Motivating example

$$f(t) = \int_a^b \frac{dx}{x^2 - t} \qquad a,b \in \mathbb{R} \quad 0<a<b$$

- Method A: be studious

Solve it: $f(t) = \frac{1}{2\sqrt{t}} \log \frac{(a+\sqrt{t})(b-\sqrt{t})}{(a-\sqrt{t})(b+\sqrt{t})}$

$\Rightarrow$ f has logarithmic branch pts at $t = a^2$ and b^2

<u>not</u> singular at $t=0$ since for $t \to 0$

$\log(\cdots) = O(2\sqrt{t})$, log vanishes faster than $\sqrt{t}$

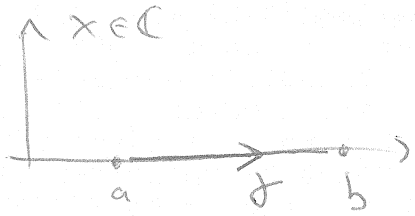
<u>but</u> on non-principal sheets of log it <u>is</u> ~~present~~ singular:

$$\frac{1}{2\sqrt{t}}\left(\log(\cdots) + 2\pi i \cdot n \right) \xrightarrow{n \neq 0} \infty \quad \text{as } t \to 0$$

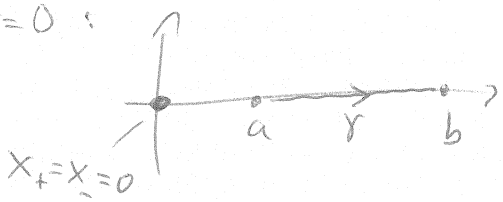
- Method B: be "smart" (goal of this course is to make this precise...)

For any t $g(x,t) = \frac{1}{x^2 - t}$ has singularities (poles) at $x_+ = \sqrt{t}$, $x_- = -\sqrt{t}$

now look at integration contour γ

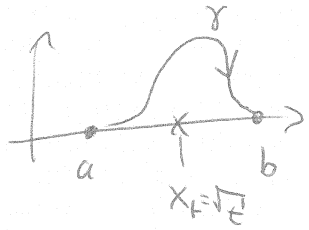


$t=0$:



no problem!

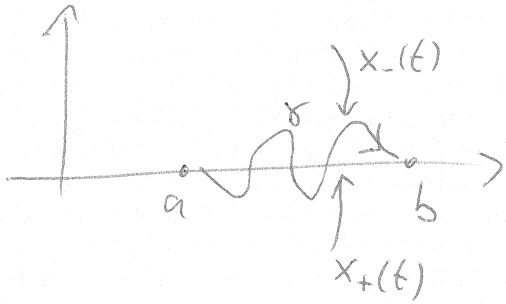
$a^2 < t < b^2$:



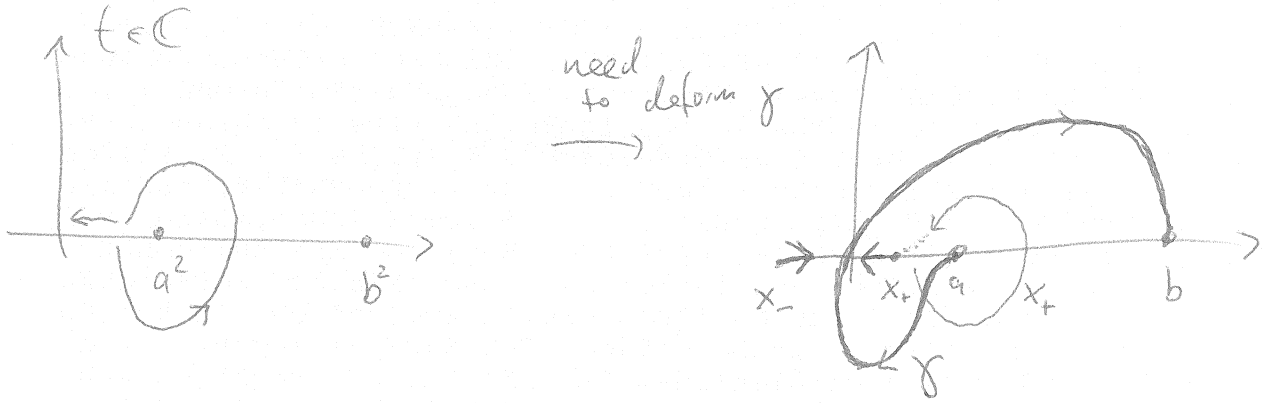
can deform γ

This doesn't work if $x_+, x_- \in \{a, b\}$, i.e. $t = a^2$ or b^2

or if γ gets "pinched"



$\rightarrow$ explains also the singularity at $t=0$ on non-principal sheet: Let t encircle a^2 to change the sheet, then approach 0

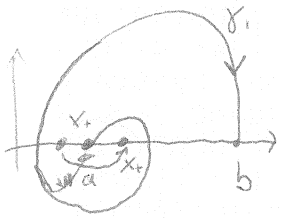


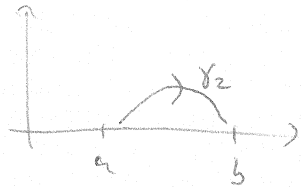
Now for $t \to 0$ γ gets trapped by x_+ and x_-, we cannot avoid the singularity!

Moreover, we find the discontinuity of f along

$$\lim_{\tau \to 2\pi} f(a^2 + re^{i\tau}) - \lim_{\tau \searrow 0} f(a^2 + re^{i\tau}) = \int_{-B_\rho(\sqrt{t_0})} \frac{dx}{x^2 - t}$$

$t_0 = a^2 + r$





$$= 2\pi i \operatorname{Res}\left(\frac{1}{x^2 - t_0}, \sqrt{t_0}\right) \cdot (-1)$$

$$= 2\pi i \cdot \frac{1}{2\sqrt{t_0}} \cdot (-1)$$

difference $\int_{\gamma_1} f - \int_{\gamma_2} f = \int_{-B_\rho(\sqrt{t_0})} f$

$\gamma_1 - \gamma_2 \simeq -B_\rho(\sqrt{t_0})$

looks weird, but is ok

a^2 plays here the role of ∞ for $\log z$ while b^2 the role of 0 ...