Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
Summer 2014
Humboldt-Universität zu Berlin

Problem sheet 1

1. (a) Let I be some general index set. A function $C : I \times I \to \mathbb{R}$ is called *positive (semi)-definite,* if

$$\forall n \geq 1;\, t_1, \ldots, t_n \in I;\, \lambda_1, \ldots, \lambda_n \in \mathbb{R} : \sum_{i,j=1}^{n} C\left(t_i, t_j\right) \lambda_i \lambda_j \geq 0.$$

Consider now $I = \mathbb{R}^+$. Show that $C : I \times I \to \mathbb{R}, (t, s) \mapsto t \wedge s$ is a positive semi-definite function and sketch how a process $\left(\overline{B}_t, t \in I\right)$ with the properties (i)-(iii) of a Brownian motion can be constructed.

(b) Let $(X_t, t \geq 0)$ be a process with independent increments and $X_t \in L^2$ for all $t \geq 0$. Show that $Cov\left(X_t, X_s\right) = a\left(t\right) \wedge a\left(s\right) = a\left(t \wedge s\right)$ holds for all $t, s \geq 0$ and a non-decreasing function $a : \mathbb{R}^+ \to \mathbb{R}^+$.

(c) Show that any centered Gaussian process $X = (X_t, t \geq 0)$ with independent increments can be constructed by setting $X_t = B_{a(t)}$ where a is the function from part (b) with respect to X and $B = (B_t, t \geq 0)$ a Brownian motion.

2. Suppose $(Y_t, t \in [0, 1])$ is a Gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}\left[Y_t\right] = 0,$$
$$Cov\left(Y_t, Y_s\right) = \begin{cases} 0, & t \neq s \\ 1, & t = s \end{cases}.$$

(a) Why does such a process Y exist?

(b) Show that $(\omega, t) \mapsto Y_t\left(\omega\right)$ cannot be jointly measurable with respect to $\mathcal{F} \otimes \mathscr{B}_{[0,1]}$ ($\mathscr{B}_{[0,1]}$ is the Borel-σ-algebra on $[0, 1]$).
Hint for (b): If the statement is not true, it would follow that $\mathbb{E}[(\int_0^1 Y_t\, dt)^2] = 0$, thereby contradicting $\mathbb{P}\left(\exists n \geq 1 : Y_{1/n} \neq 0\right) = 1$.

3. Read in a textbook on real analysis about functions $g : \mathbb{R}^+ \to \mathbb{R}$ of *bounded* or *finite variation* and the *Stieltjes-integral.* Prove for functions $f \in C^1\left(\mathbb{R}\right)$ and $g \in C\left(\mathbb{R}\right)$ with g of bounded variation that

$$f(g(t)) = f(g(0)) + \int_0^t f'(g(s))\, dg(s), \quad \forall t \geq 0.$$

Give an example of a function that is not of bounded variation.

4. Consider the *Haar system* $\mathscr{H} := \{\varphi_0\} \cup \{\psi_{j,k} : j \in \mathbb{N}_0,\ 0 \le k \le 2^j - 1\}$ of real valued functions on the unit interval $[0,1]$ where

$$\begin{aligned}\varphi_0(t) &= \mathbf{1}_{[0,1]}(t),\\ \psi_{0,0}(t) &= \mathbf{1}_{[0,1/2)}(t) - \mathbf{1}_{[1/2,1)}(t),\\ \psi_{j,k}(t) &= 2^{j/2}\psi_{0,0}\left(2^j t - k\right),\end{aligned}$$

for $0 \le t \le 1$, $j \in \mathbb{N}_0$, $0 \le k \le 2^j - 1$.

Our goal is to show that $\mathscr{H}$ is a complete orthonormal system in the Hilbert space $L^2([0,1])$ with standard scalar product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$ and norm $||x|| = \sqrt{\langle x, x\rangle}$ for any $x, y \in L^2([0,1])$, i.e.

i. $\mathscr{H} \subseteq L^2([0,1])$,
ii. $||e|| = 1$ for all $e \in \mathscr{H}$,
iii. $\langle e, f\rangle = 0$ for all $e \ne f,\ e, f \in \mathscr{H}$,
iv. $L^2([0,1]) = \overline{lin(\mathscr{H})}$ where $lin(\mathscr{H})$ is the linear span of $\mathscr{H}$, i.e. the set of all (finite) linear combinations.

Prove that the properties i., ii. and iii. hold directly by definition of the Haar system. For property *iv.* there are many possible proofs. We consider the following strategy:

(a) Consider the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ with

$$\begin{aligned}\mathcal{F}_0 &= \sigma(\varphi_0),\\ \mathcal{F}_n &= \sigma\left(\varphi_0, \psi_{j,k} :\ j \in \mathbb{N}_0, 0 \le k \le 2^j - 1,\ 2^j + k \le n\right)\end{aligned}$$

for $n \ge 1$. For any $g \in L^2([0,1])$ define the process $M = (M_n, n \in \mathbb{N}_0)$ by $M_n := \mathbb{E}[g|\,\mathcal{F}_n]$. Show that M is an $L^2([0,1])$-bounded martingale that converges in $L^2([0,1])$. What is the limit?

(b) Remind yourself that the conditional expectation $\mathbb{E}[X|\,\mathcal{G}]$ for a square integrable random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ is the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-orthogonal projection of X onto $L^2(\Omega, \mathcal{G}, \mathbb{P})$. Deduce $M \subseteq lin(\mathscr{H})$ from this (here the probability measure is just the uniform distribution on $[0,1]$).

(c) Observe that $D_{j,k} \subseteq \sigma(\mathscr{H})$ for every dyadic interval $D_{j,k} = [k/2^j, (k+1)/2^j)$, $j \in \mathbb{N}_0, 0 \le k \le 2^j - 1$ and $[0,1] \subseteq \sigma(\mathscr{H})$ and thus $\sigma(\mathscr{H}) = \mathscr{B}_{[0,1]}$.

(d) Show that parts (a), (b) and (c) imply property iv. from above.

We can therefore conclude that $\mathscr{H}$ is indeed a complete orthonormal system in $L^2([0,1])$.

Problems on this sheet give you extra points (four points for each problem). Submit before the first lecture on Thursday, 24 April 2014.

Prof. Markus Reiß
Randolf Altmeyer

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Problem sheet 2

1. Let $X = (X_t, t \in T)$, $\tilde{X} = (\tilde{X}_t, t \in T)$ be two processes on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Show that if $\tilde{X}$ is a version of X, then both processes have the same finite dimensional distributions.

   (b) Let $T = \mathbb{R}_+$. Assume now that X and $\tilde{X}$ are processes with right-continuous paths. Show that if $\tilde{X}$ is a version of X, then X and $\tilde{X}$ are indistinguishable.

2. Let $X = (X_t, t \geq 0)$ be a right-continuous process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Show that the map $(\omega, t) \mapsto X_t(\omega)$ is measurable with respect to $\big(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathscr{B}_{\mathbb{R}_+}\big)$ (X *is progressively measurable*).

   (b) Let $\tau : \Omega \to [0, \infty]$ be a finite stopping time with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of X, i.e. $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Show that X_τ is a random variable, i.e. $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathcal{F}$-measurable.

3. Prove Blumenthal's 0-1 law: For a Brownian motion $B = (B_t, t \geq 0)$ and its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ the σ-algebra $\mathcal{F}_{0+} = \bigcap_{t>0} \mathcal{F}_t$ is $\mathbb{P}$-trivial (i.e. $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{F}_{0+}$). For this prove:

   (a) For $n \in \mathbb{N}$, define σ-algebras $\mathcal{A}_n = \sigma(B_{2^{-n}+t} - B_{2^{-n}}, t \in [0, 2^{-n}])$. Show that $(\mathcal{A}_n)_{n \in \mathbb{N}}$ is an independent family of σ-algebras and that $\sigma(\mathcal{A}_m, m \geq n) = \mathcal{F}_{2^{-n+1}}$.

   (b) Review Kolmogorov's 0-1 law and apply it to deduce that $\bigcap_{n \in \mathbb{N}} \sigma(\mathcal{A}_m, m \geq n)$ is $\mathbb{P}$-trivial.

   (c) Obtain the claim by (a) and (b).

4. Define the zero set $Z = \{(\omega, t) \in \Omega \times [0, \infty) : B_t = 0\}$ of a Brownian motion $B = (B_t, t \geq 0)$ and define the sections $Z_\omega = \{t \in [0, \infty) : (\omega, t) \in Z\}$ for $\omega \in \Omega$.

   (a) Infer $Z \in \mathcal{F} \otimes \mathscr{B}_{\mathbb{R}_+}$ from exercise 2(a). Then show that $\mathbb{P}(\omega \in \Omega : \lambda(Z_\omega) = 0) = 1$ where λ is the Lebesgue measure on $\mathbb{R}$.

(b) Prove that Z_ω has for almost all $\omega \in \Omega$ an accumulation point at 0.

Hint:

i. Let $A_+ = \{B_{1/n} > 0 \text{ for infinitely many } n\}$, $A_- = \{B_{1/n} < 0 \text{ for infinitely many } n\}$. Then $\mathbb{P}(A_+) > 0$, $\mathbb{P}(A_-) > 0$.

ii. $A_+, A_- \in \mathcal{F}_{0+}$ holds.

iii. Apply Blumenthal's 0-1 law.

Submit before 13:00, Friday, 2 May 2014, in Room 1.226 (Randolf Altmeyer) or send by email to altmeyrx@math.hu-berlin.de.

Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
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Problem sheet 3

1. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space with an $(\mathcal{F}_t)_{t\geq 0}$-stopping time τ. Define $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for every } t \geq 0\}$.

   (a) $\mathcal{F}_\tau$ is a σ-algebra. If $\tau = t$ is deterministic, then $\mathcal{F}_\tau = \mathcal{F}_t$.

   (b) There exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \searrow \tau$ almost surely and such that each τ_n takes only finitely many values. Moreover, in this case $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau_n}$, if $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.

   (c) Let $X = (X_t, t \geq 0)$ be a right-continuous process adapted to $(\mathcal{F}_t)_{t\geq 0}$ and assume that τ is bounded. Then $X_\tau \in \mathcal{F}_\tau$.

2. Let $X = (X_t, t \geq 0)$ be a right-continuous process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $X_t \in L^1$ for all $t \geq 0$. Then the following statements are equivalent:

   (a) X is a martingale.

   (b) For all bounded stopping times τ we have $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

   (c) For all bounded stopping times $\sigma \leq \tau$ we have $\mathbb{E}\big[X_\tau \big| \mathcal{F}_\sigma\big] = X_\sigma$.

   Prove these equivalences using the following steps:

   - (a) $\Rightarrow$ (b): For $\tau \leq c$, $c \geq 0$, find a sequence $(\tau_n)_{n\in\mathbb{N}}$ as in 1(b) and show $X_{\tau_n} = \mathbb{E}\big[X_c \big| \mathcal{F}_{\tau_n}\big]$, in particular, $\mathbb{E}[X_{\tau_n}] = \mathbb{E}[X_c]$. Argue that $(X_{\tau_n})_{n\in\mathbb{N}}$ is uniformly integrable and obtain the statement by right-continuity of X.
   - (b) $\Rightarrow$ (c): For $A \in \mathcal{F}_\sigma$ consider the stopping time (!)

   $$\tilde{\sigma}(\omega) := \begin{cases} \sigma(\omega), & \omega \in A, \\ \tau(\omega), & \omega \in A^c \end{cases}$$

   and apply (b) to obtain $\mathbb{E}[X_{\tilde{\sigma}}] = \mathbb{E}[X_0] = \mathbb{E}[X_\tau]$.

Submit before the first lecture on Thursday, 8 May 2014.

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Problem sheet 4

1. Let $(B_t^1, 0 \leq t \leq 1)$ and $(B_t^2, 0 \leq t \leq 1)$ be two independent Brownian motions and let $(A_t, 0 \leq t \leq 1)$ be a continuous and bounded process with $V_{[0,1]}(A(\omega)) \leq C < \infty$ for some $C > 0$ and all $\omega \in \Omega$.

   (a) Show that the quadratic variation of A on $[0,1]$ vanishes, i.e. we have $S_1^n \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ for $S_t^n := \sum_{t_i \in \tau_n, t_i \leq t} (A_{t_{i+1}} - A_{t_i})^2$, $t \in [0,1]$, and a sequence of partitions $(\tau_n)_{n \in \mathbb{N}}$ of $[0,1]$, with $\max_{t_i \in \tau_n} |t_{i+1} - t_i| \xrightarrow{n \to \infty} 0$.

   (b) For $\sigma_1, \sigma_2 \in \mathbb{R}$ show that the quadratic variation of $\sigma_1 B_t^1 + \sigma_2 B_t^2$ satisfies $S_t^n \xrightarrow{\mathbb{P}} \left(\sigma_1^2 + \sigma_2^2\right) t$ as $n \to \infty$.

2. (a) Let $(X_t, t \geq 0)$ be a process with independent increments and $X_t \in L^1$ for all $t \geq 0$. Show that $\left(X_t - \mathbb{E}\left[X_t\right], t \geq 0\right)$ is a martingale with respect to its natural filtration.

   (b) Let $(N_t, t \geq 0)$ be a Poisson process of intensity $\lambda > 0$ (definition in the lecture notes of Stochastic processes I). Infer from (a) that $M_t = N_t - \lambda t$ is a martingale with respect to its natural filtration.

   (c) Find a continuous increasing process $(A_t, t \geq 0)$ such that $Y_t = M_t^2 - A_t$ is a martingale with respect to the natural filtration of M.

3. Consider the process $X_t = \mu t + \sigma B_t$, $t \geq 0$, for a Brownian motion B, *drift* $\mu \in \mathbb{R}$ and *volatility* $\sigma > 0$.

   (a) For $a < 0 < b$ calculate the probability of X hitting b before a.
   Hint: For $\mu \neq 0$ choose $\alpha \in \mathbb{R}$ such that $(\exp(\alpha X_t), t \geq 0)$ is a martingale.

   (b) Show for $\mu < 0$ that $Y := \sup_{t \geq 0} X_t$ is a.s. finite and exponentially distributed with parameter $\lambda = -2\mu/\sigma^2$.

4. Let $(X_t, t \geq 0)$ be a continuous process with $X_0 = 0$.

(a) Assume that $Y_t^\alpha = \exp(\alpha X_t - \alpha^2 t/2)$, $t \geq 0$, is a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ for every $\alpha \in \mathbb{R}$. Show that X is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Hints: If Z is a random variable, then

- $Z \sim N(0,1)$ if and only if $\mathbb{E}\left[e^{\lambda Z}\right] = e^{\lambda^2/2}$ for all $\lambda \in \mathbb{R}$
- if $\mathbb{E}\left[e^{\lambda Z} \middle| \mathcal{G}\right] = \mathbb{E}\left[e^{\lambda Z}\right] < \infty$ for λ in a neighborhood of 0, then $Z \perp \mathcal{G}$ for a sub-σ-algebra $\mathcal{G}$

(b) For $\alpha \neq 0$ show that Y_t^α converges a.s. to $Y_\infty^\alpha = 0$ as $t \to \infty$. Does $Y_t^\alpha \xrightarrow{L^1} Y_\infty^\alpha$ hold?

Submit before the first lecture on Thursday, 15 May 2014.

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Problem sheet 5

1. Let $(B_t, t \geq 0)$ be a Brownian motion and consider the process $\tau_b = \inf\{t > 0 : B_t = b\}$ for $b \geq 0$.

   (a) Show that $\tau_b - \tau_a = \inf\{t > 0 : B_{t+\tau_a} - B_{\tau_a} = b - a\}$ holds for $a \leq b$.

   (b) Use the strong Markov property of Brownian motion to show that $(\tau_b, b \geq 0)$ has stationary and independent increments.

   Hint: Use the fact that the Laplace transform of a non-negative random variable uniquely determines its distribution.

   (c) Show that almost all paths $b \mapsto \tau_b$ are increasing and left-continuous. Are they also right-continuous?

   (d) *2 extra points*: Simulate the process $(\tau_b, b \geq 0)$.

2. Let $(B_t, t \geq 0)$ be a Brownian motion. Prove that the zero set $Z = \{(\omega, t) \in \Omega \times [0, \infty) : B_t = 0\}$ of problem 2.4 is for almost all ω a *perfect set*, i.e. Z_ω is closed and contains no isolated points.

   Hint: Review the results of problem 2.4 and use the strong Markov property of Brownian motion for stopping times (!) $\sigma_q = \inf\{t > q : B_t = 0\}$, $q \in \mathbb{Q}_+$.

3. Let $(M_t, t \geq 0)$ be a continuous local martingale. Show that $(M_t, t \geq 0)$ is a martingale if and only if for every $a > 0$ the family

$$\{M_\tau : \tau \leq a \quad \text{is a bounded stopping time}\}$$

   is uniformly integrable.

4. Let $X_t = \sum_{k=0}^{\infty} \alpha_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t)$, $t > 0$, with $X_0 = 0$ be a simple process, where $(\tau_k)_{k \in \mathbb{N} \cup \{\tau_0\}}$, $\tau_0 = 0$, is a sequence of $(\mathcal{F}_t)_{t \geq 0}$-stopping times such that $\tau_k \to \infty$ a.s. as $k \to \infty$ and where the α_k are $\mathcal{F}_{\tau_k}$-measurable random variables.

   (a) Let $(M_t, t \geq 0)$ be a continuous and L^2-integrable $(\mathcal{F}_t)_{t \geq 0}$-martingale and let $(X_t, t \geq 0)$ be bounded. Show that the *stochastic integral*

$$(X \circ M)_t = \sum_{k=0}^{\infty} \alpha_k (M_{\tau_{k+1} \wedge t} - M_{\tau_k \wedge t}), \quad t \geq 0,$$

   is again a continuous L^2-integrable $(\mathcal{F}_t)_{t \geq 0}$-martingale.

(b) If $(M_t, t \geq 0)$ is only a continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingale and if $(X_t, t \geq 0)$ is not necessarily bounded, then $((X \circ M)_t, t \geq 0)$ is again a continuous $(\mathcal{F}_t)_{t\geq 0}$-local martingale.

Submit before the first lecture on Thursday, 22 May 2014.

Prof. Markus Reiß
Randolf Altmeyer

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Problem sheet 6

1. Let $(\mathcal{F}_t)_{0\leq t\leq T}$, $T>0$, be a complete filtration and let $\mathcal{M}^2_{c,T}$ be the set of all continuous $(\mathcal{F}_t)_{0\leq t\leq T}$-martingales $(M_t, 0\leq t\leq T)$ with $M_0=0$, $M_t\in L^2$ for all $0\leq t\leq T$.

   (a) Show that $\mathcal{M}^2_{c,T}$ with scalar product $\langle M,N\rangle_{\mathcal{M}^2_{c,T}} = \mathbb{E}[M_T N_T]$ for $M,N\in\mathcal{M}^2_{c,T}$ is a Hilbert space. In particular, it holds that $\|M\|_{\mathcal{M}^2_{c,T}} = (\mathbb{E}[\langle M\rangle_T])^{1/2}$ for $M\in\mathcal{M}^2_{c,T}$, where $\langle M\rangle$ is the quadratic variation of M.

   (b) Prove for $M,N\in\mathcal{M}^2_{c,T}$ that the following properties are equivalent:

      i. $\mathbb{E}[M_t N_s]=0$ for all $0\leq s,t\leq T$ (M and N are *weakly orthogonal*),

      ii. $\mathbb{E}[M_s N_s]=0$ for all $0\leq s\leq T$,

      iii. $\mathbb{E}[M_\tau N_s]=0$ for all $0\leq s\leq T$ and all stopping times $s\leq\tau\leq T$.

2. Prove: For every continuous local martingale M with $M_0=0$ there exists a unique (up to indistinguishability) increasing continuous process $(\langle M\rangle_t, t\geq 0)$ such that $\langle M\rangle_0=0$ and $(M_t^2-\langle M\rangle_t, t\geq 0)$ is a local martingale.

3. Let $X:[0,1]\to\mathbb{R}$ be a function. For any partition $\pi=\{0=t_0<t_1<\cdots<t_m=1\}$ of $[0,1]$ and any $h\in C([0,1])$ define

$$S_\pi(h)=\sum_{k=1}^m h(t_k)\left(X_{t_k}-X_{t_{k-1}}\right).$$

   (a) Prove that the map $h\mapsto S_\pi(h)$ is a continuous linear form on $C([0,1])$ with norm $\|S_\pi\|=\sum_{k=1}^m |X_{t_k}-X_{t_{k-1}}|$.

   (b) Prove: If $(S_{\pi_n}(h))_n\in\mathbb{N}$ converges to a finite limit for every $h\in C([0,1])$ and any sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ with $\max_{t_k\in\pi_n}|t_k-t_{k-1}|\xrightarrow{n\to\infty}0$, then X is of bounded variation. This shows why the stochastic integral with respect to a continuous local martingale cannot be defined in the ordinary way.

Hint: Apply the Banach-Steinhaus theorem (see e.g. *Werner, Funktionalanalysis*).

4. Let $(B_t, t \geq 0)$ be a Brownian motion. Prove the following identities for any $t \geq 0$:

   (a) $\int_0^t B_s \, dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$,

   (b) $\int_0^t B_s^2 \, dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s \, ds$.

   Moreover, it holds that $\int_0^t B_s^3 \, dB_s = \frac{1}{4}B_t^4 - \frac{3}{2}\int_0^t B_s^2 \, ds$. Can you guess a formula for $\int_0^t B_s^n \, dB_s$, $n \in \mathbb{N}$?

Submit before 13:00, Friday, 30 May 2014, in Room 1.226 (Randolf Altmeyer) or send by email to altmeyrx@math.hu-berlin.de.

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Problem sheet 7

1. Show the following properties of the stochastic integral for $M \in \mathcal{M}_c^2$, $X \in \mathscr{L}(M)$, by approximation with simple processes:

   (a) *(linearity)* $\forall \alpha, \beta \in \mathbb{R}, Y \in \mathscr{L}(M)$: $\int_0^\cdot (\alpha X + \beta Y)_s \, dM_s = \alpha \int_0^\cdot X_s \, dM_s + \beta \int_0^\cdot Y_s \, dM_s$,

   (b) *(Itô-isometry)* $\mathbb{E}\big[\big(\int_0^t X_s \, dM_s\big)^2\big] = \mathbb{E}\big[\int_0^t X_s^2 \, d\langle M\rangle_s\big] = \|X\|_{M,t}^2$ and $\|\int_0^\cdot X_s \, dM_s\|_{\mathcal{M}_c^2} = \|X\|_M$,

   (c) *(quadratic variation)* $\big\langle \int_0^\cdot X_s \, dM_s \big\rangle_t = \int_0^t X_s^2 \, d\langle M\rangle_s$, $t \geq 0$.

2. Let $(B_t, t \geq 0)$ be a Brownian motion. Define for $h \in C^1(\mathbb{R})$ the process

$$I_t(h) = h(t) B_t - \int_0^t B_s h'(s) \, ds, \quad t \geq 0.$$

   (a) Show for each $t \geq 0$: $\|I_t(h)\|_{L^2(\mathbb{P})} = \|h \mathbf{1}_{[0,t]}\|_{L^2(\mathbb{R})} = \|h\|_{L^2([0,t])}$.

   (b) How can we define I_t for $h \in L^2(\mathbb{R})$?

   Does $\mathbb{P}(\int_0^t h(s) \, dB_s = I_t(h), \forall t \geq 0) = 1$ hold for all $h \in L^2(\mathbb{R})$, where $\int_0^t h(s) \, dB_s$ is the stochastic integral from the lecture?

Submit before the first lecture on Thursday, 5 June 2014.

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Problem sheet 8

1. For $\lambda \in \mathbb{R}$, an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t, t \geq 0)$ and an $\mathcal{F}_0$-measurable random variable X_0 define the *Ornstein-Uhlenbeck* process

$$X_t = X_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\, dB_s, \quad t \geq 0.$$

   (a) Show that $(X_t, t \geq 0)$ satisfies the *stochastic differential equation* $dX_t = -\lambda X_t dt + dB_t$, i.e. it satisfies the equation

$$X_t = X_0 - \lambda \int_0^t X_s\, ds + B_t, \quad t \geq 0, \text{ a.s.}$$

   (b) For $\lambda > 0$, $X_0 \sim N(0, \frac{1}{2\lambda})$ and X_0 independent of $(B_t, t \geq 0)$ show that $(X_t, t \geq 0)$ is a stationary Gaussian process.

   (c) *2 extra points*: Simulate 100 trajectories of $(X_t, t \geq 0)$ for $\lambda \in \{-1; 0.01; 1\}$.

2. Let $(B_t, t \geq 0)$ be a Brownian motion. Assume there exists a sequence of processes $(B_t^{(n)}, t \geq 0)$ such that $t \mapsto B_t^{(n)}$ is a C^1-function and such that $\lim_{n\to\infty} \sup_{0\leq t\leq T} |B_t^{(n)} - B_t| = 0$ for all $T \geq 0$.

   (a) Show for a continuous function f and all $n \in \mathbb{N}$, $t \geq 0$ that

$$\int_0^t f(B_s^{(n)})\, dB_s^{(n)} = F(B_t^{(n)}) - F(B_0^{(n)}),$$

   where $F(x) = \int_0^x f(y)\, dy$. Conclude that there exists a continuous process $\int_0^t f(B_s) \circ dB_s := \lim_{n\to\infty} \int_0^t f(B_s^{(n)})\, dB_s^{(n)}$, $t \geq 0$, which does not depend on the approximating sequence $(B^{(n)}, t \geq 0)$, $n \in \mathbb{N}$.

   (b) Show that if $f \in C^1$, then

$$\int_0^t f(B_s) \circ dB_s = \int_0^t f(B_s)\, dB_s + \frac{1}{2}\int_0^t f'(B_s)\, ds.$$

3. For two continuous semimartingales X and Y define the *quadratic covariation* as

$$\langle X, Y \rangle_t := \frac{1}{4}\left(\langle X+Y \rangle_t - \langle X-Y \rangle_t\right), \quad t \geq 0.$$

   Show the following properties:

(a) $\langle X, X\rangle_t = \langle X\rangle_t$, $t \geq 0$.

(b) For all sequences of partitions $(\pi_n)_{n\in\mathbb{N}}$ of $[0,T]$ with $|\pi_n| \to 0$ we have for all $0 \leq t \leq T$:

$$\langle X, Y\rangle_t = \lim_{n\to\infty} \sum_{t_k\in\pi_n} \left(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)\left(Y_{t_k\wedge t} - Y_{t_{k-1}\wedge t}\right) \text{ (in probability).}$$

(c) $|\langle X, Y\rangle_t| \leq \langle X\rangle_t^{1/2}\langle Y\rangle_t^{1/2}$. In particular, if A is continuous and of bounded variation, then $\langle X, A\rangle_t = 0$, $t \geq 0$.

4. Let $(B_t, t \geq 0)$ be a Brownian motion. For $\varepsilon_0 > \varepsilon > 0$ let $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a continuous approximation of $x \mapsto \varepsilon^{-1}\mathbf{1}_{[-\varepsilon,\varepsilon]}(x)$ such that $0 \leq \varphi_\varepsilon(x) \leq \varepsilon^{-1}$ for all $x \in \mathbb{R}$, $\varphi_\varepsilon|_{[-\varepsilon,\varepsilon]} = \varepsilon^{-1}$ and $\operatorname{supp}(\varphi_\varepsilon) \subseteq [-\varepsilon_0, \varepsilon_0]$, where $\varepsilon_0 - \varepsilon = o(\varepsilon)$. We further define $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $f_\varepsilon(0) = f_\varepsilon'(0) = 0$ and $f_\varepsilon'' = \varphi_\varepsilon$.

(a) Show $f_\varepsilon(x) \to |x|$ and $f_\varepsilon'(x) \to \operatorname{sgn}(x)$ as $\varepsilon \to 0$ for $x \in \mathbb{R}$ for an appropriate choice of ε_0, where $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(x) = 1$ if $x > 0$.

(b) Conclude for all $t \geq 0$ that $|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t$ a.s. with $L_t = \lim_{\varepsilon\to 0} \frac{1}{2}\int_0^t f_\varepsilon''(B_s)\, ds$ in $L^2(\mathbb{P})$.

(c) Find conditions on ε_0 such that $L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\lambda(\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\})$ in $L^2(\mathbb{P})$.

Submit before the first lecture on Thursday, 12 June 2014.

Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
Summer 2014
Humboldt-Universität zu Berlin

Problem sheet 9

1. Let $(M_t, t \geq 0)$ be a non-negative continuous local martingale.

   (a) Prove that M is a supermartingale.
   Hint: Use Fatou's lemma.

   (b) Show that M is a martingale, if and only if $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t \geq 0$.

2. Let $(B_t, t \geq 0)$ be a Brownian motion and let $X \in \mathscr{L}_{loc}(B)$. Consider the *stochastic exponential*

$$Z_t = \exp\left(\int_0^t X_s \, dB_s - \frac{1}{2}\int_0^t X_s^2 \, ds\right), \quad t \geq 0.$$

   (a) Apply Itô's formula to $M_t = \int_0^t X_s \, dB_s$ and show that

$$e^{M_t} = 1 + \int_0^t e^{M_s} X_s \, dB_s + \frac{1}{2}\int_0^t e^{M_s} X_s^2 \, ds, \quad t \geq 0, \textit{ a.s.}$$

   (b) Argue that $Z_t e^{-M_t} = 1 - \frac{1}{2}\int_0^t X_s^2 Z_s e^{-M_s} ds$, $t \geq 0$, *a.s.* and show by partial integration that

$$Z_t = 1 + \int_0^t Z_s X_s \, dB_s, \quad t \geq 0.$$

   Conclude that Z is a local martingale and by problem 1 a supermartingale.

   (c) Is there a (non-trivial) process $X \in \mathscr{L}_{loc}(B)$ such that Z is even a martingale?

3. Let $(B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^3$, $0 \neq x \in \mathbb{R}^3$ and define the process $M_t = \frac{1}{|x+B_t|}$, $t \geq 0$. M_t is a.s. well defined because the Brownian motion in $\mathbb{R}^3$ does not hit points a.s. (see lecture).

   (a) Show that M is a continuous local martingale.
   Hint: Apply Itô's formula.

   (b) Prove that M is L^2-bounded, i.e. $\sup_{t\geq 0} \mathbb{E}[|M_t^2|] < \infty$.
   Hint: For $t \geq 0$ show

$$\mathbb{E}\left[|M_t|^2 \mathbf{1}_{\left\{|M_t| \geq \frac{2}{|x|}\right\}}\right] \leq (2\pi t)^{-\frac{3}{2}} \int_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{(|y| - |x|)^2}{2t}\right) dy.$$

(c) Show that M is not a martingale.

Hint: Prove that $\mathbb{E}[M_t] \to 0$ as $t \to \infty$.

4. Let $M \in \mathcal{M}_c^2$, $X \in \mathscr{L}(M)$ and $Y \in \mathscr{L}(X \circ M)$ with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$.

   (a) Show $YX \in \mathscr{L}(M)$.

   (b) Let $\tau_1 \leq \tau_2$ be $(\mathcal{F}_t)_{t \geq 0}$-stopping times and let ξ be a bounded $\mathcal{F}_{\tau_1}$-measurable random variable. Show for $Y_t = \xi \mathbf{1}_{(\tau_1, \tau_2]}(t)$, $t \geq 0$, that

   $$\int_0^t Y_s X_s \, dM_s = \xi \int_0^t X_s \mathbf{1}_{(\tau_1, \tau_2]}(s) \, dM_s, \quad a.s.$$

   Conclude in this case that

   $$(Y \circ (X \circ M)) = ((YX) \circ M), \quad a.s. \tag{*}$$

   Hint: Show (*) first for X simple and bounded and then by approximation for general $X \in \mathscr{L}(M)$.

   (c) Prove (*) for general $Y \in \mathscr{L}(X \circ M)$.

Submit before the first lecture on Thursday, 19 June 2014.

Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
Summer 2014
Humboldt-Universität zu Berlin

Problem sheet 10

1. Prove that the 2D-Brownian motion a.s. does not hit a fixed point $x \in \mathbb{R}^2 \setminus \{0\}$, i.e. $\mathbb{P}(\tau_x < \infty) = 0$ for $\tau_x = \inf\{t > 0 : B_t = x\}$ where $(B_t, t \geq 0)$ is a Brownian motion in R^2.

2. Let $G \subseteq \mathbb{R}^d$ be a bounded and open set and let $f : \partial G \to \mathbb{R}$ be continuous. Let $h : \bar{G} \to \mathbb{R}$ be a solution to the Dirichlet problem on G with boundary value f, i.e. $h \in C^2(G) \cap C(\bar{G})$, $\triangle h = 0$ on G and $h = f$ on ∂G.

   (a) Let $(B_t, t \geq 0)$ be a d-dimensional Brownian motion and let τ=inf $\{t > 0 : x + B_t \in G^c\}$ be a stopping time for any $x \in G$. Why does $\mathbb{P}(\tau < \infty) = 1$ hold?

   (b) Show for every $x \in G$ that h satisfies $h(x) = \mathbb{E}\left[f\left(x + B_\tau\right)\right]$. In particular, the solution of the Dirichlet problem with boundary value f is unique.
   Hint: Consider open sets $G_n = \{y \in G : \inf_{z \in \partial G} \|y - z\| > 1/n\} \subseteq G$ and corresponding stopping times $\tau_n = \inf\{t > 0 : x + B_t \in G_n^c\}$. Use Itô's formula for the stopped processes $(x + B_{t \wedge \tau_n}, t \geq 0)$ and take expectations.

   (c) Describe a stochastic algorithm to determine $h(x)$ numerically.

   (d) *2 extra points*: Check if your algorithm works by approximating the solution of the Dirichlet problem on $D_{r,R} = \left\{x \in \mathbb{R}^2 : r < |x| < R\right\}$ with boundary value f, where $f|_{\bar{B}_r(0)} = 1$ and $f|_{\bar{B}_R(0)} = 0$ for closed balls $\bar{B}_r(0)$, $\bar{B}_R(0)$ and any $r, R > 0$. Compare your approximation to the exact solution from the lecture (cf. example 3.15).

3. Prove the lower bound of the Burkholder-Davis-Gundy-inequality (BDG): For any continuous local martingale $(M_t, t \geq 0)$ with $M_0 = 0$ and any $p \geq 4$ there exists a universal constant $c_p > 0$ (depending only on p) such that for all $t \geq 0$

$$c_p \mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq \mathbb{E}\left[(M_t^*)^p\right],$$

   where $M_t^* = \sup_{0 \leq s \leq t} |M|_s$. Use the following steps:

   (a) Assume first that M and $\langle M \rangle$ are bounded. Use the equality $M_t^2 = 2\int_0^t M_s \, dM_s + \langle M \rangle_t$ to show

$$\mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq \tilde{c}_p \left(\mathbb{E}\left[(M_t^*)^p\right] + \mathbb{E}\left[\left|\int_0^t M_s \, dM_s\right|^{p/2}\right]\right)$$

   for some constant $\tilde{c}_p > 0$ and apply the upper bound of the BDG-inequality to the local martingale $\int_0^\cdot M_s \, dM_s$.

(b) Conclude the general result by localisation.

4. Let $(B_t, t \geq 0)$ be a Brownian motion and let $(\mathcal{F}_t^0)_{t\geq 0}$ be its natural filtration. For each of the three processes (i) $M_t = B_t^2 - t$, (ii) $M_t = e^{\lambda B_t - \frac{\lambda^2}{2}t} - 1$, $\lambda \in \mathbb{R}$, and (iii) $M_t = B_{t\wedge\tau}$ where τ is any $(\mathcal{F}_t^0)_{t\geq 0}$-stopping time, $t \geq 0$,

   (a) find a process $Y \in \mathscr{L}_{loc}(B)$ adapted to $(\mathcal{F}_t^0)_{t\geq 0}$ such that $M_t = \int_0^t Y_s \, dB_s$ a.s., $t \geq 0$, and

   (b) determine the *Dambis-Dubins-Schwarz (DDS) Brownian motion of* M, i.e. determine $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\tau_t = \inf\{s \geq 0 : \langle M \rangle_s > t\}$, $t \geq 0$, and construct an $(\tilde{\mathcal{F}}_t)_{t\geq 0} = (\mathcal{F}_{\tau_t})_{t\geq 0}$-Brownian motion $(\tilde{B}_t, t \geq 0)$ such that $M_t = \tilde{B}_{\langle M \rangle_t}$. Do we have $\mathbb{P}(B_t = \tilde{B}_t \, \forall t \geq 0) = 1$?

Submit before the first lecture on Thursday, 26 June 2014.

Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
Summer 2014
Humboldt-Universität zu Berlin

Problem sheet 11

1. Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures (i.e. $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$) on a measurable space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_t)_{0\le t\le T}$, $T > 0$, and denote by $\mathbb{P}_t$ and $\mathbb{Q}_t$ the probability measures restricted to $\mathcal{F}_t$.

   (a) Show that $\mathbb{P}_t$ and $\mathbb{Q}_t$, $0 \le t \le T$, are also equivalent and that the Radon-Nikodym-derivatives satisfy $\mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right] = \frac{d\mathbb{Q}_t}{d\mathbb{P}_t}$ $\mathbb{P}_t$-a.s. In particular, $\left(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right)_{0\le t\le T}$ is a $\mathbb{P}$-martingale.

   (b) *(Bayes rule)* For any $\mathcal{F}_T$-measurable random variable X with $\mathbb{E}_{\mathbb{Q}}[|X|] < \infty$ prove that

   $$\mathbb{E}_{\mathbb{Q}}\left[X\middle|\mathcal{F}_t\right] = \left(\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}\right)^{-1}\mathbb{E}_{\mathbb{P}}\left[X\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\middle|\mathcal{F}_t\right], \quad \mathbb{P}\text{-}a.s. \text{ and } \mathbb{Q}_T\text{-}a.s.$$

   (c) Show that a process $(M_t, 0 \le t \le T)$ is a $\mathbb{Q}$-martingale if and only if $(M_t\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}, 0 \le t \le T)$ is a $\mathbb{P}$-martingale.

2. Let $(B_t, t \ge 0)$ be a Brownian motion and let $(\mathcal{F}_t^0)_{t\ge 0}$ be its natural filtration completed by events of probability zero in $\sigma(B_t, t \ge 0)$. Let further ξ be an $L^2(\Omega, \mathcal{F}_T^0, \mathbb{P})$-random variable for $T > 0$ and let $(X_t, 0 \le t \le T)$ be a progressively measurable process adapted to $(\mathcal{F}_t^0)_{t\ge 0}$ such that $\mathbb{E}[\int_0^T |X_s|^2\, ds] < \infty$.

   (a) Find $(\mathcal{F}_t^0)_{0\le t\le T}$-adapted processes $(Y_t, 0 \le t \le T)$ and $(Z_t, 0 \le t \le T)$ with $Z \in \mathscr{L}(B)$ satisfying the *backward stochastic differential equation (BSDE)*

   $$\begin{cases} dY_t &= -X_t dt + Z_t dB_t, \\ Y_T &= \xi, \end{cases}$$

   with terminal value ξ such that $\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^2\right] < \infty$.

   Hint: Apply the martingale representation theorem to $\xi + \int_0^T X_s\, ds$ to find Z. Apply the Doob inequality for proving the bound on $\sup_{0\le t\le T}|Y_t|$.

   (b) Is the solution unique?

3. Let $(B_t, t \geq 0)$ and $(\mathcal{F}_t^0)_{t\geq 0}$ be as in problem 2 and let $\mu \in \mathbb{R}\setminus\{0\}$. Let further $\mathbb{Q}_T$ be the unique probability measure such that $\tilde{B}_t = B_t - \mu t$, $t \geq 0$, is a Brownian motion under $\mathbb{Q}_T$ for $T > 0$. Show for the stopping time $\tau_b = \inf\left\{t \geq 0 : \tilde{B}_t + \mu t = b\right\}$, $b \in \mathbb{R}$:

   (a) τ_b has Lebesgue-density $f_b(t) = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{(b-\mu t)^2}{2t}} \mathbf{1}_{[0,\infty)}(t)$ under $\mathbb{Q}_T$,

   (b) $\mathbb{Q}_T(\tau_b < \infty) = \exp\left(\mu b - |\mu b|\right)$,

   (c) $\mathbb{E}_{\mathbb{Q}_T}\left[e^{-\lambda \tau_b}\right] = e^{\mu b - |b|\sqrt{\mu^2 + 2\lambda}}$ for $\lambda > 0$.

4. Let $(B_t, t \geq 0)$ and $(\mathcal{F}_t^0)_{t\geq 0}$ be as in problem 2. Construct a measurable and $(\mathcal{F}_t^0)_{t\geq 0}$-adapted process $(X_t, t \geq 0)$ such that for some $0 < \varepsilon < \frac{1}{2}$

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{2} - \varepsilon\right) \int_0^\infty X_s^2 \, ds\right)\right] < \infty,$$

   and such that the stochastic exponential $Z_t = \exp\left(\int_0^t X_s \, dB_s - \frac{1}{2}\int_0^t X_s^2 \, ds\right)$ is not a closed martingale. Use the following steps:

   (a) Show using problem 3 that the stopping time $\tau_a = \inf\{t \geq 0 : B_t - (1-\varepsilon)t = -a\}$ for $a > 0$ satisfies $\mathbb{E}\left[e^{\left(\frac{1}{2}-\varepsilon\right)\tau_a}\right] = e^{(1-2\varepsilon)a}$.
   Hint: Use that $\tau_a \overset{d}{=} \sigma_a := \inf\{t \geq 0 : B_t + (1-\varepsilon)\, t = a\}$.

   (b) Establish the inequality $\mathbb{E}\left[e^{B_{\tau_a} - \frac{1}{2}\tau_a}\right] < 1$.

   (c) Construct X.

   2 extra points: Construct a process $(X_t, 0 \leq t \leq 1)$ satisfying for some $0 < \varepsilon < \frac{1}{2}$

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{2} - \varepsilon\right) \int_0^1 X_s^2 \, ds\right)\right] < \infty,$$

   such that the corresponding stochastic exponential is *not* a martingale on $[0, 1]$.

Submit before the first lecture on Thursday, 03 July 2014.

Prof. Markus Reiß
Randolf Altmeyer

Course *Stochastic Analysis*
Summer 2014
Humboldt-Universität zu Berlin

Problem sheet 12

1. Consider the process $X_t = B_t + \mu t$, $t \in [0, T]$, $T > 0$, for a a Brownian motion $(B_t, t \geq 0)$ and $\mu \in \mathbb{R}$.

   (a) For unknown μ show that the *maximum likelihood estimator* satisfies $\hat{\mu}_T = \frac{X_T}{T}$.

   (b) Discuss expectation and variance of the estimator $\hat{\mu}_T$ (i) for growing sample size but fixed time horizon $T > 0$ and (ii) for fixed sample size but with $T \to \infty$.

2. Let $T > 0$ and let $u, a, b : [0, T] \to \mathbb{R}_+$ be measurable functions such that u is bounded and b is integrable.

   (a) Prove *Gronwall's lemma*: If

   $$u(t) \leq a(t) + \int_0^t u(s)b(s)\,ds, \quad t \in [0, T], \tag{1}$$

   then

   $$u(t) \leq a(t) + \int_0^t a(s)b(s) \exp\left(\int_s^t b(r)\,dr\right) ds, \quad t \in [0, T].$$

   (b) Show for constant $a(t) = c \geq 0$ under condition (1) that

   $$u(t) \leq c \exp\left(\int_0^t b(s)\,ds\right).$$

3. Let $a, x_0 \in \mathbb{R}$ and let $b, \sigma : \mathbb{R}_+ \to \mathbb{R}$ be deterministic and (Borel-)measurable.

   (a) Show that

   $$X_t = e^{at}\left(x_0 + \int_0^t e^{-as} b_s\,ds + \int_0^t e^{-as} \sigma_s\,dB_s\right)$$

   is the unique strong solution of the SDE $dX_t = (aX_t + b_t)\,dt + \sigma_t\,dB_t$, $t \geq 0$, with $X_0 = x_0$.

   (b) *2 extra points*: How can we extend the solution if a depends on time?

4. Consider the SDE $dX_t = (1-t)^{-1} X_t\, dt + dB_t$, $0 \le t < 1$, $X_0 = 0$. Show that the unique strong solution on $[0,T]$, $T < 1$, is given by a *Brownian Bridge*. Do we have $\lim_{t\to 1} X_t = 0$ in any mode of convergence?

   Hint: Consider $X_t = (1-t)\int_0^t (1-s)^{-1}\, dB_s$.

Submit before the first lecture on Thursday, 10 July 2014.