

# I: Complex functions & integrals

## Holomorphic functions

1. Def

Let  $U \subset \mathbb{C}$  domain, i.e. open & connected,  
 $f: U \rightarrow \mathbb{C}$  continuous is holomorphic

if  $f$  is complex differentiable in every  $z \in U$ :

$$\forall z \in U : f'(z) = \lim_{\xi \rightarrow z} \frac{f(\xi) - f(z)}{\xi - z} \text{ exist}$$

2. Then

$f$  holomorphic in  $U$

$\Leftrightarrow f$  complex differentiable  $\infty$ -many times

$\Leftrightarrow f$  is real diff'able and  $\frac{\partial f}{\partial \bar{z}} = 0$

$$\text{for } \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

$\Leftrightarrow f$  is real diff'able and  $u = \operatorname{Re}(f)$ ,  $v = \operatorname{Im}(f)$

$$\text{satisfy CR-eq's : } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

( $\Rightarrow f$  conformal,  $u$  and  $v$  harmonic ...)

$\Leftrightarrow f$  is analytic, i.e. has a series representation in a neighborhood of each  $z_0 \in U$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$(cf. f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases})$$

$\Leftrightarrow V \subset U$  simply connected and  $\gamma: [a, b] \rightarrow V$  smooth with  $\gamma(a) = \gamma(b)$ , then  $\int_{\gamma} f = 0$

$\Leftrightarrow D \subset U$  disk,  $\gamma = \partial D$ . Then Cauchy's integral formula

states  $\forall z \in D: f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw$

(Ex)

•  $f(z) = \bar{z}$  nowhere holomorphic e.g.

at  $z=0$   $\lim_{\xi \rightarrow 0} \frac{\bar{\xi}}{\xi}$  doesn't exist.

$$\bullet f(z) = |z|^2$$

Consider  $(x, y) \mapsto (x^2 + y^2)$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$\Rightarrow$  only complex differentiable in 0  $\Rightarrow$  not holomorphic

$$f(z) = \frac{1}{z} \quad f'(z) = -\frac{1}{z^2} \quad \Rightarrow \text{holomorphic in } U = \mathbb{C} \setminus \{0\}$$

$$\therefore f(z) = z^{\frac{1}{2}} \quad f'(z) = \frac{1}{2} z^{-\frac{1}{2}}$$

holomorphic in  $\mathbb{C} \setminus \{0\}$ ?

$$\text{not contin. e.g. } f(1) = \lim_{\varepsilon \searrow 0} f(e^{i\varepsilon}) = \lim_{\varepsilon \searrow 0} e^{\frac{i\varepsilon}{2}} = 1$$

$$\lim_{\varepsilon \searrow 0} f(e^{i(2\pi - \varepsilon)}),$$

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$$\lim_{\epsilon \rightarrow 0} e^{i\pi} \cdot e^{i\frac{\epsilon}{2}} = -1$$

In fact, holomorphic on  $\mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \operatorname{Im}(z) = 0\}$

$z=0$  is a "branch point". More later (cf.  $\log z$ )

## Meromorphic functions

3. Def

$U \subset \mathbb{C}$  domain,  $D \subset U$  discrete subset,  $f: U \setminus D \rightarrow \mathbb{C}$  holomorphic.  $f$  is meromorphic<sup>in  $U$</sup>  if every singularity  $d \in D$  is a pole, i.e.

$$\forall d \in D : \exists k \in \mathbb{N} : \lim_{z \rightarrow d} (z-d)^k f(z) \text{ exists.}$$

smallest such  $k$  is the order of the pole  $d$ .

4. Thm

$f$  meromorphic

$\Leftrightarrow$  around every  $d \in D$   $f$  has a representation as Laurent series

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-d)^n$$

$\nwarrow$  order of pole

$\Leftrightarrow$  Residue thm

$$\gamma: [a, b] \rightarrow U, \quad \gamma(a) = \gamma(b)$$

$$\frac{1}{2\pi i} \oint_{\gamma} f = \sum_{d \in \text{Int}(\gamma)} \text{ind}_{\gamma}(d) \text{res}(f, d)$$

$\uparrow$  Winding nr       $\uparrow$   $a_i$  in Laurent series

(Ex)

$$f(z) = \frac{1}{z}, \quad \gamma_k: [0, 1] \rightarrow \mathbb{C} \setminus \{0\} \quad t \mapsto e^{2\pi i \cdot kt} \quad k \neq 0$$

$$\begin{aligned} \oint_{\gamma} f &= \int_0^1 e^{-2\pi i kt} \cdot 2\pi i k e^{2\pi i kt} dt \\ &= 2\pi i \cdot k \quad \Leftrightarrow \quad \text{ind}_{\gamma_k}(0) = k \\ &\qquad\qquad\qquad \text{res}\left(\frac{1}{z}, 0\right) = 1 \end{aligned}$$

$$f(z) = e^{\frac{1}{z}} = 1 + \sum_{n=-\infty}^{-1} \frac{z^n}{n!} \quad \Rightarrow 0 \text{ is an "essential singularity"}$$

$\Rightarrow$  not meromorphic on  $\mathbb{C}$ , but on  $\mathbb{C} \setminus \{0\}$ .

every rational function is meromorphic on  $\mathbb{C}$ .

## Contour integrals

5. Def

$U \subset \mathbb{C}$  domain. A contour in  $U$  is a piecewise smooth curve  $\gamma: [a,b] \rightarrow U$ , i.e.

a sequence of smooth curves  $\gamma_i: [a,b] \rightarrow U$   $i=1,\dots,n$

$$\text{w/ } \gamma_i(b) = \gamma_{i+1}(a) \quad \forall i=1,\dots,n-1.$$

We write  $\gamma = \gamma_1 + \dots + \gamma_n$  and  $-\gamma$  for

the opposite orientation on  $\gamma$ :  $-\gamma: [a,b] \rightarrow U$

$$t \mapsto \gamma(b+a-t)$$

Recall the definition of complex integration

6. Def

$$\int_{\gamma} f = \int_{\gamma} f(z) dz := \int_{[a,b]} f(\gamma(t)) \cdot \gamma'(t) dt$$

$$f \text{ holomorphic} \xrightarrow{\text{Thm 2}} \int_{\gamma} f = \int_{\tilde{\gamma}} f \quad \text{if } \gamma \approx \tilde{\gamma} \text{ rel } \{a,b\}$$

Keeping it "real" we can make two more observations

1. As with functions on  $\mathbb{R}$ , it is usually better to study integrals over real-valued functions as contour integrals,

$f: \mathbb{R} \rightarrow \mathbb{R} \Rightarrow$  Study  $\int_{\alpha}^{\beta} f(x) dx$  as

$\alpha, \beta \in \mathbb{R}$

$\int_{\gamma} f(z) dz$  where  $\gamma: [a,b] \rightarrow \mathbb{C}$ ,  $\gamma(a)=\alpha$ ,  $\gamma(b)=\beta$ .  
 and  $f$  holomorphic/meromorphic on  $U \subset \mathbb{C}$   
 ( $\rightarrow$  exercises ...)

w/  $[\alpha, \beta] \subset U$ .

2. For  $z = x + iy$ ,  $dz = dx + idy$ ,  $f = u + iv$

$$\int_{\gamma} f = \int_{\gamma} (u + iv)(dx + idy)$$

$$= \int_{\gamma} \underbrace{u dx - v dy}_{\omega_1} + i \int_{\gamma} \underbrace{v dx + u dy}_{\omega_2}$$

integrals  
over  
1-forms on  $\mathbb{R}^2$   
along  $\gamma$

$\Gamma$   $\omega = p dx + q dy$  1-form on  $\mathbb{R}^2$

$$d\omega = \left( -\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} \right) dx \wedge dy \quad 2\text{-form on } \mathbb{R}^2$$

$$\omega \text{ closed} \stackrel{\text{Def}}{\Leftrightarrow} d\omega = 0 \Leftrightarrow \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$

More  
on  
forms  
(over  $\mathbb{C}$ )  
next time...

$$\omega \text{ exact} \stackrel{\text{Def}}{\Leftrightarrow} \exists h: \mathbb{R}^2 \rightarrow \mathbb{R} : \omega = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

$f$  holomorphic  $\Rightarrow$   $\omega_1$  and  $\omega_2$  are closed /  $\omega = f(z) dz$  is closed

Thus, if  $\gamma$  is closed (and sufficiently "nice"), then

Stoke's theorem implies  $\int_{\gamma} f = 0 !$

(or, if  $\text{int}(\gamma)$  is contractible, use Poincaré-Lemma ...)

## II: Complex Manifolds & differential forms

We start with the case  $\mathbb{C}^n$ :

I. Def

$U \subset \mathbb{C}^n$  open,  $f: U \rightarrow \mathbb{C}^m$  is complex differentiable in  $z_0 \in U$  if

$$\exists A \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m): \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0) - A(z - z_0)}{\|z - z_0\|} = 0. *$$

$f$  is holomorphic if it complex differentiable  $\forall z_0 \in U$ .

Identifying  $\mathbb{C}^k$  with  $\mathbb{R}^{2k}$  we see that  $f$  is complex differentiable  $\Leftrightarrow f$  is real differentiable and the linear map  $df(z_0)$  is  $\mathbb{C}$ -linear.

There are, as in the case  $m=n=1$ , many equivalent characterisations of holomorphic maps such as being analytic, a multidimensional Cauchy Integral formula or Cauchy-Riemann equations:

For  $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ ,  $f = (f_1, \dots, f_m)$  write

$f_j = u_j + i v_j$  and  $z_k = x_k + i y_k$ . Then

\* We write  $df \mapsto C$ . H. L. Lewin, Trigonometric Functions

$$df_{\mathbb{R}} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$$

$$df(x_0, y_0) = \begin{pmatrix} \frac{\partial u_i}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \hline \frac{\partial v_j}{\partial x_k} & \frac{\partial v_i}{\partial y_k} \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

because each  $f_j$  is holomorphic and thus

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k} \quad \text{and} \quad \frac{\partial u_j}{\partial y_k} = - \frac{\partial v_j}{\partial x_k}$$

Moreover,

$$\begin{aligned}
 \frac{\partial f_j}{\partial z_k} &= \frac{1}{2} \left( \frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) (u_j + i v_j) \\
 &= \frac{1}{2} \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial v_j}{\partial x_k} + i \left( \frac{\partial v_j}{\partial x_k} - \frac{\partial u_j}{\partial y_k} \right) \right) \\
 &= A_{jk} + i B_{jk} \quad \Rightarrow \quad df(z_0) = A + i B
 \end{aligned}$$

2. Prop

1)  $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$  holomorphic if and only if

$$J \cdot f_R \cdot J^{-1} = f_R \quad \text{for } J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \leftarrow \text{"Multiplication by } i\text{"}$$

$$2) f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ holomorphic} \Rightarrow \det(df_R) \geq 0.$$

Proof: 1) Let  $X \in M_{2n}(\mathbb{R})$ ,  $X = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$  and  $JXJ^{-1} = X$

$$\text{Then } JX = \begin{pmatrix} -S & -T \\ Q & R \end{pmatrix} = XJ = \begin{pmatrix} R & -Q \\ T & -S \end{pmatrix} \Rightarrow \begin{array}{l} Q=T \\ S=-R \end{array}$$

2) Consider  $N = \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix} \in M_{2n}(\mathbb{C})$

$$\text{with } N' = \frac{1}{2} \begin{pmatrix} I_n & I_n \\ -iI_n & iI_n \end{pmatrix}$$

$$\text{Then } N \text{ df}_{\mathbb{R}} N' = \dots = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} = \begin{pmatrix} \text{df}_C & 0 \\ 0 & \overline{\text{df}_C} \end{pmatrix}$$

and thus

$$\det(\text{df}_{\mathbb{R}}) = \det(N \text{ df}_{\mathbb{R}} N') = \det(\text{df}_C) \cdot \det(\overline{\text{df}_C}) \geq 0$$

□

### Complex manifolds

In the following let  $X$  be a topological manifold of dimension  $2n$ .

3. Def A complex chart  $(U, \varphi)$  is an open subset

$U \subset X$  together with a homeomorphism

$$\varphi: U \rightarrow V = \varphi(U) \subset \mathbb{C}^n.$$

Two such charts  $(U_1, \varphi_1)$  and  $(U_2, \varphi_2)$  are

compatible if the transition map  $t_{12} := \varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$

is holomorphic.

A holomorphic atlas is a collection of charts  $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$  such that all transitions are biholomorphic.

A complex analytic structure is a maximal holomorphic atlas. If such a structure is given on  $X$ , then  $X$  is called a complex (analytic) manifold.



- $X = \mathbb{C}^n$

- $X = \overline{\mathbb{C}}$  the "Riemann sphere"  $\mathbb{C} \cup \{\infty\} \approx S^2$  with charts  $(U_i, \varphi_i)_{i=1,2}$  ,  $U_1 = \overline{\mathbb{C}} \setminus \{\infty\}$   $\varphi_1: z \mapsto z$   $U_2 = \overline{\mathbb{C}} \setminus \{0\}$   $\varphi_2: z \mapsto \begin{cases} 0 & z=\infty \\ \frac{1}{z} & \text{else} \end{cases}$

- $X = \mathbb{P}^n$  complex projective space .

Using homogeneous coordinates  $[z_0 : \dots : z_n]$

the charts are defined by

$$j=0, \dots, n: U_j = \left\{ [z_0 : \dots : \underset{j}{1} : \dots : z_n] \right\}, \varphi_j: [z_0 : \dots : z_n] \mapsto (z_0, \dots, \hat{z}_j, \dots, z_n) \in \mathbb{C}^n$$

4. Def. •  $U \subset X$  open.  $f: U \rightarrow \mathbb{C}$  is holomorphic if

for all charts  $(U_i, \varphi_i)$  st.  $U \cap U_i \neq \emptyset$  the function  $f \circ \varphi_i^{-1}: \varphi_i(U \cap U_i) \rightarrow \mathbb{C}$  is holomorphic.

• A map  $f: X \rightarrow Y$  between two complex manifolds is holomorphic if

for all charts  $(U_i, \varphi_i)$  of  $X$ ,  $(V_j, \psi_j)$  of  $Y$ :

$$\psi_j \circ f \circ \varphi_i^{-1}: \varphi_i(U_i \cap f^{-1}(V_j)) \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$$

is holomorphic.



Complex manifolds are quite different from smooth manifolds. For instance, if  $X$  is compact, then any holomorphic function on  $X$  is constant.

Complex manifolds usually have infinitely many complex structures (as opposed to smooth structures if  $\dim X \neq 4$ )  $\rightsquigarrow$  {moduli space of curves, Teich-

Moreover, they are {Müller theory etc...  
canonically oriented (Prop. 2) so not every 2n-dim. top. mf. admits a cplx.

## The complex (co-)tangent bundle

For  $X$  a complex manifold let  $X_0$  denote the underlying real manifold. We consider now  $T_p X := T_p X_0 \otimes_{\mathbb{R}} \mathbb{C}$ , the complexification of the tangent space of  $X_0$  at  $p$ , the space of  $\mathbb{R}$ -linear derivations on (germs of) smooth functions in a neighborhood of  $p$ . It is a complex vector space of dimension  $\dim_{\mathbb{C}} (T_p X) = 2n$ . Its elements are  $v+iw$ ,  $v, w \in T_p X_0$ , acting on  $f = g+ih$  by  $(v+iw)(f) = v(f) + iw(f) = v(g) - w(h) + i(v(h) + w(g))$

Note that we haven't yet used the complex structure on  $X$ . This will allow us to pick the "correct" tangent space for  $X$ , i.e. of the right dimension.

Recall: A complex structure on a  $\mathbb{R}$ -vector space  $V$  is a map  $J \in \text{Hom}(V, V)$  with  $J^2 = -\text{Id}$ . (cf. Multiplication by  $i$  on  $\mathbb{C} \cong \mathbb{R}^2$ )  
Then  $z \cdot v = (x+iy) \cdot v = x \cdot v + y \cdot Jv$

Using the complex structure on  $X$  we find coordinates

for  $T_p X_0$  by  $(z_j = x_j + iy_j) \quad \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p, \left. \frac{\partial}{\partial y_1} \right|_p, \dots, \left. \frac{\partial}{\partial y_n} \right|_p$ .

So define  $J_p$  by  $\frac{\partial}{\partial x_j}|_p \mapsto \frac{\partial}{\partial y_i}|_p$ ,  $\frac{\partial}{\partial y_i}|_p \mapsto -\frac{\partial}{\partial x_j}|_p$

and extend it  $\mathbb{C}$ -linear to  $T_p X = T_p X_0 \otimes_{\mathbb{R}} \mathbb{C}$ .

5. Def

$T_p^{1,0} X = \{ w \in T_p X \mid J_p w = iw \}$ , the holomorphic tangent space.

$T_p^{0,1} X = \{ w \in T_p X \mid J_p w = -iw \}$ , the antiholomorphic —.

Using the basis  $\left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}$  for  $T_p X$  we find

$$\frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

that  $T_p X = T_p^{1,0} X \oplus T_p^{0,1} X = \left\langle \frac{\partial}{\partial z_j} \right\rangle_{\mathbb{C}} \oplus \left\langle \frac{\partial}{\partial \bar{z}_j} \right\rangle_{\mathbb{C}}$

If  $f \in C^\infty(X, \mathbb{C})$ , then  $\frac{\partial f}{\partial \bar{z}_j} = 0 \quad \forall j \Leftrightarrow \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)(g + ih) = 0$

$$\Leftrightarrow \frac{\partial g}{\partial x_j} - \frac{\partial h}{\partial y_j} = 0 \quad \& \quad \frac{\partial g}{\partial y_j} + \frac{\partial h}{\partial x_j} = 0$$

$T_p^{0,1} X = \{ w \in T_p X \mid w(f) = 0 \quad \forall f \text{ holom.} \} \Rightarrow f \text{ holomorphic}$

Thus  $T_p^{0,1} X$  kills holomorphic functions. Since we usually study holomorphic functions on  $X$  we choose  $T_p^{1,0} X$  as the "correct" tangent space (hence the name...\*).

Its dual  $(T_p^{1,0} X)^*$  is the holomorphic cotangent space at  $p \in X$ , its canonical dual basis denoted by  $\{dz_j\}$ .

The key point about complex manifolds is that this whole construction depends "smoothly" on  $p$ ; we patch these local pictures together to form

6. Def

- the complex tangent bundle  $TX = \{(p, T_p X) \mid p \in X\}$
- the holomorphic tangent bundle  $T^{1,0}X = \{(p, T_p^{1,0}X) \mid p \in X\}$
- the antiholomorphic tangent bundle  $T^{0,1}X = \{(p, T_p^{0,1}X) \mid p \in X\}$

and similar for the dual constructions, i.e. the corresponding cotangent bundles,  $TX^*$ ,  $T^{1,0}X^*$ ,  $T^{0,1}X^*$ .

For conditions and obstructions on even-dim smooth manifolds to admit complex structures (which is a very interesting, but long story!) we refer to the literature...

### Differential forms

Elements of  $TX^*$  are usually referred to as complex 1-forms on  $X$  and are part of a much more general picture:

Let  $U \subset X$  be open and let  $\mathcal{E}^k(U)$  denote the space of (real)  $k$ -forms on  $U$

$$\mathcal{E}^k(U) := \left\{ \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k} du_{i_1} \wedge \dots \wedge du_{i_k} \right\}$$

where  $u = (u_1, \dots, u_n)$  are (real) coordinates on  $U$  and all

$f_{i_1 \dots i_k} : U \rightarrow \mathbb{R}$  are smooth.

Now let  $\mathbb{A}^k(U) = \mathcal{E}^k(U) \otimes \mathbb{C}$  be the space of complex k-forms  $\omega = \omega_1 + i\omega_2$ ,  $\omega_1$  and  $\omega_2$  in  $\mathcal{E}^k(U)$ .

In fact, elements  $\omega \in \mathbb{A}^k(U)$  decompose further

$$\omega = \omega_{k,0} + \omega_{k-1,1} + \dots + \omega_{0,k}$$

where  $\omega_{p,q}$  denotes a form in  $\mathbb{A}^k(U)$  that is expressible as

$$\omega_{p,q} = \sum f_{i_1, \dots, i_p, j_1, \dots, j_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

(or shorter  $\omega_{p,q} = \sum f_{I,J} dz_I \wedge d\bar{z}_J$ )

using the complex coordinates  $z = (z_1, \dots, z_n)$  of a holomorphic chart. Thus,  $\mathbb{A}^k(U) = \bigoplus_{\substack{s,t \geq 0 \\ s+t=k}} \mathbb{A}^{s,t}(U)$ .

! For  $k=1$  we retain the decomposition of  $TX^*$  from the last chapter

7. Def

The space of holomorphic k-forms on  $U \times \mathbb{C}$  is

$$\Omega^k(U) = \left\{ \omega \in \mathbb{A}^{k,0}(U) : \omega = \sum f_I dz_I \text{ with } f_I \text{ holomorphic on } U \right\}$$

The exterior differential naturally generalizes to a map  $d: \mathbb{A}^k(U) \rightarrow \mathbb{A}^{k+1}(U)$  and, using the projections  $\pi^{p,q}: \mathbb{A}^k(U) = \bigoplus_{\substack{s,t \geq 0 \\ s+t=k}} \mathbb{A}^{s,t}(U) \rightarrow \mathbb{A}^{p,q}(U)$  we define

$$\partial: \mathbb{A}^{p,q}(U) \rightarrow \mathbb{A}^{p+1,q}(U) \quad \text{by} \quad \partial := \pi^{p+1,q} \circ d$$

$$\text{and } \bar{\partial}: \mathbb{A}^{p,q}(U) \rightarrow \mathbb{A}^{p,q+1}(U) \quad \text{by} \quad \bar{\partial} := \pi^{p,q+1} \circ d$$

Reality check:

$$f \in \mathbb{A}^0(X) = C^\infty(X, \mathbb{C}) \text{ . Then}$$

$$df = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j = \underset{\mathbb{A}^{0,0}(X)}{\partial f} + \underset{\mathbb{A}^{0,1}(X)}{\bar{\partial} f}$$

and same for  $\omega \in \mathbb{A}^{p,q}(U)$ , we find  $d\omega \in \mathbb{A}^{p+1,q}(U) \oplus \mathbb{A}^{p,q+1}(U)$   
so that

$$d = \partial + \bar{\partial}$$

and from  $d^2 = 0$  we find

$$\left\{ \begin{array}{l} \partial^2 = 0, \bar{\partial}^2 = 0 \\ \partial \circ \bar{\partial} = -\bar{\partial} \circ \partial = 0 \end{array} \right.$$

8. Prop

$f: X \rightarrow Y$  holomorphic. Then  $f^* \pi_Y^{p,q} = \pi_X^{p,q} \circ f^*$  and

$$f^* \partial_Y = \partial_X f^*$$

Proof: 1) Case  $p+q=1$  clear. For general  $\omega = \sum_{\substack{|I \cup J|=1 \\ I=I_1, \dots, I_p \\ J=J_1, \dots, J_q}} g_{I,J} dz_I \wedge d\bar{z}_J$   
we calculate

$$f^*(\pi_Y^{p,q} \omega) = f^* \left( \sum_{|I|=p, |J|=q} g_{I,J} dz_I \wedge d\bar{z}_J \right) = \sum_{\substack{I=\{i_1, \dots, i_p\} \\ J=\{j_1, \dots, j_q\}}} (g_{I,J} \circ f) f^* dz_{i_1} \wedge \dots \wedge f^* dz_{i_p} \wedge f^* d\bar{z}_{j_1} \wedge \dots \wedge f^* d\bar{z}_{j_q}$$

$$\pi_X^{p,q}(f^* \omega) = \pi_X^{p,q} \left( \sum_{|I \cup J|=p+q} (g_{I,J} \circ f) f^* dz_I \wedge f^* d\bar{z}_J \right)$$

$$= \sum_{|I|=p, |J|=q} (g_{I,J} \circ f) f^* dz_{i_1} \wedge \dots \wedge f^* dz_{i_p} \wedge f^* d\bar{z}_{j_1} \wedge \dots \wedge f^* d\bar{z}_{j_q}$$

$$2) \partial_X f^* \omega = (\pi_X^{p+1,q} \circ d) f^* \omega = \pi_X^{p+1,q} (df^* \omega) = \pi_X^{p+1,q} (f^* dw)$$

$$= (\pi_X^{p+1,q} \circ f^*) dw = (f^* \pi_Y^{p+1,q}) dw = \dots = f^* \partial_Y \omega.$$

### III Homology & cohomology of manifolds

#### Singular homology

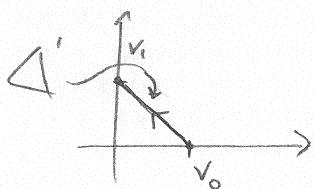
Let  $\Delta^n$  denote the standard  $n$ -simplex

$$\begin{aligned}\Delta^n &= \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\} \\ &= [v_0, \dots, v_n] \quad \text{mit } v_i = e_i = (0, \dots, \underset{i}{1}, \dots, 0) \in \mathbb{R}^{n+1}\end{aligned}$$

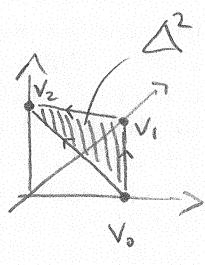
and recall the definition of its "boundary"

$$\begin{aligned}\partial \Delta^n &= \sum_{i=0}^n (-1)^i \Delta_i^n \quad \text{with} \quad \Delta_i^n := [v_0, \dots, \hat{v}_i, \dots, v_n] \\ &= \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \begin{array}{l} x_i = 0 \\ x_j \geq 0 \quad \forall j \neq i \\ \sum x_j = 1 \end{array} \right\}\end{aligned}$$

ex



$$\partial \Delta^1 = \Delta_0^1 - \Delta_1^1 = [v_1] - [v_0]$$



$$\partial \Delta^2 = \Delta_0^2 - \Delta_1^2 + \Delta_2^2 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$(\partial \Delta^2 = \Delta_0^2 - \Delta_1^2 + \Delta_2^2)$$

1. Def

- Let  $X$  be a smooth manifold. An  $n$ -dimensional chain element on  $X$  is a  $\overset{\text{continuous/smooth}}{\text{map}} \sigma: \Delta^n \rightarrow X$ , denoted by  $[\sigma]$ .
- An  $n$ -dimensional (singular) chain is a finite

linear combination of  $n$ -dimensional chain elements with  $\mathbb{Z}$ -coefficients,

$$\gamma \in C_n(X) \Leftrightarrow \gamma = \sum_{i=1}^k \lambda_i [\sigma^i] \quad \lambda_i \in \mathbb{Z}$$

- The boundary of a chain element is

$$\partial[\sigma] = \sum_{i=0}^n (-1)^i [\sigma_i] \quad \text{where}$$

$[\sigma_i]$  corresponds to the map

$$\sigma|_{\Delta^n_i} : \Delta^n_i \rightarrow X$$

The boundary of a chain  $\gamma = \sum \lambda_i [\sigma^i]$  is defined as  $\partial\gamma = \sum \lambda_i \partial[\sigma^i]$ .

2. Prop

$$\partial^2 = 0$$

Proof: Calculate...

Thus, we have a chain complex  
 $C_{\text{sing}} = (C_i, \partial_i)$  with  $C_i = i\text{-dim chains}$   
 $\partial_i = \text{boundary}$

Given an  $n$ -form  $\omega$  on  $X$ , we can pair it with an  $n$ -chain  $\gamma$  by means of integration:

$$\int_{\gamma} \omega = \sum_{i=1}^k \lambda_i \int_{[\sigma^i]} \omega \quad \text{and} \quad \int_{[\sigma]} \omega := \int_{\Delta^n} \sigma^* \omega$$

3. Thm

Stokes formula:

$$\int_{\gamma} dw = \int_{\partial\gamma} \omega \quad \text{for } \gamma \in C_{n+1}(X) \quad \text{and } w \in \Omega^n(X).$$

4. Prop

Let  $f: X \rightarrow Y$  be smooth. Define a map

$$f_*: C_*(X) \rightarrow C_*(Y) \text{ by}$$

$$f_*\gamma = \sum \lambda_i f_*[\sigma^i] \quad \text{and} \quad f_*[\sigma] := f \circ \sigma: \Delta^n \rightarrow Y$$

Then  $f_*\partial = \partial f_*$ , i.e.  $f_*$  is a chain map.

Proof:

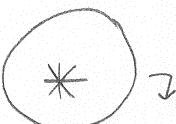
$$1. \int_{f_*[\sigma]} \omega = \int_{\Delta^n} (f \circ \sigma)^* \omega = \int_{\Delta^n} \sigma^* f^* \omega = \int_{[\sigma]} f^* \omega$$

$$2. \Rightarrow \int_{f_*\gamma} \omega = \int_Y f^* \omega \quad \text{by linearity}$$

$$3. \int_{\partial f_*\gamma} \omega = \int_{\gamma} d\omega = \int_{\gamma} f^* d\omega = \int_{\gamma} df^* \omega = \int_{\partial \gamma} f^* \omega = \int_{f_*\partial \gamma} \omega$$

$$f^* d = df^*!$$

■



5. Def

$$C_*(X) = \bigoplus_{i=0}^{\infty} C_i(X) \stackrel{1.}{\rightarrow} Z_*(X) := \ker \partial = \left\{ \gamma \in C_*(X) \mid \partial \gamma = 0 \right\}$$

2.  $\rightarrow B_*(X) := \text{im } \partial$  (group of) cycles in  $X$ , graded by dimension

$$= \left\{ \gamma \in C_*(X) \mid \exists \eta \in C_*(X) : \gamma = \partial \eta \right\}$$

(group of) boundaries in  $X$ , graded by dimension.

Since  $\partial^2 = 0$ ,  $B_*(X)$  is a subgroup of  $Z_*(X)$ .

3.  $\rightarrow H_*(X) := Z_*(X)/B_*(X)$  homology group of  $X$ ,

graded by dimension, i.e.  $H_*(X) = \bigoplus_{i=0}^{\infty} H_i(X)$

with  $H_i(X) = Z_i(X)/B_i(X)$

5

Moreover, one can show that a homotopy between two continuous maps  $f_0, f_1: X \rightarrow Y$  gives rise to a chain homotopy between  $f_{0*}$  and  $f_{1*}$ . ]

Ex

Show:

1.  $f: X \rightarrow Y$  induces a homomorphism  $f_*: H_*(X) \rightarrow H_*(Y)$ .
2. If  $X$  homotopy equivalent to  $Y$ , then  $H_*(X) \cong H_*(Y)$ .

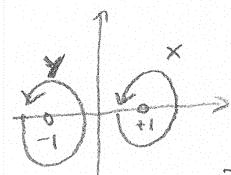
Ex

$$1. \text{ Torus } T^2 = \begin{array}{c} \xrightarrow{\quad} \\ \boxed{x} \\ \xrightarrow{\quad} \end{array} = \text{doughnut}$$

$$\text{Fact: Simplicial homology} \cong \text{Singular homology} \Rightarrow H_i(T^2) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} \oplus \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & \text{else} \end{cases}$$

$$3. X = (\cup \{-1, +1\}: \pi_1(X) \cong F_2, \text{ free group on two generators } x, y$$

whereas  $H_1(X) \cong \mathbb{Z}^2$  is the free abelian group generated by  $x, y$ !



$$2. \text{ Klein bottle } K: \text{ From simplicial homology (exercise) we find } H_0(K) \cong \mathbb{Z}, H_i(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2, H_i(K) \cong 0 \text{ for } i \geq 2.$$

Remark: If we use, instead of  $\mathbb{Z}$ , coefficients in a field  $k$ , then we get vector spaces instead of groups and no "torsion" (e.g. the  $\mathbb{Z}_2$  in  $H_1(K)$  above) appears. So the  $H_i(X)$  are vector spaces as well, the rank of  $H_i(X)$  is then called the i-th Betti number of X,  $b_i := \dim_k(H_i(X))$ .

## Relative homology

For a pair  $(X, A)$ , i.e.  $A \subset X$  topological spaces, the inclusion  $i: A \hookrightarrow X$  induces  $i_*: C_*(A) \rightarrow C_*(X)$ .  $i_*$  is injective (why?), so we identify  $C_*(A)$  with the subgroup  $i_* C_*(A)$  of  $C_*(X)$  and form the quotient group  $C_*(X, A) := C_*(X)/C_*(A)$ .

Furthermore,  $\partial$  induces a homomorphism  $\partial: C_*(X, A) \rightarrow C_{*-1}(X, A)$  which is graded by  $-1$  and  $\partial^2 = 0$ .

### 6. Def

The relative homology groups of the pair  $(X, A)$  are defined as  $H_*(X, A) := Z_*(X, A)/B_*(X, A)$ .  
(cf. Definition 5)

### Remarks

1. In "good cases"  $H_*(X, A) \cong H_*(X/A)$
2. There is a homomorphism  $\partial_*: H_i(X, A) \rightarrow H_{i-1}(A)$  defined by:

Let  $[y] \in H_i(X, A)$  be represented by  $y \in Z_i(X, A)$ ,  
i.e.  $y \in C_i(X)$  s.t.  $\partial y \in C_{i-1}(A)$ .

$\partial^2 = 0 \Rightarrow \partial y \in Z_{i-1}(A)$ .  $\rightsquigarrow$  Define  $\partial_*[y] := [\partial y] \in H_{i-1}(A)$



3. The above properties of  $H_*$  with respect to maps  $f: X \rightarrow Y$  hold also for relative homology and maps of pairs  $f: (X, A) \rightarrow (Y, B)$ , i.e.  $f: X \rightarrow Y$  s.t.  $f(A) \subset B$ .

## The Co-world

Given a chain complex  $C = (C_i, \partial_i)$  we can always consider its dual by "applying the Hom-functor":

Let  $C^i := \text{Hom}(C_i, G)$  for  $G$  a group (the "coefficients") and define  $\delta^i: C^i \rightarrow C^{i+1}$  by  $\delta^i := \partial_{i+1}^*$

$$C^i \ni f \mapsto \delta^i f: C_{i+1} \rightarrow G, x \mapsto f(\partial_{i+1} x)$$

Pictorially,

$$\dots \rightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \rightarrow \dots$$

$$\downarrow \text{Hom}(\cdot, G)$$

$$\dots \leftarrow C_{i+1}^* \xleftarrow{\partial_{i+1}^*} C_i^* \xleftarrow{\partial_i^*} C_{i-1}^* \leftarrow \dots$$

Furthermore, for any chain map  $\varphi_i: C_i \rightarrow D_i$  we

get  $\varphi_i^*: D_i^* \rightarrow C_i^*$  by  $f \in D_i^* \mapsto \varphi_i^* f: x \mapsto f(\varphi_i(x))$ .

Working in the "Co-world" has certain advantages (e.g. product structure ... see Hatcher) as has the ability to compare dual objects with their non-dual counterparts (see below, "Duality")

## Singular cohomology

Applying the dualisation process to the singular chain complex of a topological space/manifold  $X$  we obtain

7. Def.

- The singular cochain complex of  $X^*$  is

$$C^*(X) = \bigoplus_{i \geq 0} C^i(X) \quad \text{with } C^i(X) := C_i(X)^* = \left\{ f : C_i(X) \rightarrow \mathbb{R} \text{ linear} \right\}$$

the group of cochains in  $X$ , graded by dimension,  
with the coboundary homomorphism  $\delta$ , defined

$$\text{by } \delta = \partial^* \quad (\Rightarrow \delta^2 = 0 !)$$

- $Z^*(X) = \ker \delta$ , the group of cocycles, graded by dimension
- $B^*(X) = \text{im } \delta$ , the group of coboundaries, graded by dimension
- $H^*(X) = Z^*(X) / B^*(X)$ , the cohomology group  
of  $X$ , graded by dimension.

### Remarks

1.  $H^i(X; k) = H_i(X)^* = \text{Hom}(H_i(X), k)$  if  $k$  is a field
2. If  $X \approx Y$ , then  $H^*(X) = H^*(Y)$ ;  
if  $f_0$  and  $f_1$  are homotopic, then  $f_0^* = f_1^*$  on cohomology.

7 \* with coefficients/values in  $\mathbb{R}$

3. For a pair  $(X, A)$  we can define relative co-chains by

$$C^*(X, A) := \ker(i^*: C^*(X) \rightarrow C^*(A)) \quad \text{for } i: A \hookrightarrow X$$

$\delta$  restricts/projects to  $C^*(X, A)$  s.t. we can form the relative cohomology group

$$H^*(X, A) := \frac{Z^*(X, A)}{B^*(X, A)} = \ker \delta|_{C^*(X, A)} / \text{im } \delta|_{C^*(X, A)}$$

Further, there is  $\delta^*: H^i(A) \rightarrow H^{i+1}(X, A)$ :

Let  $f \in Z^i(A)$  cocycle, representing a class  $[f] \in H^i(A)$

There is a cochain  $g \in C^i(X)$  with  $i^* g = f$ , (!)

so that  $\delta g$  is a cocycle in  $X$  whose restriction

to  $A$  vanishes,  $\delta g|_A = i^* \delta g = \delta i^* g = \delta f = 0$ ,

i.e.  $\delta g \in Z^{i+1}(X, A) \rightarrow$  Define  $\delta^* h = [\delta g] \in H^{i+1}(X, A)$

### De Rham cohomology

There are also cochain complexes which do not (directly) arise from dualising known chain complexes. For instance, the de Rham complex of a smooth manifold  $X$ :

- $\Omega(X) = \bigoplus_{i \geq 0} \Omega^i(X)$ ,  $d$  = exterior differential

$$\omega = \sum_I f_I dx_I \mapsto d\omega = \sum_I \left( \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \right) \wedge dx_I$$

- $\ker d = \{ \text{closed forms} \}$ ,  $\text{im } d = \{ \text{exact forms} \}$

$$H_{dR}^*(X) = \ker d / \text{im } d, \text{ the de Rham cohomology of } X.$$

Natural question: What is the relation between

$$H^*(X) \text{ and } H_{dR}^*(X) ?$$

For this define a pairing (bilinear form)

$$\langle \cdot, \cdot \rangle : C^i(X) \otimes C_i(X) \rightarrow \mathbb{R}, \langle f, x \rangle := f(x).$$

Since  $\langle f, \partial x \rangle = \langle \delta f, x \rangle$  by definition (!), we see that  $\langle f, x \rangle$  only depends on the (co-)homology classes of  $f$  and  $x$ .  $\rightarrow$  can view  $\langle \cdot, \cdot \rangle$  as a map on  $H^i(X) \otimes H_i(X)$ .

Now consider another pairing

$$I : \Omega^i(X) \otimes C_i(X) \rightarrow \mathbb{R}, I(\omega, \sigma) := \int_{\sigma} \omega.$$

It also depends only on (co-)homology classes since

$$I(dw, \sigma) = I(\omega, d\sigma) \quad (\text{Stokes})$$

**[8. Thm]** De Rham's Theorem: The map  $\Omega^i(X) \ni \omega \mapsto I(\omega, \cdot) \in C^i(X)$  defines an isomorphism between  $H_{dR}^i(X)$  and  $H^i(X) = H_i(X)^*$ .

### Families of supports

**[5 Def]** A family of supports in  $X$  is a set  $\Phi$  of closed subsets of  $X$  with

- 1)  $A, B \in \Phi \Rightarrow A \cup B \in \Phi$
- 2)  $A \in \Phi, B$  closed in  $A \Rightarrow B \in \Phi$
- 3) every  $A \in \Phi$  has a neighborhood belonging to  $\Phi$

**[Ex]**

- $F = \{ \text{all closed sets in } X \}$
- $C = \{ \text{all compact sets in } X \}$
- $X \subset Y, \Phi$  a family of supports in  $Y$ . Then
  - $\Phi|_X = \{ A \subset X \mid A \in \Phi \}$  (need  $X$  to be locally closed)
  - $\Phi \cap X = \{ A = B \cap X \mid B \in \Phi \}$

are families of support in  $X$ .

In particular,  $C_Y|_X = C_X$ ,  $F_Y \cap X = F_X$ .

There are rather natural notions for the support of chains and cochains (see literature for a precise definition...) so that one may form  $C_*(\underline{\Phi} X)$  and  $C^*(\underline{\Phi} X)$ , (co-)chain complexes of (co-)chains with support in  $\underline{\Phi}$  and thus  $H_*(\underline{\Phi} X)$  and  $H^*(\underline{\Phi} X)$  are the (co-)homology of  $X$  with supports in the family  $\underline{\Phi}$ :

(Ex)

Show

$$1. H^*(F X) = H^*(X) \quad 2. H_*(c X) = H_*(X)$$

[10. Thm]

Poincaré's isomorphism: Let  $X$  be an oriented  $n$ -dim. manifold and  $\underline{\Phi}$  a family of supports. Then

$$H_i(\underline{\Phi} X) \cong H^{n-i}(F X)$$

(11. Def)

For  $\mathbb{Z}$ -coefficients consider

$$\langle , \rangle: H^i(X) \otimes H_i(X) \rightarrow \mathbb{Z}, \quad f \otimes x \mapsto f(x).$$

Thm 10 with  $\underline{\Phi} = F \rightsquigarrow \langle , \rangle: H_{n-i}(F X) \otimes H_i(X) \rightarrow \mathbb{Z}$ .

For  $\sigma \in Z_{n-i}(F X), \tau \in Z_i(X)$  we call  $\stackrel{\text{I}}{\langle} (\text{"Poincaré duality"})$

$\langle \sigma, \tau \rangle \in \mathbb{Z}$  the intersection index of the cycles  $\sigma$  and  $\tau$ .

[12. Corollary]

$X$  compact, then  $F = c$  and Poincaré duality is between  $H_{n-i}(X)$  and  $H_i(X)$ .  $\Rightarrow$  If  $X$  orientable, then  $H_n(X) \cong \mathbb{Z}$ .