5. Aufgabenblatt zum Stochastik–Praktikum

Variance reduction - The stratification method

The idea behind stratification is to constrain observations to belong to specific subsets (strata). The method could be efficient, but full stratification of a multi-dimentional random vector could be very costly. For computing option prices in finance (given by suitable expectations), much of the variability of the option's payoff can often be reduced by stratifying the terminal value, even if the payoff (the random variable whose expectation we are to compute) is dependent on the whole path of the asset price. This may be seen as a motivation for the questions of this sheet. Let $B = (B_t)_{t\in[0,T]}$ be a standard Brownian Motion, starting at $B_0 = 0$. We are going to look for an Monte-Carlo estimator of $\theta = E[\Phi(B)]$ for the expectation of the random variable $\Phi(B) := \max_{l=0,\dots,N} B_{t_l}$, with a time grid $\{t_l\}$ as defined in part 2).

Question 1) (Stratification of a Gaussian r.v.)

As preparation for b), we aim to stratify a standard Gaussian random variable. Let X be a random variable with law $N(0,1)$. Let $(\Delta_i)_{1\le i\le m}$ be a finite partition of $\mathbb{R}$. This partition will represent our strata. Let $\widetilde{X}_i \sim \mathcal{L}(X|X \in [a_i, b_i])$, i.e. let $\widetilde{X}_i$ be distributed according to the conditional law of X given that X belongs to the stratum $\Delta_i := [a_i, b_i]$. For any i then M_i i.i.d. simulations of $\widetilde{X}_i$ can be generated as follows:

- Generate M_i i.i.d. random variables $(\widetilde{U}_{i,k})_{1\le k\le M_i}$ according to the law $U(F(a_i), F(b_i))$, where F is the cumulative distribution function (CDF) of $X \sim N(0,1)$.
- For each k in $\{1,\dots,M_i\}$, set $\widetilde{X}_{i,k} := F^{-1}(\widetilde{U}_{i,k})$.

(a) Write a Matlab script wich simulates the random variable X by stratification, using 500 overall simulations ($\sum_i M_i = 500$) and 100 equiprobable strata, with all $M_i = 5$.

(a) Plot a histogram showing the resulting empirical distribution of the stratified sample.

Question 2) (Terminal stratification)

In this part, we aim to implement a variance reduction by stratification when computing $\theta = E[\Phi(B)]$. To this end, we stratify the terminal value B_{t_N} of the underlying stochastic process B. The method of Question 1) is easily adapted to stratify an $N(0,T)$-distributed random variable, where $T \in (0,+\infty)$ is a time horizon.
We use the following result on the conditional law of the Brownian motion: Let $\pi := \{t_0, \dots, t_N\}$ be a finite partition of the time interval $[0,T]$. For simplicity, we take a uniform time grid $t_l := \frac{l}{N}T$, for $l = 0, \dots, N$. Since for each t_l $(0 < l < N)$, the conditional law of B_{t_l}, given $B_{t_{l-1}}$ and B_{t_N}, is

$$N\Big(\frac{(t_N - t_l)}{t_N - t_{l-1}} B_{t_{l-1}} + \frac{(t_l - t_{l-1})}{t_N - t_{l-1}} B_{t_N}, \frac{(t_N - t_l)(t_l - t_{l-1})}{t_N - t_{l-1}}\Big),$$

one can simulate one (discrete, in π) path of the Brownian motion B by the following scheme: Given $B_{t_0} = 0$ and B_{t_N} (simulated by stratification!), let for $l = 1, \dots, N-1$

$$B_{t_l} := \frac{(t_N - t_l)}{t_N - t_{l-1}} B_{t_{l-1}} + \frac{(t_l - t_{l-1})}{t_N - t_{l-1}} B_{t_N} + \sqrt{\frac{(t_N - t_l)(t_l - t_{l-1})}{t_N - t_{l-1}}} Z_l,$$

where Z_l are i.i.d. $N(0,1)$ random variables[1].

We are seeking to compute $E[\Phi(B)]$ for $\Phi(B) := \max_{0 \le l \le N}(B_{t_l})$ based on simulated (discrete, in π) paths of the Brownian motion B, by using stratification of the terminal value $B_{t_N} = B_T$:
We will use equiprobable strata. A fixed number M_i of paths generated by this algorithm belongs to each stratum.
Let F denote the cumulative distribution function (CDF) of a standard Gaussian random variable, let M be the number of paths (simulations) to be generated, n_{st} be the number of strata and M_{st} the (same, hence $= M_i$) number of paths associated to each stratum:

[1]The Z_l's are to be taken as independent for different paths.

- For $i = 1, \dots, n_{st}$
  - generate M_{st} i.i.d. r.v. $(U_{i,k})_{1\le k\le M_{st}}$ according to the law $U(0,1)$.
  - For $k = 1, \dots, M_{st}$, set $V_{i,k} := (i - 1 + U_{i,k})/n_{st}$.
  - For $k = 1, \dots, M_{st}$
    * Set $B^{i,k}_{t_N} := \sqrt{T}F^{-1}(V_{i,k})$ and $B^{i,k}_0 := 0$.
    * Generate N i.i.d. random variables[2] $(Z^{i,k}_l)_{1\le l\le N}$ according to the law $N(0,1)$.
    * For $l = 1, \dots, N$
      · Set $B^{i,k}_{t_l} := \frac{(t_N - t_l)}{t_N - t_{l-1}} B^{i,k}_{t_{l-1}} + \frac{(t_l - t_{l-1})}{t_N - t_{l-1}} B^{i,k}_{t_N} + \sqrt{\frac{(t_N - t_l)(t_l - t_{l-1})}{t_N - t_{l-1}}} Z^{i,k}_l$.

(a) Compute in this way an Monte Carlo estimate of $\theta = E[\Phi(B)]$ by using the stratification method, based on the following parameters: $T = 5$, $N = 20$, $M = 10000$ simulations of the Brownian motion B and $n_{st} = 2000$ equiprobable strata. From each stratum, we will simulate $M_{st} = \frac{M}{n_{st}} = 5$ paths, with terminal value in the respective stratum.

(b) To check your results from a) and investigate the variance reduction by stratification, use the classical Monte Carlo method to give an estimate of θ with the same parameters: $T = 5$, $N = 20$, $M = 10000$ simulations.
The paths $(B^k)_{1\le k\le M}$ of the Brownian motion can be generated using the following scheme (already used on a previous sheet)

  - For $k = 1, \dots, M$
    * Set $B^k_0 := 0$.
    * Generate N i.i.d. random variables $(Z^k_l)_{1\le l\le N}$ according to the law $N(0,1)$.[3]
    * For $l = 1, \dots, N$
      · Set $B^k_{t_l} := B^k_{t_{l-1}} + \sqrt{\frac{T}{N}} Z^k_l$.

(c) Compute the empirical standard deviation of the Monte Carlo estimator of each method. Discuss your results.

[2]These $Z^{i,k}_l$'s are to be independent for different i, k (paths).
[3]Independent of those for other k (paths).