sum_{j=0}^{\infty} h^{(j+3)}(b_i - b_{i-1}) \sum_{a=0}^j (-1)^{j-a} \alpha_{i-1}^a \alpha_i^{j-a} \\ &= \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, 0, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, 0, b_{i+1}, \dots, b_\ell \end{matrix} \right) \sum_{m,n=0}^{\infty} h^{(m+n+3)}(b_i - b_{i-1}) (-1)^n \alpha_{i-1}^m \alpha_i^n , \end{aligned} \quad (\text{E.11})$$

which gives rise to the last line of eq. (3.4).

For the contributions  $C_{i,\pm}$  we have

$$\begin{aligned}
C_{i,+} &= \sum_{j,k=0}^{\infty} \sum_{p=0}^k \binom{k}{p} \alpha_{i-1}^{k+j-p-2} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j)}(b_i - b_{i-1}) \\
&= \sum_{j,k=0}^{\infty} \sum_{p=0}^{k-1} \binom{k}{p} \alpha_{i-1}^{k+j-p-1} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j+1)}(b_i - b_{i-1}) \\
&\quad + \sum_{k=0}^{\infty} \sum_{p=0}^k \binom{k+1}{p} \alpha_{i-1}^{k-p-1} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+2, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(0)}(b_i - b_{i-1}) \\
&\quad + \alpha_{i-1}^{-2} \sum_{a=0}^{\infty} h^{(0)}(b_i - b_{i-1}) \alpha_i^a \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, a+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) .
\end{aligned} \tag{E.12}$$

The sums proportional to  $\alpha_{i-1}^{-2}$  cancels among the corresponding contributions from  $C_{i,+}$  and  $C_{i-1,-}$ . For the cases  $i = 2$  and  $i = r$  it is canceled by the corresponding sums in the last line of eq. (E.4). We then find for the remaining part of  $C_{i,+}$

$$\begin{aligned}
&\sum_{j,k=0}^{\infty} \sum_{p=0}^{k-1} \binom{k}{p} \alpha_{i-1}^{k+j-p-1} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j+1)}(b_{i-1} - b_i) \\
&\quad + \sum_{k=0}^{\infty} \sum_{p=0}^k \binom{k+1}{p} \alpha_{i-1}^{k-p-1} \alpha_i^p \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, k+2, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(0)}(b_{i-1} - b_i) \\
&= \sum_{m,n=0}^{\infty} \alpha_{i-1}^{m-1} \alpha_i^n \sum_{j=0}^m \binom{m+n-j}{n} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, m+n+1-j, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(j+1)}(b_{i-1} - b_i) \\
&\quad + \sum_{m,n=0}^{\infty} \alpha_{i-1}^{m-1} \alpha_i^n \binom{m+n+1}{n} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, m+n+2, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(0)}(b_{i-1} - b_i) \\
&= \sum_{m,n=0}^{\infty} \alpha_{i-1}^{m-1} \alpha_i^n \sum_{k=0}^{m+1} \binom{n+k}{k} \omega \left( \begin{matrix} n_1, \dots, n_{i-2}, n+k+1, n_{i+1}, \dots, n_\ell \\ b_1, \dots, b_{i-2}, b_i, b_{i+1}, \dots, b_\ell \end{matrix} \right) h^{(m-k+1)}(b_{i-1} - b_i) ,
\end{aligned} \tag{E.13}$$

which, in combination with an analogous contribution from  $C_{i,-}$ , gives rise to the second and third line of eq. (3.4).

## F Differential equation for proper rational twists

The derivation of the differential equation given in appendix E was a priori only valid for generic twists. In this appendix, we provide an argument why it also holds for proper rational twists. The only difference is that in eq. (E.1) we now need to carefully keep track of the effect of deforming the domain of integration  $[0, 1]$  infinitesimally to obtain  $[0, 1]_\varepsilon$  (cf. figure 3). The key point is that the integral along  $[0, 1]_\varepsilon$  “is the same as along  $[0, 1]$  up to terms which vanish in the limit”, and the differential equation (E.1) goes through without change.

As in the case of generic twists, it will be convenient to define a generating function

$$\mathbb{T}^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] = \lim_{\varepsilon \rightarrow 0} \mathbb{T}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] , \tag{F.1}$$

$$\mathbb{T}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] = \int_{0 \leq t_i \leq t_{i+1} \leq 1} (\gamma_R^* \Omega(z_1 - b_1, \alpha_1, \tau) dz_1) \dots (\gamma_R^* \Omega(z_\ell - b_\ell, \alpha_\ell, \tau) dz_\ell) , \tag{F.2}$$

where  $\gamma_R = [0, 1]_\varepsilon$  and  $\gamma_R^*$  denotes its pullback. Here and throughout this appendix, we will denote by  $t_i$  the pullback of the coordinate  $z_i$  along  $\gamma_R$ . We note that in the case where all  $b_i$  are generic twists, we may pass to the limit  $\varepsilon \rightarrow 0$  immediately and integrate along the line  $\gamma_R(t) = t$ , which leads to eq. (3.1). Pulling back  $\Omega(z, \alpha, \tau)dz$  along  $\gamma_R$ , we obtain

$$\gamma_R^* \Omega(z_i - b_i, \alpha_i, \tau) dz_i = \gamma_R^* \underbrace{e^{-2\pi i r_i \alpha_i} F(z_i - b_i, \alpha_i, \tau)}_{= \tilde{\Omega}(z_i - b_i, \alpha_i, \tau)} dz_i + \mathcal{O}(\varepsilon), \quad (\text{F.3})$$

since  $\text{Im}(z_i)$  is of order  $\varepsilon$  on  $[0, 1]_\varepsilon$ .

The resulting form  $\tilde{\Omega}(z, \alpha, \tau)dz$  is now meromorphic; therefore the integral over  $\tilde{\Omega}(z, \alpha, \tau)dz$  along any path depends exclusively on its homotopy class. In particular, since the paths  $[0, 1]_\varepsilon$  are all homotopic for sufficiently small  $\varepsilon$ , for every such  $\varepsilon$  we get<sup>12</sup>

$$\begin{aligned} \mathbb{T}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] &= \lim_{\varepsilon \rightarrow 0} \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] = \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right], \\ \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] &= \int_{0 \leq t_i \leq t_{i+1} \leq 1} (\gamma_R^* \tilde{\Omega}(z_1 - b_1, \alpha_1, \tau) dz_1) \dots (\gamma_R^* \tilde{\Omega}(z_\ell - b_\ell, \alpha_\ell, \tau) dz_\ell). \end{aligned} \quad (\text{F.4})$$

Furthermore, we define the intermediate object

$$\tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_i \\ b_1, \dots, b_i \end{matrix}; z_{i+1} \right] = \int_{0 \leq t_1 \leq \dots \leq t_i \leq t_{i+1}} (\gamma_R^* \tilde{\Omega}(z_1 - b_1, \alpha_1, \tau) dz_1) \dots (\gamma_R^* \tilde{\Omega}(z_i - b_i, \alpha_i, \tau) dz_i) \quad (\text{F.5})$$

$$= \int_0^{t_{i+1}} \gamma_R^* \left( \tilde{\Omega}(z_i - b_i, \alpha_i, \tau) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_i \\ b_1, \dots, b_i \end{matrix}; z_i \right] dz_i \right) \quad (\text{F.6})$$

with  $0 < t_{i+1} < 1$ . It satisfies

$$\partial_{z_{i+1}} \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_i \\ b_1, \dots, b_i \end{matrix}; z_{i+1} \right] = \tilde{\Omega}(z_{i+1} - b_i, \alpha_i, \tau) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_{i-1} \\ b_1, \dots, b_{i-1} \end{matrix}; z_{i+1} \right]. \quad (\text{F.7})$$

Using the above setup, we now show that all essential aspects of the computation (E.1) in appendix E remain unchanged. Firstly, we compute the  $\tau$ -derivative of  $\gamma_R^* \tilde{\Omega}$  and obtain

$$\begin{aligned} 2\pi i \partial_\tau \gamma_R^* \left( \tilde{\Omega}(z_i - b_i, \alpha_i, \tau) dz_i \right) &= 2\pi i \partial_\tau \left( e^{-2\pi i r_i \alpha_i} F(\gamma_R(t_i) - b_i, \alpha_i, \tau) d\gamma_R(t_i) \right) \\ &= e^{-2\pi i r_i \alpha_i} \left[ (-2\pi i r_i \partial_{\gamma_R(t_i)} + \partial_{\gamma_R(t_i)} \partial_{\alpha_i}) F(\gamma_R(t_i) - b_i, \alpha_i, \tau) \right] d\gamma_R(t_i) \\ &= \gamma_R^* \left( \partial_{z_i} \partial_{\alpha_i} \tilde{\Omega}(z_i - b_i, \alpha_i, \tau) dz_i \right), \end{aligned} \quad (\text{F.8})$$

using the mixed heat equation (B.4) for  $F(z, \alpha, \tau)$  in the second step. In particular,  $\gamma_R^* \tilde{\Omega}$  itself satisfies a mixed-heat type equation. Secondly, we may exchange the  $\tau$ -derivative with the integration

$$2\pi i \partial_\tau \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{matrix} \alpha_1, \dots, \alpha_\ell \\ b_1, \dots, b_\ell \end{matrix} \right] = \sum_{i=1}^{\ell} \int_{0 \leq t_{j-1} \leq t_j \leq 1} \prod_{j>i} (\gamma_R^* \tilde{\Omega}(z_j - b_j, \alpha_j, \tau) dz_j) \quad (\text{F.9})$$

<sup>12</sup>Recall that the same argument has already been used in subsection 2.1 to show that our version of teMZVs for proper rational twists is well-defined (more precisely that the limit in eq. (2.7) exists).

$$\times \int_0^{t_{i+1}} \gamma_R^* \left( (\partial_{z_i} \partial_{\alpha_i} \tilde{\Omega}(z_i - b_i, \alpha_i, \tau)) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{smallmatrix} \alpha_1, \dots, \alpha_{i-1} \\ b_1, \dots, b_{i-1} \end{smallmatrix}; z_i \right] dz_i \right),$$

since the  $\tau$ -derivative of the integrand is bounded on the domain of integration. Rewriting the  $i$ -th integration using integration by parts yields

$$\begin{aligned} & \int_0^{t_{i+1}} \gamma_R^* \left( (\partial_{z_i} \partial_{\alpha_i} \tilde{\Omega}(z_i - b_i, \alpha_i, \tau)) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{smallmatrix} \alpha_1, \dots, \alpha_{i-1} \\ b_1, \dots, b_{i-1} \end{smallmatrix}; z_i \right] dz_i \right) \\ &= \gamma_R^* \left( \partial_{\alpha_i} \tilde{\Omega}(z_{i+1} - b_i, \alpha_i, \tau) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{smallmatrix} \alpha_1, \dots, \alpha_{i-1} \\ b_1, \dots, b_{i-1} \end{smallmatrix}; z_{i+1} \right] \right) \\ & \quad - \int_0^{t_{i+1}} \gamma_R^* \left( (\partial_{\alpha_i} \tilde{\Omega}(z_i - b_i, \alpha_i, \tau)) \tilde{\Omega}(z_{i-1} - b_{i-1}, \alpha_{i-1}, \tau) \tilde{\mathbb{T}}_\varepsilon^R \left[ \begin{smallmatrix} \alpha_1, \dots, \alpha_{i-2} \\ b_1, \dots, b_{i-2} \end{smallmatrix}; z_i \right] dz_i \right), \end{aligned} \quad (\text{F.10})$$

as in the case of generic twists. Therefore, we may proceed further as in the computation (E.1) and arrive at the same result simply by virtue of the replacements  $\mathbb{T} \rightarrow \mathbb{T}_\varepsilon^R$  and  $\Omega \rightarrow \tilde{\Omega}$ .

## G Properties of teMZVs

The purpose of this appendix is to gather constant terms and  $q$ -expansions of teMZVs as well as selected relations relevant to the one-loop open-string amplitude in section 4.

### G.1 Constant terms for generic twists

We start by listing a few simple examples for constant terms of teMZVs with generic twists. These constant terms are obtain from eq. (3.18) by comparing the coefficients of words in the non-commutative variables  $\text{ad}_{x_b}^n(y)$  on both sides, see eq. (3.14) for the change of alphabet between the two sides. At length one, specializing eqs. (3.11) and (3.12) to  $r = \frac{1}{2}$  yields

$$\omega_0 \left( \begin{smallmatrix} n \\ \tau/2 \end{smallmatrix} \right) = \begin{cases} \frac{2^{n-1} - 1}{2^{n-2}} \zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}, \quad (\text{G.1})$$

see eq. (2.24) for the  $\omega_0(\dots)$  notation. This immediately implies that

$$\omega_0 \left( \dots, \begin{smallmatrix} 2k-1 \\ \tau/2 \end{smallmatrix}, \dots \right) = 0, \quad k \in \mathbb{N}, \quad (\text{G.2})$$

regardless on the position of the combined letter  $\begin{smallmatrix} 2k-1 \\ \tau/2 \end{smallmatrix}$ . Similarly, higher-length examples include

$$\begin{aligned} \omega_0 \left( \begin{smallmatrix} n, 0 \\ \tau/2, 0 \end{smallmatrix} \right) &= \begin{cases} \frac{2^{n-1} - 1}{2^{n-1}} \zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}, & \omega_0 \left( \begin{smallmatrix} 1, n \\ 0, \tau/2 \end{smallmatrix} \right) &= \begin{cases} (-i\pi) \frac{2^{n-1} - 1}{2^{n-1}} \zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases} \\ \omega_0 \left( \begin{smallmatrix} 2, 0, 0 \\ 0, 0, 0 \end{smallmatrix} \right) &= -\frac{\zeta_2}{3}, & \omega_0 \left( \begin{smallmatrix} 0, 1, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix} \right) &= \frac{3\zeta_3}{4\pi^2}, & \omega_0 \left( \begin{smallmatrix} 0, 3, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix} \right) &= 0 \\ \omega_0 \left( \begin{smallmatrix} 1, 0, 2 \\ 0, 0, \tau/2 \end{smallmatrix} \right) &= -\frac{i\pi^3}{24}, & \omega_0 \left( \begin{smallmatrix} 0, 2, 2 \\ 0, 0, \tau/2 \end{smallmatrix} \right) &= -\frac{\pi^4}{108}, & \omega_0 \left( \begin{smallmatrix} 0, 1, 0, 2 \\ 0, 0, 0, \tau/2 \end{smallmatrix} \right) &= \frac{\zeta_3}{8}, \end{aligned} \quad (\text{G.3})$$

and we obtain the following examples with more general twists  $b \in (\Lambda_n + \Lambda_n \tau) \setminus \Lambda_n^\times$ :

$$\omega_0 \left( \frac{1}{1/2 + \tau/3} \right) = -\frac{i\pi}{3}, \quad \omega_0 \left( \frac{2}{\tau/2}, \frac{1}{\tau/4}, \frac{0}{0}, \frac{1}{0} \right) = -\frac{i\pi}{48} \zeta_3 + \frac{5}{8} \zeta_4, \quad (\text{G.4})$$

$$\omega_0 \left( \frac{1}{\tau/3}, \frac{1}{\tau/5} \right) = -\frac{3}{5} \zeta_2, \quad \omega_0 \left( \frac{3}{2\tau/5}, \frac{1}{0}, \frac{0}{0}, \frac{1}{\tau/4} \right) = -\frac{9}{125} \zeta_2 \zeta_3. \quad (\text{G.5})$$

Given a twist  $b = s + r\tau$  with  $r \neq 0$ , the constant term does not depend on  $s$  (cf. eqs. (3.11) and (3.12)).

Up to weight five and three at length three and four, respectively, we have checked the constant-term procedure for consistency with Fay relations among teMZVs which can be derived along the lines of [10]. As already noted above, the constant-term procedure discussed in subsection 3.2 covers all eMZVs occurring in the one-loop open-string amplitude at the orders considered in section 4. The examples presented here address all teMZVs relevant to the calculations in section 4.3.

## G.2 Constant terms for proper rational twists

For the proper rational twist  $b = \frac{1}{2}$  we arrive at the following examples of constant terms

$$\omega_0 \left( \frac{1}{1/2} \right) = -i\pi, \quad \omega_0 \left( \frac{2}{1/2}, \frac{0}{0}, \frac{1}{1/2} \right) = \frac{i\pi^3}{24} - \zeta_2 \log(2), \quad (\text{G.6})$$

$$\omega_0 \left( \frac{0}{0}, \frac{1}{1/2} \right) = -\frac{i\pi}{2} + \log(2), \quad \omega_0 \left( \frac{1}{1/2}, \frac{0}{0}, \frac{0}{0} \right) = -\frac{i\pi}{8} - \frac{\log(2)}{2}, \quad (\text{G.7})$$

while twists  $b \in \Lambda_3^\times$  give rise to

$$\omega_0 \left( \frac{1}{1/3} \right) = -i\pi, \quad \omega_0 \left( \frac{1}{1/3}, \frac{1}{2/3} \right) = i\pi \left( \zeta \left( \frac{1}{e^{4\pi i/3}} \right) - \zeta \left( \frac{1}{e^{2\pi i/3}} \right) \right) - 3\zeta_2, \quad (\text{G.8})$$

$$\omega_0 \left( \frac{1}{0}, \frac{1}{0}, \frac{2}{1/3} \right) = \frac{5}{2} \zeta_4, \quad \omega_0 \left( \frac{1}{1/3}, \frac{1}{0}, \frac{0}{0} \right) = -\frac{i\pi}{2} \zeta \left( \frac{1}{e^{2\pi i/3}} \right) + \frac{1}{2} \zeta \left( \frac{2}{e^{2\pi i/3}} \right) - \zeta \left( \frac{1}{e^{4\pi i/3}}, \frac{1}{e^{2\pi i/3}} \right), \quad (\text{G.9})$$

$$\begin{aligned} \omega_0 \left( \frac{1}{1/3}, \frac{1}{0}, \frac{0}{0}, \frac{1}{0} \right) &= \frac{i\pi}{4} \zeta_2 + \zeta_2 \zeta \left( \frac{1}{e^{2\pi i/3}} \right) + \frac{i\pi}{4} \zeta \left( \frac{2}{e^{2\pi i/3}} \right) + \frac{1}{4} \zeta \left( \frac{3}{e^{2\pi i/3}} \right) \\ &\quad - \frac{i\pi}{2} \zeta \left( \frac{1}{e^{4\pi i/3}}, \frac{1}{e^{2\pi i/3}} \right) - \frac{1}{2} \zeta \left( \frac{1}{e^{4\pi i/3}}, \frac{2}{e^{2\pi i/3}} \right) - \frac{1}{2} \zeta \left( \frac{2}{e^{4\pi i/3}}, \frac{1}{e^{2\pi i/3}} \right), \end{aligned} \quad (\text{G.10})$$

see eq. (D.2) for the definition of cyclotomic MZVs  $\zeta \left( \frac{n_1}{\sigma_1}, \frac{n_2}{\sigma_2}, \dots, \frac{n_r}{\sigma_r} \right)$ .

Again, consistency of the constant-term procedure with Fay relations among teMZVs has been checked up to weight five and three at length three and four, respectively.

## G.3 teMZV relations for the string amplitude

The simplification of the string amplitude in section 4.3 requires several relations among teMZVs and eMZVs. The subsequent identities involving teMZVs can be proven by comparing both the constant terms encoded in eq. (3.18) and the  $\tau$ -derivatives eq. (3.4) of both sides. The relations among eMZVs follow from a combination of Fay and shuffle identities [10] and are listed at <https://tools.aei.mpg.de/emzv/>. At the second order in  $\alpha'$ , we make use of

$$2\omega \left( \frac{1}{0}, \frac{1}{0}, \frac{0}{0} \right) - \omega \left( \frac{1}{0}, \frac{0}{0} \right)^2 = \omega \left( \frac{2}{0}, \frac{0}{0}, \frac{0}{0} \right) + \frac{5\zeta_2}{6} \quad (\text{G.11})$$

$$2\omega \left( \frac{1}{\tau/2}, \frac{1}{\tau/2}, \frac{0}{0} \right) - \omega \left( \frac{1}{\tau/2}, \frac{0}{0} \right)^2 = \omega \left( \frac{2}{0}, \frac{0}{0}, \frac{0}{0} \right) + \frac{\zeta_2}{3}, \quad (\text{G.12})$$

while the simplifications at the order  $\alpha'^3$  are based on the relations

$$\begin{aligned} & \omega\left(\begin{smallmatrix} 1, 0 \\ 0, 0 \end{smallmatrix}\right)^3 - 3\omega\left(\begin{smallmatrix} 1, 0 \\ 0, 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1, 1, 0 \\ 0, 0, 0 \end{smallmatrix}\right) + 3\omega\left(\begin{smallmatrix} 1, 1, 1, 0 \\ 0, 0, 0, 0 \end{smallmatrix}\right) \\ &= \frac{1}{6}\omega\left(\begin{smallmatrix} 0, 3, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix}\right) - \frac{1}{4}\zeta_3 - 4\zeta_2\omega\left(\begin{smallmatrix} 0, 1, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix}\right) \end{aligned} \quad (\text{G.13})$$

$$\begin{aligned} & \omega\left(\begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix}\right)^3 - 3\omega\left(\begin{smallmatrix} 1, 0 \\ \tau/2, 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{smallmatrix}\right) + 3\omega\left(\begin{smallmatrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{smallmatrix}\right) \\ &= \frac{1}{6}\omega\left(\begin{smallmatrix} 0, 3, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix}\right) - \frac{1}{4}\zeta_3 + 2\zeta_2\omega\left(\begin{smallmatrix} 0, 1, 0, 0 \\ 0, 0, 0, 0 \end{smallmatrix}\right). \end{aligned} \quad (\text{G.14})$$

#### G.4 $q$ -expansions for the string amplitude

The eMZVs seen in the final results eqs. (4.55) and (H.26) for the integrals in the string amplitude have the following  $q$ -expansions

$$\begin{aligned} \omega(0, 0, 2) &= -\frac{\zeta_2}{3} + 2\sum_{n,m=1}^{\infty} \frac{m}{n^2} q^{mn} \\ \omega(0, 1, 0, 0) &= \frac{3\zeta_3}{4\pi^2} + \frac{3}{2\pi^2}\sum_{n,m=1}^{\infty} \frac{1}{n^3} q^{mn} \\ \omega(0, 3, 0, 0) &= -3\sum_{n,m=1}^{\infty} \frac{m^2}{n^3} q^{mn}. \end{aligned} \quad (\text{G.15})$$

These expressions follow from repeatedly integrating the Eisenstein series eq. (3.6) in the  $\tau$ -derivatives eq. (3.4) of the eMZVs in question. Analogous  $q$ -expansions for teMZVs with twists  $b_i \in \{0, \tau/2\}$  can be determined based on

$$f^{(k)}(\tau/2) = \begin{cases} \frac{2^{k-1} - 1}{2^{k-2}}\zeta_k - \frac{2(2\pi i)^k}{(k-1)!}\sum_{n,m=1}^{\infty} \left(n - \frac{1}{2}\right)^{k-1} q^{m(n-1/2)} & : k \text{ even} \\ 0 & : k \text{ odd} \end{cases}. \quad (\text{G.16})$$

## H The non-planar integral along with $\text{Tr}(t^1 t^2 t^3)\text{Tr}(t^4)$

In this appendix, we investigate the low-energy expansion of the integral eq. (4.6) associated with the color factor  $\text{Tr}(t^1 t^2 t^3)\text{Tr}(t^4)$  in the one-loop four-point open-string amplitude. The representations eqs. (4.12) and (4.13) of the Green functions allow to cast the integral into the form

$$I_{123|4}(q) = \int_{123}^4 \prod_{1 \leq i < j}^3 \sum_{n_{ij}=0}^{\infty} \frac{(s_{ij} \Gamma(\frac{1}{2}; x_{ij}))^{n_{ij}}}{n_{ij}!} \prod_{j=1}^3 \sum_{n_{j4}=0}^{\infty} \frac{(s_{j4} \Gamma(\frac{1}{\tau/2}; x_{j4}))^{n_{j4}}}{n_{j4}!} \quad (\text{H.1})$$

analogous to eq. (4.23), where  $c(q)$  cancels by momentum conservation eq. (4.5), and the integration measure is defined by

$$\begin{aligned} \int_{123}^4 &= \int_0^1 dx_4 \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^1 dx_1 \delta(x_1) \\ &= \int_0^1 dx_4 \delta(x_4) \left( \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 + \text{cyc}(x_1, x_2, x_3) \right). \end{aligned} \quad (\text{H.2})$$

One can check through the change of variables  $x_i \rightarrow 1 - x_i$  and symmetry properties of the Green function that the measure  $\int_{132}^4$  with  $x_2$  and  $x_3$  interchanged yields the same result for

the integral eq. (H.1). Hence, one can equivalently employ the simpler measure eq. (4.22) with  $x_i \in [0, 1]$  tailored to the color structure  $\text{Tr}(t^1 t^2) \text{Tr}(t^3 t^4)$  and rewrite

$$I_{123|4}(q) = \frac{1}{2} \int_{12}^{34} \prod_{1 \leq i < j}^3 \sum_{n_{ij}=0}^{\infty} \frac{(s_{ij} \Gamma(\frac{1}{0}; x_{ij}))^{n_{ij}}}{n_{ij}!} \prod_{j=1}^3 \sum_{n_{j4}=0}^{\infty} \frac{(s_{j4} \Gamma(\frac{1}{\tau/2}; x_{j4}))^{n_{j4}}}{n_{j4}!}. \quad (\text{H.3})$$

In the remainder of this appendix, we will follow the steps of subsection 4.3 to expand the integral  $I_{123|4}(q)$  to the order  $\alpha'^3$ , where the representation eq. (H.3) is most convenient for practical purposes. While the elliptic iterated integrals in this expansion lead to teMZVs for each monomial in  $s_{ij}$ , the final results for the orders  $\alpha'^{\leq 3}$  turn out to comprise eMZVs only.

## H.1 Structure of the leading orders $\alpha'^{\leq 3}$

We start by classifying the inequivalent integrals w.r.t. the cycle structure of  $\text{Tr}(t^1 t^2 t^3) \text{Tr}(t^4)$  which occur at the orders  $\alpha'^{\leq 3}$  of eq. (H.1): With the shorthands  $P_{ij} = P(x_{ij})$  and

$$\hat{Q}_{ij} = Q(x_{ij}) - c(q) = \Gamma\left(\frac{1}{\tau/2}; x_{ij}\right), \quad (\text{H.4})$$

we have two inequivalent cases at the first order,

$$e_1^1 = \int_{123}^4 P_{12}, \quad e_2^1 = \int_{123}^4 \hat{Q}_{14}, \quad (\text{H.5})$$

six cases at the second order,

$$\begin{aligned} e_1^2 &= \frac{1}{2} \int_{123}^4 P_{12}^2, & e_2^2 &= \int_{123}^4 P_{12} P_{13}, & e_3^2 &= \int_{123}^4 P_{12} \hat{Q}_{34} \\ e_4^2 &= \frac{1}{2} \int_{123}^4 \hat{Q}_{14}^2, & e_5^2 &= \int_{123}^4 \hat{Q}_{14} \hat{Q}_{24}, & e_6^2 &= \int_{123}^4 P_{12} \hat{Q}_{14} \end{aligned} \quad (\text{H.6})$$

and fourteen cases at the third order:

$$\begin{aligned} e_1^3 &= \frac{1}{6} \int_{123}^4 P_{12}^3, & e_2^3 &= \frac{1}{2} \int_{123}^4 \hat{Q}_{14}^2 \hat{Q}_{24}, & e_3^3 &= \int_{123}^4 P_{12} P_{13} \hat{Q}_{14} \\ e_4^3 &= \frac{1}{6} \int_{123}^4 \hat{Q}_{14}^3, & e_5^3 &= \frac{1}{2} \int_{123}^4 P_{12}^2 \hat{Q}_{14}, & e_6^3 &= \int_{123}^4 P_{12} P_{13} \hat{Q}_{24} \\ e_7^3 &= \frac{1}{2} \int_{123}^4 P_{12}^2 P_{13}, & e_8^3 &= \frac{1}{2} \int_{123}^4 P_{12}^2 \hat{Q}_{34}, & e_9^3 &= \int_{123}^4 P_{12} \hat{Q}_{14} \hat{Q}_{24} \\ e_{10}^3 &= \int_{123}^4 P_{12} P_{13} P_{23}, & e_{11}^3 &= \frac{1}{2} \int_{123}^4 P_{12} \hat{Q}_{14}^2, & e_{12}^3 &= \int_{123}^4 P_{12} \hat{Q}_{14} \hat{Q}_{34} \\ e_{13}^3 &= \int_{123}^4 \hat{Q}_{14} \hat{Q}_{24} \hat{Q}_{34}, & e_{14}^3 &= \frac{1}{2} \int_{123}^4 P_{12} \hat{Q}_{34}^2. \end{aligned} \quad (\text{H.7})$$

## H.2 teMZVs at orders $\alpha'^{\leq 3}$

As a next step, we evaluate the above  $e_j^i$  in terms of teMZVs, using the equivalence of the measures  $\int_{123}^4$  and  $\frac{1}{2} \int_{12}^{34}$  noted in eqs. (H.1) and (H.3). By largely recycling the results of section 4.3, we obtain the following expressions at the first order,

$$e_1^1 = \frac{1}{2} d_1^1 = \frac{1}{2} \omega\left(\frac{1}{0}, 0\right), \quad e_2^1 = \frac{1}{2} d_2^1|_{c(q) \rightarrow 0} = \frac{1}{2} \omega\left(\frac{1}{\tau/2}, 0\right), \quad (\text{H.8})$$



the following ones at the second order,

$$e_1^2 = \frac{1}{2} d_1^2 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ 0, 0, 0 \end{pmatrix}, \quad e_3^2 = \frac{1}{2} d_4^2 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}^2 \quad (\text{H.9})$$

$$e_2^2 = \frac{1}{2} d_2^2 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{pmatrix}, \quad e_4^2 = \frac{1}{2} d_5^2 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^2 \quad (\text{H.10})$$

$$e_5^2 = \frac{1}{2} d_3^2 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix} = e_6^2, \quad (\text{H.11})$$

and the following ones at the third order:

$$e_1^3 = \frac{1}{2} d_1^3 = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 1, 0 \\ 0, 0, 0, 0 \end{pmatrix} \quad (\text{H.12})$$

$$e_2^3 = \frac{1}{2} d_2^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{pmatrix} \quad (\text{H.13})$$

$$e_3^3 = \frac{1}{2} d_5^3 = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ 0, 0, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} \quad (\text{H.14})$$

$$e_4^3 = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}^3 - \zeta_2 \omega(0, 1, 0, 0) + \frac{1}{6} \omega(0, 3, 0, 0) \quad (\text{H.15})$$

$$e_5^3 = \frac{1}{2} d_{12}^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^3 \quad (\text{H.16})$$

$$e_6^3 = \frac{1}{2} d_7^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix} \quad (\text{H.17})$$

$$e_7^3 = \frac{1}{2} d_3^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ 0, 0, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix} = e_8^3 \quad (\text{H.18})$$

$$e_9^3 = \frac{1}{2} d_4^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} = e_{10}^3 \quad (\text{H.19})$$

$$e_{11}^3 = \frac{1}{2} d_8^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}^2 \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix} = e_{12}^3 \quad (\text{H.20})$$

$$e_{13}^3 = \frac{1}{2} d_{11}^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^2 \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} + \frac{1}{6} \omega(0, 3, 0, 0) \quad (\text{H.21})$$

$$e_{14}^3 = \frac{1}{2} d_9^3 |_{c(q) \rightarrow 0} = \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^2 \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}. \quad (\text{H.22})$$

Expressions for  $d_{11}^3$  (cf. eq. (4.45)) and  $e_4^3$  are derived in appendix I.

### H.3 Assembling the orders $\alpha'^{\leq 3}$

Similar to eq. (4.47), momentum conservation<sup>13</sup> leaves the following contributions to the integral eq. (H.1) at the orders  $\alpha'^{\leq 3}$ ,

$$I_{123|4}(q) = \frac{1}{2} + (s_{12}^2 + s_{12}s_{23} + s_{23}^2)(2e_1^2 + 2e_2^2 - e_3^2 - e_4^2) \quad (\text{H.23}) \\ + s_{12}s_{23}(s_{12} + s_{23})(-3e_1^3 - 3e_2^3 + 3e_3^3 - e_4^3 - e_5^3 + 3e_6^3 - 3e_{13}^3 + 3e_{14}^3) + \mathcal{O}(\alpha'^4),$$

<sup>13</sup>We have already exploited the equalities  $e_5^2 = e_6^2$ ,  $e_7^2 = e_8^2$ ,  $e_9^2 = e_{10}^2$  and  $e_{11}^2 = e_{12}^2$  in simplifying  $I_{123|4}(q)$  to the expression in eq. (H.23).

in agreement with the  $\alpha'^{\leq 1}$  results of ref. [29]. The combinations of  $e_j^i$  can be expressed in terms of eMZVs

$$\begin{aligned} I_{123|4}(q) \Big|_{s_{12}^2+s_{12}s_{23}+s_{23}^2} &= \omega \begin{pmatrix} 1, 1, 0 \\ 0, 0, 0 \end{pmatrix} - \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}^2 + \omega \begin{pmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{pmatrix} - \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^2 \\ &= \frac{7\zeta_2}{12} + \omega(0, 0, 2) \end{aligned} \quad (\text{H.24})$$

$$\begin{aligned} I_{123|4}(q) \Big|_{s_{12}s_{23}(s_{12}+s_{23})} &= \frac{3}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 1, 0 \\ 0, 0, 0 \end{pmatrix} - \frac{3}{2} \omega \begin{pmatrix} 1, 1, 1, 0 \\ 0, 0, 0, 0 \end{pmatrix} - \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ 0, 0 \end{pmatrix}^3 \\ &\quad + \frac{3}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix} \omega \begin{pmatrix} 1, 1, 0 \\ \tau/2, \tau/2, 0 \end{pmatrix} - \frac{3}{2} \omega \begin{pmatrix} 1, 1, 1, 0 \\ \tau/2, \tau/2, \tau/2, 0 \end{pmatrix} - \frac{1}{2} \omega \begin{pmatrix} 1, 0 \\ \tau/2, 0 \end{pmatrix}^3 \\ &\quad + \zeta_2 \omega(0, 1, 0, 0) - \frac{2}{3} \omega(0, 3, 0, 0) \\ &= 2\zeta_2 \omega(0, 1, 0, 0) - \frac{5}{6} \omega(0, 3, 0, 0) + \frac{\zeta_3}{4}, \end{aligned} \quad (\text{H.25})$$

where we have used the teMZV relations of appendix G.3.

#### H.4 Summary of the orders $\alpha'^{\leq 3}$

Once we insert the simplified expressions eqs. (H.24) and (H.25) for the combinations of  $e_j^i$ , the leading low-energy orders of the integral eq. (H.23) boil down to the following eMZVs:

$$\begin{aligned} I_{123|4}(q) &= \frac{1}{2} + (s_{12}^2 + s_{12}s_{23} + s_{23}^2) \left( \frac{7\zeta_2}{12} + \omega(0, 0, 2) \right) \\ &\quad + s_{12}s_{23}(s_{12} + s_{23}) \left( 2\zeta_2 \omega(0, 1, 0, 0) - \frac{5}{6} \omega(0, 3, 0, 0) + \frac{\zeta_3}{4} \right) + \mathcal{O}(\alpha'^4). \end{aligned} \quad (\text{H.26})$$

From the constant terms of the eMZVs gathered in eq. (G.3), our result eq. (H.26) is consistent with the tree-level expression

$$\begin{aligned} I_{123|4}(q) &= -\frac{1}{\pi^2} \left[ \frac{\Gamma(s_{12})\Gamma(s_{23})}{\Gamma(1+s_{12}+s_{23})} + \text{cyc}(1, 2, 3) \right] + \mathcal{O}(q) \\ &= \frac{1}{2} + \frac{1}{4} \zeta_2 (s_{12}^2 + s_{12}s_{23} + s_{23}^2) + \frac{1}{2} \zeta_3 s_{12}s_{23}(s_{12} + s_{23}) + \mathcal{O}(q, \alpha'^4), \end{aligned} \quad (\text{H.27})$$

which is known to arise at the  $q^0$  order of the integral  $I_{123|4}(q)$  [30].

## I Sample integrals in non-planar string amplitudes

In this appendix, we will derive the results eqs. (4.45) and (H.15) for  $d_{11}^3$  and  $e_4^3$ , respectively. These integrals are the most difficult cases at the orders  $\alpha'^{\leq 3}$  because the Green functions form closed subcycles such as  $P_{ij}P_{ik}P_{jk}$  and  $P_{ij}Q_{ik}Q_{jk}$ . Identities between elliptic iterated integrals will be seen to yield answers in terms of teMZVs, and the extensions of these manipulations to all weights and lengths [28] guarantee that each term in the low-energy expansion of the integrals eqs. (4.23) and (H.1) can be expressed in terms of teMZVs.

### I.1 The $d_{11}^3$ integral from $P_{ij}Q_{ik}Q_{jk}$

The contributions to  $d_{11}^3$  with at least one factor of  $c(q)$  are equivalent to those in  $d_9^3$ , so it is sufficient to study

$$\begin{aligned}
\int_{12}^{34} P_{34}Q_{13}Q_{14} |_{c(q) \rightarrow 0} &= 2 \int_0^1 dx_4 \int_0^{x_4} dx_3 \Gamma\left(\frac{1}{\tau/2}; x_3\right) \Gamma\left(\frac{1}{\tau/2}; x_4\right) \left( - \int_{x_3}^{x_4} du f^{(1)}(u - x_4) \right) \\
&= -2 \int_0^1 dx_4 \Gamma\left(\frac{1}{\tau/2}; x_4\right) \int_0^{x_4} du f^{(1)}(u - x_4) \int_0^u dx_3 \Gamma\left(\frac{1}{\tau/2}; x_3\right) \quad (\text{I.1}) \\
&= -2 \int_0^1 dx_4 \Gamma\left(\frac{1}{\tau/2}; x_4\right) \Gamma\left(\frac{1}{x_4} \frac{0}{0} \frac{1}{\tau/2}; x_4\right) .
\end{aligned}$$

In the first step, we have exploited that the integration regions with  $0 \leq x_3 \leq x_4 \leq 1$  and  $0 \leq x_4 \leq x_3 \leq 1$  yield the same contributions by the symmetry of the integrand  $P_{34}Q_{13}Q_{14}$  under exchange of  $x_3$  and  $x_4$ . Moreover, we have used the integral representation  $P_{34} = - \int_{x_3}^{x_4} du f^{(1)}(u - x_4)$  of the Green function, reparametrized the integration domain with  $0 \leq x_3 \leq u \leq x_4$  and applied the definition eq. (2.1) of elliptic iterated integrals.

The elliptic iterated integral  $\Gamma\left(\frac{1}{x_4} \frac{0}{0} \frac{1}{\tau/2}; x_4\right)$  in the last line of eq. (I.1) is not yet suitable for integration over  $x_4$  in its present form due to the twofold appearance of the integration variable. As explained in [28], Fay relations among the weighting functions allow to derive a differential equation in  $x_4$  whose integration yields the alternative representation

$$\begin{aligned}
\Gamma\left(\frac{1}{x_4} \frac{0}{0} \frac{1}{\tau/2}; x_4\right) &= \Gamma\left(\frac{0}{0} \frac{0}{0} \frac{2}{0}; x_4\right) + \Gamma\left(\frac{0}{0} \frac{0}{0} \frac{2}{\tau/2}; x_4\right) + \Gamma\left(\frac{0}{0} \frac{2}{\tau/2} \frac{0}{0}; x_4\right) \\
&\quad - \Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{1}{\tau/2}; x_4\right) - \Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{1}{0}; x_4\right) . \quad (\text{I.2})
\end{aligned}$$

After shuffle multiplication with  $\Gamma\left(\frac{1}{\tau/2}; x_4\right)$ , the integral over  $x_4$  in eq. (I.1) can be readily performed, e.g.

$$\begin{aligned}
&\int_0^1 dx_4 \Gamma\left(\frac{1}{\tau/2}; x_4\right) \Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{1}{0}; x_4\right) \\
&= \int_0^1 dx_4 \left[ \Gamma\left(\frac{1}{\tau/2} \frac{0}{0} \frac{1}{\tau/2} \frac{1}{0}; x_4\right) + 2\Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{1}{\tau/2} \frac{1}{0}; x_4\right) + \Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{1}{0} \frac{1}{\tau/2}; x_4\right) \right] \\
&= \Gamma\left(\frac{0}{0} \frac{1}{\tau/2} \frac{0}{0} \frac{1}{\tau/2} \frac{1}{0}; 1\right) + 2\Gamma\left(\frac{0}{0} \frac{0}{0} \frac{1}{\tau/2} \frac{1}{\tau/2} \frac{1}{0}; 1\right) + \Gamma\left(\frac{0}{0} \frac{0}{0} \frac{1}{\tau/2} \frac{1}{0} \frac{1}{\tau/2}; 1\right) \\
&= \omega\left(\frac{1}{\tau/2}\right) \omega\left(\frac{1}{0}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{0}\right) - \omega\left(\frac{1}{0}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{\tau/2}\right) \quad (\text{I.3})
\end{aligned}$$

for the rightmost term in eq. (I.2). The shuffle operation in the last step of eq. (I.3) together with  $\omega\left(\frac{1}{\tau/2}\right) = 0$  reduces the number of terms by four and leads to the following end result

$$\begin{aligned}
\int_{12}^{34} P_{34}Q_{13}Q_{14} |_{c(q) \rightarrow 0} &= 2\omega\left(\frac{2}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) + 2\omega\left(\frac{2}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) + 2\omega\left(\frac{0}{0}, \frac{2}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) \\
&\quad - 2\omega\left(\frac{1}{0}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) - 2\omega\left(\frac{1}{\tau/2}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) . \quad (\text{I.4})
\end{aligned}$$

Finally, the expression for  $d_{11}^3$  in eq. (4.45) results from the eMZV relation

$$\begin{aligned}
2\omega\left(\frac{2}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) + 2\omega\left(\frac{2}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) + 2\omega\left(\frac{0}{0}, \frac{2}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) \\
- 2\omega\left(\frac{1}{0}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) - 2\omega\left(\frac{1}{\tau/2}, \frac{1}{\tau/2}, \frac{0}{0}, \frac{0}{0}, \frac{1}{\tau/2}\right) = \omega\left(\frac{1}{0}, \frac{0}{0}\right) \omega\left(\frac{1}{\tau/2}, \frac{0}{0}\right)^2 + \frac{1}{3} \omega(0, 3, 0, 0) . \quad (\text{I.5})
\end{aligned}$$

## I.2 The $e_4^3$ integral from $P_{ij}P_{ik}P_{jk}$

A similar strategy applies to  $e_4^3$  in eq. (H.7),

$$\begin{aligned} \int_{123}^4 P_{12}P_{13}P_{23} &= \int_0^1 dx_3 \int_0^{x_3} dx_2 \Gamma\left(\frac{1}{0}; x_2\right) \Gamma\left(\frac{1}{0}; x_3\right) \left( - \int_{x_2}^{x_3} du f^{(1)}(u - x_3) \right) \\ &= - \int_0^1 dx_3 \Gamma\left(\frac{1}{0}; x_3\right) \Gamma\left(\frac{1}{x_3} \frac{0}{0} \frac{1}{0}; x_3\right) , \end{aligned} \quad (\text{I.6})$$

where the relevant identity among elliptic iterated integrals reads

$$\Gamma\left(\frac{1}{x_3} \frac{0}{0} \frac{1}{0}; x_3\right) = 2\Gamma\left(\frac{0}{0} \frac{0}{0} \frac{2}{0}; x_3\right) + \Gamma\left(\frac{0}{0} \frac{2}{0} \frac{0}{0}; x_3\right) - 2\Gamma\left(\frac{0}{0} \frac{1}{0} \frac{1}{0}; x_3\right) + \zeta_2 \Gamma\left(\frac{0}{0}; x_3\right) . \quad (\text{I.7})$$

Note the extra term  $\zeta_2 \Gamma\left(\frac{0}{0}; x_3\right)$  in comparison to the analogous identity eq. (I.2). Upon insertion into eq. (I.6), we obtain

$$\int_{123}^4 P_{12}P_{13}P_{23} = 2\omega(2, 0, 0, 0, 1) + \omega(0, 2, 0, 0, 1) - 2\omega(1, 1, 0, 0, 1) - \frac{1}{2}\zeta_2 \omega(1, 0) , \quad (\text{I.8})$$

which translates into the expression eq. (H.15) for  $e_4^3$  by the eMZV relation

$$\begin{aligned} &2\omega\left(\frac{2}{0}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{0}\right) + \omega\left(\frac{0}{0}, \frac{2}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{0}\right) - 2\omega\left(\frac{1}{0}, \frac{1}{0}, \frac{0}{0}, \frac{0}{0}, \frac{1}{0}\right) - \frac{1}{2}\zeta_2 \omega\left(\frac{1}{0}, \frac{0}{0}\right) \\ &= \frac{1}{2} \omega\left(\frac{1}{0}, \frac{0}{0}\right)^3 - \zeta_2 \omega(0, 1, 0, 0) + \frac{1}{6} \omega(0, 3, 0, 0) . \end{aligned} \quad (\text{I.9})$$

## J Some all-order contributions

For certain contributions to the integral eq. (4.3) in the non-planar open-string amplitude, closed-form expressions in terms of teMZVs can be given to all orders in the  $\alpha'$ -expansion. In the context and notation of subsection 4.3, we find

$$d_{n;1} = \frac{1}{n!} \int_{12}^{34} P(x_{12})^n = \frac{1}{n!} \int_0^1 dx_2 \prod_{i=1}^n \int_0^{x_2} dy_i f^{(1)}(y_i) = \omega\left(\underbrace{\frac{1}{0}, \dots, 1, 0}_{n \text{ times}}\right) \quad (\text{J.1})$$

$$\begin{aligned} d_{n;2} &= \frac{1}{n!} \int_{12}^{34} Q(x_{13})^n = \frac{1}{n!} \int_{12}^{34} \left( \sum_{r=0}^n \binom{n}{r} c(q)^{n-r} \Gamma\left(\frac{1}{\tau/2}; x_3\right)^r \right) \\ &= \sum_{r=0}^n \frac{1}{(n-r)!} c(q)^{n-r} \omega\left(\underbrace{\frac{1}{\tau/2}, \dots, 1, 0}_{r \text{ times}}\right) \end{aligned} \quad (\text{J.2})$$

$$\begin{aligned} d_{n,m;1} &= \frac{1}{n!m!} \int_{12}^{34} P(x_{12})^n P(x_{34})^m = d_{n;1} d_{m;1} \\ &= \omega\left(\underbrace{\frac{1}{0}, \dots, 1, 0}_{n \text{ times}}\right) \omega\left(\underbrace{\frac{1}{0}, \dots, 1, 0}_{m \text{ times}}\right) \end{aligned} \quad (\text{J.3})$$

$$\begin{aligned} d_{n,m;2} &= \frac{1}{n!m!} \int_{12}^{34} P(x_{12})^n Q(x_{13})^m = d_{n;1} d_{m;2} \\ &= \omega\left(\underbrace{\frac{1}{0}, \dots, 1, 0}_{n \text{ times}}\right) \sum_{r=0}^m \frac{1}{(m-r)!} c(q)^{m-r} \omega\left(\underbrace{\frac{1}{\tau/2}, \dots, 1, 0}_{r \text{ times}}\right) \end{aligned} \quad (\text{J.4})$$

$$d_{n,m;3} = \frac{1}{n!m!} \int_{12}^{34} Q(x_{13})^n Q(x_{14})^m = d_{n;2} d_{m;2} \quad (\text{J.5})$$

$$d_{n,m,p;1} = \frac{1}{n!m!p!} \int_{12}^{34} P(x_{12})^n P(x_{34})^m Q(x_{13})^p = d_{n;1}d_{m,p;2} = d_{n;1}d_{m;1}d_{p;2} \quad (\text{J.6})$$

$$d_{n,m,p;2} = \frac{1}{n!m!p!} \int_{12}^{34} P(x_{12})^n Q(x_{13})^m Q(x_{14})^p = d_{n;1}d_{m,p;3} = d_{n;1}d_{m;2}d_{p;2} \quad (\text{J.7})$$

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