Motivating Example

\[ f(t) = \int_a^b \frac{dx}{x^2 - t} \quad a, b \in \mathbb{R} \quad 0 < a < b \]

- Method A: be studious

Solve it: \[ f(t) = \frac{1}{2\sqrt{t}} \log \frac{(a + \sqrt{t})(b - \sqrt{t})}{(a - \sqrt{t})(b + \sqrt{t})} \]

\[ \Rightarrow f \text{ has logarithmic branch pts at } t = a^2 \text{ and } b^2 \]

not singular at \( t = 0 \) since for \( t \to 0 \)

\[ \log(\ldots) = O(2\sqrt{t}) \text{, } \log \text{ vanishes faster than } \frac{1}{t} \]

but on non-principal sheets of \( \log \) it is present singular:

\[ \frac{1}{2\sqrt{t}} \left( \log(\ldots) + 2 \min \right) \rightarrow \infty \text{ as } t \to 0 \]

- Method B: be "smart" (goal of this course is to make this precise...)

For any \( t \) \( g(x; t) = \frac{1}{x^2 - t} \) has singularities (poles)
at \( x = \sqrt{t}, x = -\sqrt{t} \)
Now look at integration contour $\gamma$

\[ \gamma \in \mathbb{C} \]

$t = 0$:

\[ x_+ = x_- = 0 \]

no problem!

$a^2 < t < b^2$:

can deform $\gamma$

This doesn't work if $x_+ \neq x_- \in \{a, b\}$, i.e. $t = a^2$ or $b^2$

or if $\gamma$ gets "pinched"

--- explains also the singularity at $t = 0$ on non-principal sheet: Let $t$ encircle $a^2$

to change the sheet, then approach 0
Now for $t \to 0$, $f$ gets trapped by $x_1$ and $x_\infty$, we cannot avoid the singularity! 

Moreover, we find the discontinuity of $f$ along $a^2 \to b^2$:

$$\lim_{T \to 2\pi} f(a^2 + re^{i\tau}) - \lim_{T \to 2\pi} f(a^2 + re^{-i\tau}) = \int_0^{2\pi} \frac{dx}{x^2 - t} = -B_\partial(\gamma_\partial)$$

$$t = a^2 + r$$

$$= 2\pi i \text{ Res} \left( \frac{1}{x^2 - t} , \gamma_\partial \right) \quad (-1)$$

$$= 2\pi i \cdot \frac{1}{2i} \cdot (-1)$$

The difference is:

$$\int_{Y_1} f - \int_{Y_2} f = \int_{Y_1} f - B_\partial(\gamma_\partial)$$

$$Y_1 \sim Y_2 = -B_\partial(\gamma_\partial)$$

$\mathbf{a^2}$ plays the role of $00$ for $\log z$ while $\mathbf{b^2}$ is the role of $0 \ldots$