

IV Motivating example, redux

By now we have a new language at our hands to reconsider our motivating example, the function $f: \mathbb{C} \rightarrow \mathbb{C}$, defined by

$$f(t) = \int_a^b \frac{dx}{x^2 - t} \quad a, b \in \mathbb{R}, \quad a < b.$$

Of course, our motivation is to generalise this simple setting, but let us look first at how we can reformulate the problem of studying the analytic properties of f :

Recall, the integrand $g(x,t) = \frac{1}{x^2 - t}$ has singularities at $x_+ = \sqrt{t}$, $x_- = -\sqrt{t}$. Unless $t = a^2$ or b^2 the integration contour $\gamma = [a,b]$ * not meet x_- or x_+ , so that we can evaluate the integral to find

$$f(t) = \frac{1}{2\pi i} \log \frac{(a+\sqrt{t})(b-\sqrt{t})}{(a-\sqrt{t})(b+\sqrt{t})}.$$

In particular, there is no singularity at $t=0$!

But, if we first let t encircle a^2 or b^2 , crossing

* can be deformed continuously to

the branch cut between a^2 and b^2 , then γ can not be deformed to avoid x_- and x_+ as $t \rightarrow 0$. Thus, we find a singularity at $t=0$ for f evaluated on a non-principal sheet of the logarithm where

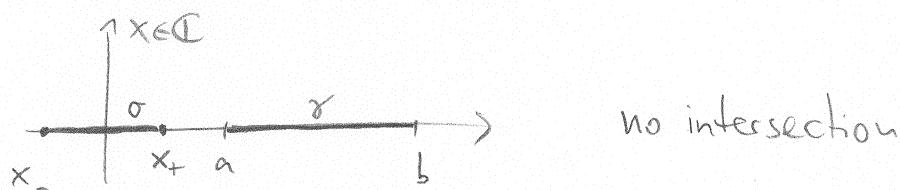
$$f(t) = \frac{1}{2\pi t} (\log(\dots) + 2\pi i n) \quad \text{for } n \in \mathbb{Z} \setminus \{0\}.$$

All this can be recast as follows:

The line from x_- to x_+ is a cycle $\sigma \in C_1(\mathbb{C}, x_- \cup x_+)$ it generates $H_1(\mathbb{C}, x_- \cup x_+) \cong \mathbb{Z}$. σ is called a vanishing cycle, since for $t \rightarrow 0$ it degenerates.

How does the integration contour γ relate to σ ?

1. On the principal sheet

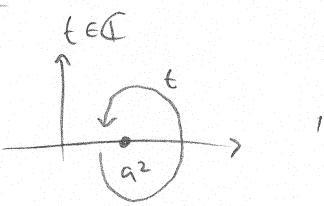


(even if we deform γ without touching x_- and x_+)

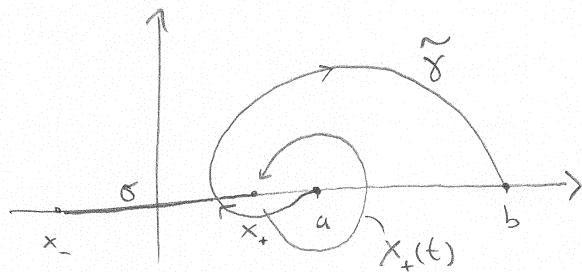
the intersection index of σ and γ is
still zero ($-1+1=0$)

2. On a non-principal sheet

Let t encircle a^2

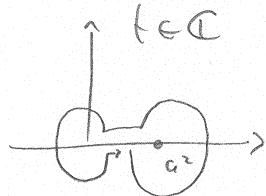


then we need to deform $\tilde{\gamma}$ away from x_+

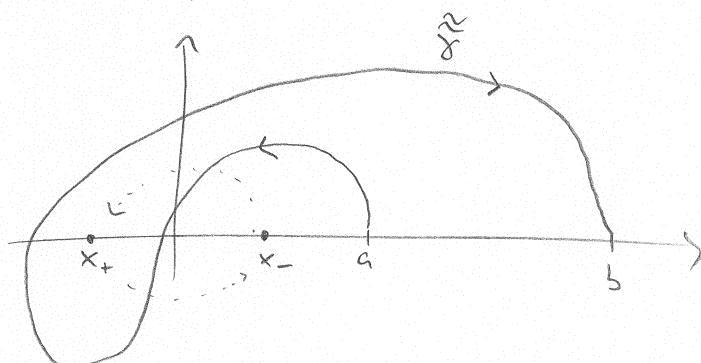


In that case, the intersection index is +1 (or -1, depending on the orientations) and therefore $\tilde{\gamma}$ gets "pinched" as σ vanishes for $t \rightarrow 0$.

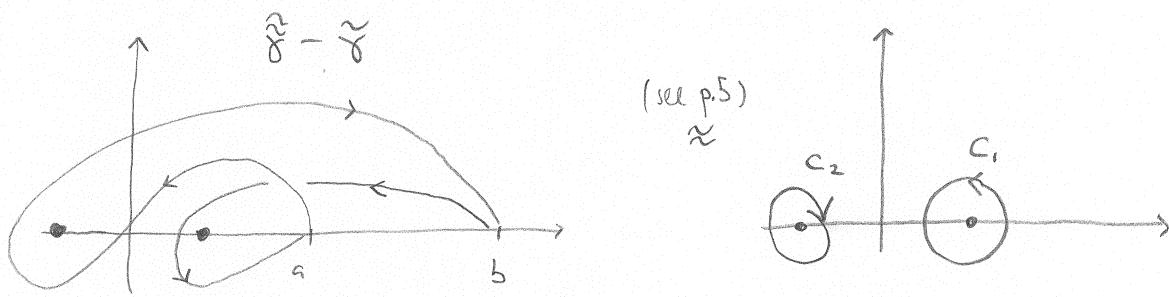
To probe f at this singularity $t=0$ we let t further encircle the origin (thereby crossing the branch cut of $\sqrt{-}$):



The corresponding deformation of $\tilde{\gamma}$ is



The difference between $\tilde{\gamma}$ and $\tilde{\gamma}'$ is then



and the discontinuity of f produced by encircling the singularity at $t=0$ while "being" on a non-principal sheet is

$$\Delta f(t) = \int_{C_1 + C_2} \frac{dx}{x^2 - t} = 2\pi i \operatorname{Res}\left(\frac{1}{x^2 - t}, +\sqrt{t}\right) - 2\pi i \operatorname{Res}\left(\frac{1}{x^2 - t}, -\sqrt{t}\right)$$

$$= 2\pi i \left(\frac{1}{2\sqrt{t}} - \left(-\frac{1}{2\sqrt{t}}\right) \right) = \frac{2\pi i}{\sqrt{t}}.$$

Here, the integration contour $C_1 + C_2$ is a Leray coboundary, constructed as follows:

- take the boundary of the vanishing cycle σ ,
 - $\partial\sigma = x_+ - x_-$
 - for $z \in \mathbb{C}$, let S_z be a small (counterclockwise oriented) circle around z .
- $\rightarrow C_1 + C_2 = \delta \partial\sigma = \delta(x_+) - \delta(x_-)$

Ex:
Check by explicit computation!

δ is called the Leray boundary operator.

Lastly, the relation $C_1 + C_2 = \tilde{\gamma} - \tilde{\gamma}$ is given
by the Picard-Lefschetz theorem

"as t encircles 0": $\tilde{\gamma} \mapsto \tilde{\gamma} + \langle \sigma, \tilde{\gamma} \rangle \cdot \delta \sigma$.

The notion of vanishing cycles, Leray's coboundary operator and the Picard-Lefschetz theorem are the necessary tools to study functions defined by integrals (with singularities) in the general setting.

Before we discuss these objects in detail, we need one more ingredient, the notion of a generalised residue, i.e. a map $\text{Res}(\cdot, \cdot)$ that acts on general differential forms.

** this is to be understood as an equation in $H_1(\mathbb{C} \setminus \{x_-, x_+\})$ (or wrt $\int f$ for f holom. on $\mathbb{C} \setminus \{x_-, x_+\}$ - cf. Exercise 6 on Sheet 2). Also note that $\omega = \frac{dx}{z^2} \in H^1_{\text{dR}}(\mathbb{C} \setminus \{x_-, x_+\})$.