

V Leray's theory of residues

Let f be a meromorphic function on \mathbb{C} with a pole at z_0 and $\omega = f(z) dz \in \Omega^{1,0}(\mathbb{C} \setminus \{z_0\})$.

Cauchy: $\text{Res}(\omega, z_0) = \frac{1}{2\pi i} \int_{\gamma} \omega \quad \text{for } \gamma \text{ counterclockwise circle around } z_0$
 $(\gamma \in H_1(\mathbb{C} \setminus \{z_0\}))$

$$= a_1 \quad \text{for } f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$$

Leray: Reformulate this as

$$\begin{aligned} \omega &= f(z) dz = \sum_{n=-k}^{\infty} a_n (z - z_0)^n dz \\ &= \underbrace{\sum_{n=-k}^{-2} a_n (z - z_0)^n dz}_{\text{exact form}} + a_1 (z - z_0)^{-1} dz + \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n dz}_{\text{form on } \mathbb{C} \setminus \{z_0\} \text{ with first order pole}} + \underbrace{\Psi(z) dz}_{\text{holomorphic form on } \mathbb{C} \setminus \{z_0\}} \end{aligned}$$

$\Rightarrow [\omega] \in H_{dR}^1(\mathbb{C} \setminus \{z_0\})$ is represented by

$$\frac{\Psi(z)}{z - z_0} dz + \Psi(z) dz \quad \text{with } \Psi, \Psi' \text{ regular at } z_0$$

and we define

"the restriction of

$\leftarrow (z-z_0) \cdot \frac{\psi(z)}{z-z_0} dz$ to z_0 "

$$\underline{\text{Res}(w, z_0) := \psi(z_0)}.$$

Let now X be a (complex) manifold, $S \subset X$ a closed (complex) submanifold of codimension one. For $y \in S$ a local equation for S around y will be denoted by $s_y(x) = 0$; if S has a global equation, this will be denoted by $s(x) = 0$. Note that in both cases ds_y or ds must not vanish on S .

Division and derivatives of forms

1. Lemma

If w_y is a regular form in U_y s.t. $ds_y \wedge w_y = 0$, then there is a regular form ψ_y with

$$w_y = ds_y \wedge \psi_y$$

Its restriction to S , $\psi_{y|S}$, depends only on w_y and s_y , and is denoted by

$$\psi_{y|S} = \frac{w_y}{ds_y}|_S.$$

Moreover, if w_y is holomorphic at y , then ψ_y is as well, and thus also $\psi_{y|S}$.

Proof: Choose local coordinates z_1, \dots, z_n on X such that

$s_y(z) = z_1$. Then $ds_y = dz_1$ and

$$ds_y \wedge \omega_y = 0 \Rightarrow \omega_y = dz_1 \wedge \underbrace{\sum_{I \neq 1} f_{I,1} dz_I \wedge d\bar{z}_I}_{=: \psi_y}$$

For another $\tilde{\psi}_y$ with $\omega_y = ds_y \wedge \tilde{\psi}_y$ we have

$$0 = ds_y \wedge (\tilde{\psi}_y - \psi_y) \Rightarrow \tilde{\psi}_y - \psi_y \text{ has } dz_1 \text{ as a factor}$$

$$dz_1|_S = 0 \Rightarrow \tilde{\psi}_y - \psi_y|_S = 0.$$

□

2. Lemma

If S has a global equation $s(x)=0$ and ω is regular on X with $ds \wedge \omega = 0$, then there exists regular ψ on X s.t. $\omega = ds \wedge \psi$ and its restriction to S , $\psi|_S = \frac{\omega}{ds}|_S$, depends only on ω and s , and is holomorphic if ω is.

Proof: Use partition of unity and Lemma 1. □

Now suppose s is a global equation and that ω is closed and $ds \wedge \omega = 0$

Lemma 2 $\exists \omega_1$, regular on X : $\omega = ds \wedge \omega_1$

$d\omega = 0 = -ds \wedge d\omega_1 \xrightarrow{\text{Lemma 2}} \exists \omega_2 : d\omega_1 = ds \wedge \omega_2$
 and $ds \wedge d\omega_2 = -dd\omega_1 = 0$

$$\dots \Rightarrow \exists \omega_k : d\omega_{k-1} = ds \wedge \omega_k$$

3. Prop.

The cohomology class of $\omega_{k|S}$ is well-defined
(depends only on ω and s).

Proof: Need to show: $\omega=0 \Rightarrow \omega_{k|S} \in [0]$ in $H_{dR}^*(S)$.

$$\omega = ds \wedge \omega_1 = 0 \Rightarrow \exists \psi_1 : \omega_1 = ds \wedge \psi_1 \\ d\omega_1 = -ds \wedge d\psi_1$$

$$d\omega_1 = ds \wedge \omega_2 \Rightarrow \quad \text{---} \quad ds \wedge (\omega_2 + d\psi_1) = 0$$

$$\Rightarrow \exists \psi_2 : \omega_2 + d\psi_1 = ds \wedge \psi_2$$

$$d\omega_2 = -ds \wedge d\psi_2$$

$$\text{and } d\omega_2 = ds \wedge \omega_3 \dots$$

$$\dots \Rightarrow \exists \psi_k : \omega_k = -d\psi_{k-1} + ds \wedge \psi_k$$

This means $\omega_{k|S} = d(-\psi_{k-1}|_S)$.



The cohomology class of $\omega_{k|S}$ is denoted by

$$\underline{\frac{d^{k-1}\omega}{ds^k}|_S \in H_{dR}^*(S)}.$$

Formally: $\omega_{k|S} \in \underline{\frac{d^{k-1}(ds \wedge \omega_1)}{ds^k}|_S} = \underline{\frac{d^{k-1}\omega_1}{ds^{k-1}}|_S}$
this allows

and if $\omega = d\omega'$, then $\omega_{k|S} \in \underline{\frac{d^{k-1}(d\omega')}{ds^k}|_S} = \underline{\frac{d^k\omega'}{ds^k}|_S}$

The residue in case of a simple pole

[4. Lemma & Def]

Let ω be a closed, regular form on $X-S$

with a pole of order one along S , i.e.

$$\forall y \in S : s_y \omega = v_y|_{X-S} \quad \text{for } v_y \text{ regular in a neighborhood of } y.$$

Then there exist regular forms ψ_y and Θ_y near y , s.t.

$$\omega_y = \frac{ds_y}{s_y} \wedge \psi_y + \Theta_y$$

where $\psi_y|_S$ is closed and depends only on ω . It's called the residue form of ω , denoted by

$$\text{Res}(\omega, S) = \left. \frac{s_y \omega}{ds_y} \right|_S.$$

It's holomorphic if ω is meromorphic.

Proof: Literature. \blacksquare

(Ex)

1) $X = \mathbb{C}^3$, S defined by $s(z) = z_1 + z_2 z_3 = 0$

$$\omega = \frac{z_1 dz_1 \wedge dz_2 \wedge dz_3}{z_1 + z_2 z_3}$$

Decompose $\omega = \frac{ds_y}{s_y} \wedge \psi_y + \Theta_y$ as follows:

$$ds = \sum_{i=1}^3 \frac{\partial s}{\partial z_i} dz_i = dz_1 + z_3 dz_2 + z_2 dz_3$$

$$so \quad \frac{z_1 dz_1 \wedge dz_2 \wedge dz_3}{z_1 + z_2 z_3} = \frac{dz_1 + z_3 dz_2 + z_2 dz_3}{z_1 + z_2 z_3} \wedge \psi_y + \Theta_y$$

$$\Rightarrow \Theta_y = 0, \quad \psi_y = z_1 dz_2 \wedge dz_3$$

$$\Rightarrow \text{Res}(\omega, S) = \psi_y|_S = -z_2 z_3 dz_2 \wedge dz_3$$

$$2) \quad X = \mathbb{C}^n, \quad S = \{s(x) = 0\}, \quad \omega = \frac{f(z) dz_1 \wedge \dots \wedge dz_n}{s(z)}$$

$$\omega = \frac{ds_y}{s_y} \wedge \psi_y + \Theta_y$$

\Updownarrow

$$\frac{f(z) dz_1 \wedge \dots \wedge dz_n}{s(z)} = \sum_{i=1}^n \frac{\frac{\partial s}{\partial z_i} dz_i \wedge \psi_y}{s(z)} + \Theta_y$$

$$\Rightarrow \Theta_y = 0 \quad \text{and} \quad \psi_y = (-1)^{j-1} \frac{f(z)}{\frac{\partial s}{\partial z_j}} dz_1 \wedge \dots \wedge \overset{\wedge}{dz_j} \wedge \dots \wedge dz_n$$

if $\frac{\partial s}{\partial z_j} \neq 0$ at y .

$$\Rightarrow \text{Res}(\omega, S) = (-1)^{j-1} \left. \frac{f(z)}{\frac{\partial s}{\partial z_j}} \right|_S dz_1 \wedge \dots \wedge \overset{\wedge}{dz_j} \wedge \dots \wedge dz_n$$

is called the Poincaré residue.

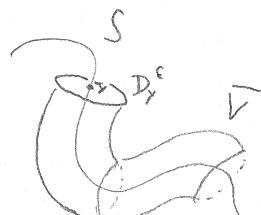
To generalize the classical residue theorem we need a general notion for a "small circle around a critical submanifold $S \subset X$ ".

For this let $S \subset X$ with $\text{codim}_X S = 1$ and denote by

\bar{V} a closed tubular neighborhood of S , i.e. the

fiber bundle $\mu_\varepsilon: \bigcup_{y \in S} D_y^\varepsilon \xrightarrow{*} S$ where D_y^ε is a small

disk in the normal plane at $y \in S$.



Let σ be a p -dimensional chain in S (w.l.o.g.

embedded). Then $\tilde{\mu}_\varepsilon(\sigma) \approx D^\varepsilon \times \sigma$ defines an

oriented chain $\mu_\varepsilon^* \sigma := D^\varepsilon \otimes \sigma$ (using the natural orientation on $D^\varepsilon \subset \mathbb{C}^{\dim D^\varepsilon}$) with boundary

$$\partial \mu_\varepsilon^* \sigma = \partial(D^\varepsilon \otimes \sigma) = \partial D^\varepsilon \otimes \sigma + (-1)^{\dim D^\varepsilon} D^\varepsilon \otimes \partial \sigma.$$

5. Def

Define a map $\delta: C_p(S) \rightarrow C_{p+1}(X \setminus S)$ by

$$\sigma \mapsto \partial D^\varepsilon \otimes \sigma. \quad \text{It satisfies}$$

$$\partial \delta \sigma = \partial(\partial D^\varepsilon \otimes \sigma) = -\partial D^\varepsilon \otimes \partial \sigma = -\delta \partial \sigma, \quad \text{that}$$

is δ is an anti-chain map.

The Leray coboundary map is then given by the in-

duced map $\delta_*: H_p(S) \rightarrow H_{p+1}(X \setminus S)$ on homology

(up to sign \rightarrow literature).

* for $\varepsilon > 0$ sufficiently small

6. Theorem

Let $\sigma \in C_p(\overset{\text{comp. supp.}}{S})$ with $d\sigma=0$ and ω a closed $p+1$ -form on $X \setminus S$ with a pole of order one along S . Then

$$\int_{\delta_* \sigma} \omega = 2\pi i \int_S \text{Res}(\omega, S).$$

Proof: For $\varepsilon > 0$ sufficiently small

$$\int_{\delta_* \sigma} \omega = \int_{\delta_*^\varepsilon \sigma} \omega. \quad \text{Also } \int_{\delta_* \sigma} \omega = \lim_{\varepsilon \searrow 0} \int_{\delta_*^\varepsilon \sigma} \omega \text{ if it exists.}$$

Working in a local chart (U, φ) st. σ and $\delta_*^\varepsilon \sigma$, representing in U and $s(x) = x_1$, we have $[\delta_*^\varepsilon \sigma] \in H^*_*(S \setminus X)$, lie

$$\omega = \frac{dx_1}{x_1} \wedge \psi + \theta \quad \text{in } U$$

and therefore

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \int_{\delta_*^\varepsilon \sigma} \omega &= \lim_{\varepsilon \searrow 0} \int_{\partial D^\varepsilon \otimes \sigma} \frac{dx_1}{x_1} \wedge \psi + \theta \stackrel{\text{Stokes}}{=} \lim_{\varepsilon \searrow 0} \int_{\partial D^\varepsilon \otimes \sigma} \frac{dx_1}{x_1} \wedge \psi \\ &= \lim_{\varepsilon \searrow 0} \int_{\partial D^\varepsilon} \frac{dx_1}{x_1} \cdot \int_S \psi = 2\pi i \int_S \text{Res}(\omega, S) \end{aligned}$$

See exercise

7. Corollary & Def.

$[\text{Res}(\omega, S)] \in H_{dR}^*(S)$ depends only on $[\omega] \in H_{dR}^*(X \setminus S)$, its called the residue class of ω .

The residue in case of a multiple pole

The notion of a residue class allows to reduce this to the case of a simple pole:

8. Theorem Every closed regular form $\omega \in \Omega^*(X \setminus S)$ is co-homologous in $X \setminus S$ to a form $\tilde{\omega}$ which has a simple pole on S .

Proof: See literature \square

9. Def. Corollary 7 & Theorem 8 \Rightarrow The residue class of ω (as above) is well-defined by

$$[\text{Res}(\omega, S)] := [\text{Res}(\tilde{\omega}, S)].$$

10. Theorem $\sigma \in C_p(cS)$, $\partial\sigma = 0$, ω closed form on $X \setminus S$.

Then

$$\int_{\partial\sigma} \omega = 2\pi i \int_{\sigma} [\text{Res}(\omega, S)] = 2\pi i \int_{\sigma} \text{Res}(\tilde{\omega}, S).$$

Proof: Put everything together. \square

Construction of $\tilde{\omega}$ (if $S = \{s(x)=0\}$ globally):

$\omega \in \Omega^*(X \setminus S)$, $s^k \omega$ regular on X for $k \geq 1$ (" ω has a pole of order k along S ").

Then using Lemma 2 one can find forms $\Psi, \Theta \in \Omega^*(X)$ with

$$\begin{aligned}\omega &= \frac{ds}{s^k} \wedge \Psi + \frac{\Theta}{s^{k-1}} \\ &= d\left(-\frac{1}{k-1} \frac{\Psi}{s^{k-1}}\right) + \frac{1}{s^{k-1}} \left(\frac{d\Psi}{k-1} + \Theta \right)\end{aligned}$$

Thus, ω is cohomologous to the form

$$\frac{\Theta}{s^{k-1}} := \frac{1}{s^{k-1}} \left(\frac{d\Psi}{k-1} + \Theta \right)$$

which has a pole of order $k-1$ along S .

Now repeat this step to find $\tilde{\omega} \in [\omega]$ with only a first order pole along S .

[II. Prop]

If additionally $ds \wedge \omega = 0$, then for $\varphi = s^k \omega$

$$[\text{Res}(\omega, S)] = \frac{1}{(k-1)!} \cdot \left. \frac{d^{k-1}\varphi}{ds^{k-1}} \right|_S \quad (\text{cf. Prop. 3}).$$

Proof: Exercise!

We demonstrate the construction with the following

(Ex)

$$X = \mathbb{C}^2, S = \{x+y=0 \mid (x,y) \in \mathbb{C}^2\}$$

$$\omega = \frac{xy \, dx \wedge dy}{(x+y)^2} \Rightarrow$$

- pole of order 2 along S
- ω closed
- $ds = dx + dy \Rightarrow ds \wedge \omega = 0$

Change variables $z = x+y, w = y$, then

$$\begin{aligned}\omega &= \frac{(z-w)w \, dz \wedge dw}{z^2} \stackrel{!}{=} \frac{ds}{s^2} \wedge \psi + \frac{\Theta}{s} \\ &= \frac{dz}{z^2} \wedge \psi + \frac{\Theta}{z}\end{aligned}$$

$$\Rightarrow \psi = -w^2 \, dw, \Theta = w \, dz \wedge dw \Rightarrow \Theta_1 = \frac{d\psi}{1} + \Theta = 0 + w \, dz \wedge dw$$

$$\text{Def. 9: } [\text{Res}(\omega, S)] = [\text{Res}(\frac{\Theta_1}{s}, S)]$$

$$= \left[\underbrace{\text{Res} \left(\frac{w \, dz \wedge dw}{z}, S \right)}_{*} \right]$$

$$*: \frac{w}{z} \, dz \wedge dw \stackrel{!}{=} \frac{d\tilde{\psi}}{z} \wedge \tilde{\psi} + \tilde{\Theta} \Rightarrow \tilde{\Theta} = 0 \\ \tilde{\psi} = w \, dw$$

$$\text{So } * = \tilde{\psi}/s = w \, dw \Big|_{z=0} = w \, dw$$

$$\text{and therefore } [\text{Res}(\omega, S)] = [w \, dw] \in H_{dR}^*(S)$$

On the other hand, let's compute $\frac{d s^2 \omega}{ds^2} \Big|_S$:

$$\varphi := s^2 \omega = (z-w)w dz \wedge dw$$

$$\varphi \text{ closed} \& \quad ds \wedge \varphi = dz \wedge \varphi = 0$$

$$\Rightarrow \exists \varphi_1 \text{ regular on } X : \quad \varphi = dz \wedge \varphi_1$$

$$(\Leftrightarrow (z-w)w dz \wedge dw \stackrel{!}{=} dz \wedge \varphi_1)$$

$$\Rightarrow \varphi_1 = (z-w)w dw$$

$$dz \wedge d\varphi_1 = 0 \Rightarrow \exists \varphi_2 : \quad d\varphi_1 = dz \wedge \varphi_2$$

$$(\Leftrightarrow w dz \wedge dw \stackrel{!}{=} dz \wedge \varphi_2)$$

$$\Rightarrow \varphi_2 = w dw$$

$$\Rightarrow \underbrace{\frac{d\varphi}{ds^2}}_{|S} = [\varphi_2|_S] = [w dw] \in H_{dR}^*(S). \quad \checkmark$$

Composed residues

Let S_1, \dots, S_m be closed complex submanifolds of X with $\text{codim}_c S_i = 1$ and suppose they intersect in general position, i.e.

$\forall y \in \bigcap_{i \in I \subset \{1, \dots, m\}} S_i : \quad \{ds_{i,y}\}_{i \in I}$ are linearly independent

(\Rightarrow every intersection is a complex submanifold),

then there exist residue homomorphisms

$$H_{\text{dR}}^p(X - (S_1 \cup \dots \cup S_m)) \xrightarrow{\text{Res}_1} H_{\text{dR}}^{p-1}(S_1 - (S_2 \cup \dots \cup S_m)) \xrightarrow{\text{Res}_2} \dots \\ \dots \xrightarrow{\text{Res}_m} H_{\text{dR}}^{p-m}(S_1 \cap \dots \cap S_m)$$

and Leray coboundary homomorphisms

$$H_{p-m}(S_1 \cap \dots \cap S_m) \xrightarrow{\delta_*^m} H_{p-(m-1)}((S_1 \cap \dots \cap S_{m-1}) \setminus S_m) \xrightarrow{\delta_*^{m-1}} \dots \\ \dots \xrightarrow{\delta_*^2} H_{p-1}(S_1 \setminus (S_2 \cup \dots \cup S_m)) \xrightarrow{\delta_*^1} H_p(X - (S_1 \cup \dots \cup S_m))$$

satisfying

$$\int \omega = (2\pi i)^m \int \left[\text{Res}_m \circ \dots \circ \text{Res}_1 (\omega) \right] \sigma \quad \begin{pmatrix} \text{just apply} \\ \text{the residue} \\ \text{theorem} \\ m\text{-times} \end{pmatrix}$$

$$\delta_*^1 \circ \dots \circ \delta_*^m \circ$$

Remarks

- The composed residue can be calculated similarly to the case $m=1$ (see literature)
- All concepts of this chapter generalize to the case of relative homology and cohomology ...