

## VI Thom's isotopy theorem & Landau varieties

We have developed a language and tools to study functions defined by parametric integrals and we have seen that generally they have branch points (and other critical points).

How do we find these critical points?

Let  $f: T \rightarrow \mathbb{C}$  be defined by  $f(t) = \int_{\sigma_0} \omega_t$   
where

- $T$  is a complex manifold
- $\{\omega_t\}_{t \in T}$  is a family of closed differential forms on a complex manifold  $X$  with singularities along  $S_t \subset X$ , depending holomorphically on  $(x, t)$ .  
( $\forall t \in T$   $\omega_t \in \Omega^{k,0}(X \setminus S_t)$  and  $t \mapsto \omega_t$  is holom.)
- $\sigma_0$  is a  $k$ -dimensional (compact) cycle on  $X \setminus S_{t_0}$ ,  $\sigma_0 \in Z_k(X \setminus S_{t_0})$ .

1. Lemma  $f$  as above is holomorphic at  $t_0$ .

Proof By continuity,  $\omega_t$  has no singularities on  $\sigma_0$  for  $t$  in a neighborhood  $U_0$  of  $t_0$ . Hence,  $\int_{\sigma_0} \omega_t$  is well-defined and we can differentiate under the integral to see that  $f$  is holomorphic in  $U_0$ .  $\square$

Q: Can we extend the domain  $U_0$  by analytic continuation?

A: Yes, by continuously deforming  $\sigma_0$  to avoid the singular set  $S_t \rightarrow$  Need to define a continuous family of cycles  $\sigma_t$  with  $\sigma_t \in Z_k(X \setminus S_t)$  and  $[\sigma_{t_0}] = [\sigma_0]$ .  
(cf. examples & exercises...)

Roughly speaking, this is possible as long as "the topology of  $X \setminus S_t$  does not change".

2. Prop.

Let  $S := \{(x, t) \mid x \in S_t\} \subset X \times T$ . If the projection  $\pi: (X \times T, S) \rightarrow T, (x, t) \mapsto t$ , is a locally trivial fibration of the pair  $(X \times T, S)$  (defined below) with fibre  $(X, S_{t_0})$ ,

then for any cycle  $\sigma_0 \in Z_p(X \setminus S_{t_0})$  the function  $f$  can be continued homomorphically along any path  $\gamma$  in  $T$ .

The remainder of this chapter is devoted to make this precise.

"Proof": From local triviality we get the following statement:

$\forall t_0 \in T \cdot \exists U_{t_0}$  neighborhood and a continuous family of homeomorphisms of pairs

$$g_t: (X, S_t) \rightarrow (X, S_{t_0}) \text{ defined for all } t \in U_{t_0}.$$

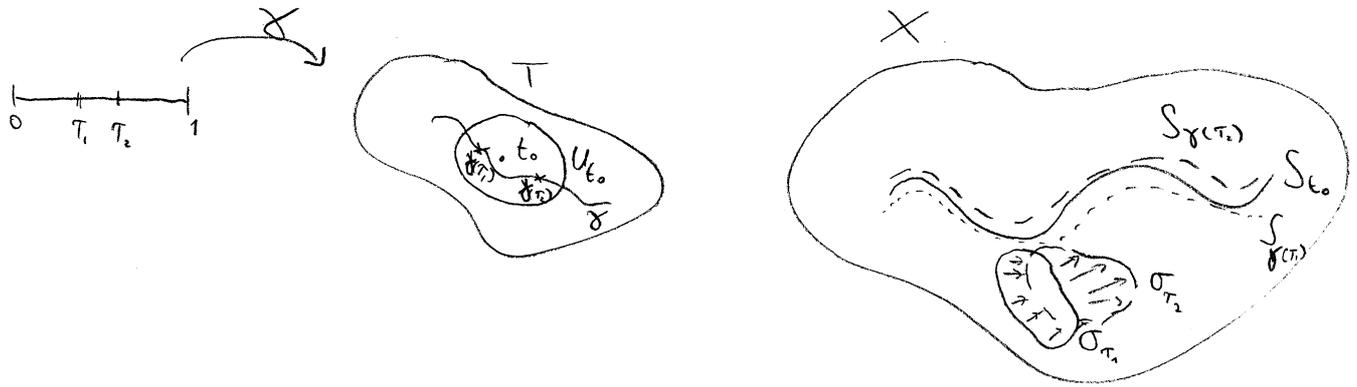
$\Rightarrow$  For  $\gamma: [0, 1] \rightarrow T, \gamma(0) = t_0$  one constructs a contin.

family  $g_\tau: (X, S_{t_0}) \rightarrow (X, S_{\gamma(\tau)}) \quad \tau \in [0, 1]$

and sets  $\sigma_\tau := g_{\tau*}(\sigma_0) \in Z_p(X \setminus S_{\gamma(\tau)}).$

By continuity, if  $\tau_1$  and  $\tau_2$  are nearby and  $\gamma(\tau_1)$  and  $\gamma(\tau_2)$  close to  $t_0 \in T$ , then  $\sigma_{\tau_1}$  and  $\sigma_{\tau_2}$  are homologous in  $X \setminus S_{t_0}$ . Thus,

$$\int_{\sigma_{\tau_1}} \omega_t = \int_{\sigma_{\tau_2}} \omega_t$$



The points where  $\pi$  is not locally trivial are thus the critical points of  $f$ . These form the so-called Landau variety or apparent/visible contour for  $\pi$ , or  $f$  respectively.

### Ambient isotopies

[3. Def] A homeomorphism <sup>(of pairs)</sup>  $g: (X, S_0) \rightarrow (X, S_1)$

is an ambient isotopy of  $S_0$  in  $X$  / from  $S_0$  to  $S_1$  in  $X$

if there exists a family of subspaces  $\{S_t\}_{t \in [0,1]}$  and a continuous family of homeomorphisms

$g_t: (X, S_0) \rightarrow (X, S_t)$  s.t.  $g_0 = \text{id}$  and  $g_1 = g$ .

$S_0$  and  $S_1$  are then called isotopic in  $X$ .

Two ambient isotopies  $g, h: (X, S_0) \rightarrow (X, S_1)$  are equivalent ( $g \equiv h$ ) if they can be realized by the same family  $\{S_t\}_{t \in [0,1]}$ .

Ex. Show that  $\equiv$  defines an equivalence relation on ambient isotopies.

4. Prop. If  $g \equiv h$ , then  $g$  and  $h$  are homotopic.

Proof: Define  $H: [0,1] \times (X, S_0) \rightarrow (X, S_1)$  by

$$H(t, x) = h \circ h_t^{-1} \circ g_t(x). \quad H \text{ is continuous } \checkmark$$

$$\text{and } H(0, x) = h \circ \text{id} \circ \text{id}(x) = h(x) \text{ and } H(1, x) = h \circ h^{-1} \circ g(x) = g(x). \quad \checkmark$$

5. Corollary  $g \equiv h \Rightarrow g_* = h_* : H_*(X, S_0) \rightarrow H_*(X, S_1),$   
 $H_*(X - S_0) \rightarrow H_*(X - S_1),$   
 $H_*(S_0) \rightarrow H_*(S_1).$

Ex  $X = \mathbb{C}$ ,  $S_0 = S_1 = \{-1, +1\}$ ,  $g = \text{id}$ ,  $h = x \mapsto -x$   
 $g$  and  $h$  are both ambient isotopies, realized by

-  $g_t = \text{id}$  ,  $S_t = S_0$

-  $h_t : x \mapsto e^{\pi i t} \cdot x$  ,  $S_t = \{ -e^{\pi i t}, e^{\pi i t} \}$

$g$  and  $h$  are not equivalent because

$g_* : H_0(S_t) \rightarrow H_0(S_0)$  is the identity

while  $h_* : H_0(S_t) \rightarrow H_0(S_0)$  swaps the two generators  $[-1]$  and  $[+1]$

Locally trivial fiber bundles of pairs

Recall that a fiber bundle is a tuple  $(\pi, Y, T, X)$

where  $T, Y, X$  are topological spaces (the "base, total and fibre" space of  $\pi$ ) and  $\pi : Y \rightarrow T$  continuous surjective

s.t. for all  $t \in T$  the fibre  $Y_t = \pi^{-1}(t)$  is homeomorphic to  $X$

A fiber bundle is locally trivial if every  $t \in T$

has a neighborhood  $U_t$  such that the restriction

$$\begin{array}{ccc} \pi^{-1}(U_t) & & U_t \times X \\ \pi \downarrow & \approx & \text{pr}_1 \downarrow \\ U_t & & U_t \end{array} \text{ is the trivial bundle.}$$

Ex

$Y = T = \mathbb{C} \setminus \{0\}$  ,  $X = \{-1, +1\}$  ,  $\pi(y) = y^2 \Rightarrow Y_t = \{-\sqrt{t}, \sqrt{t}\}$

Triviality  $\Leftrightarrow$  Is  $Y$  homeomorphic to  $T \times X$  ?

**6. Def**

A fiber bundle of pairs is a pair  $(Y, S)$  of topological spaces  $S \subset Y$  and a continuous surjective map  $\pi: Y \rightarrow T$  such that  $\forall t \in T: (Y_t, S_t)$  is homeomorphic to a given pair  $(X, S_0)$  (here  $S_t := S \cap \pi^{-1}(t)$ ).

It is locally trivial if every  $t \in T$  has a neighborhood  $U_t$  such that

$$\begin{array}{ccc} \pi^{-1}(U_t) & \xrightarrow{f} & U_t \times X \\ \pi \downarrow & \searrow \text{pr}_1 & \\ U_t & & \end{array}$$

for a homeomorphism  $f: (\pi^{-1}(U_t), \pi^{-1}(U_t) \cap S) \rightarrow (U_t \times X, S_0)$

**Ex**

$\pi$  from Proposition 2, i.e. the bundle

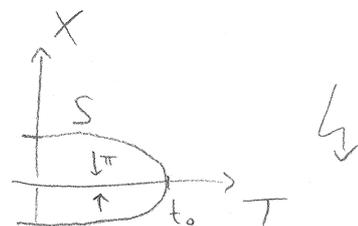
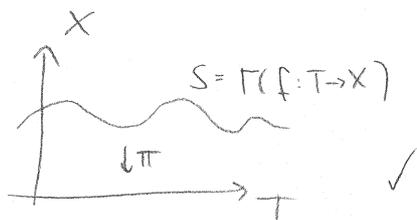
$\pi: (X \times T, S) \rightarrow T, (x, t) \mapsto t, S = \bigcup_t S_t \subset X \times T.$

Def. of bundle of pairs

1.  $\forall t \in T \exists$  homeom.  $g_t: (X, S_0) \rightarrow (Y_t, S_t) = (X, S_t)$

local triviality } 2.  $\forall t \in T \exists U_t$  and  $f$  st.  $(X \times U_t, S_t) \xrightarrow{f} (X \times U_t, S_0) \Rightarrow$  first claim in proof of Prop. 2

picture examples:



## Stratified sets

In general the singular set  $S = \{(x,t) \mid x \in S_t\} \subset X \times T$  is not a submanifold (in our examples, the zero locus of a polynomial), but it can be decomposed into "submanifold-pieces".

Ex

$$Y = \mathbb{R}^3, \quad S = \{f(x,y,z) = 0\} \quad \text{for } f(x,y,z) = x^2 - y^2 z.$$

$$df = (2x \quad -2yz \quad -y^2) \Rightarrow \text{singular at } \{x=y=z=0\} \in S \\ \text{and } \{x=y=0\} \subset S$$

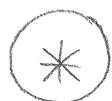
Decomposition into manifolds ("strata")

$$S = A_1^2 \cup A_2^2 \cup A_1^1 \cup A_2^1 \cup A^0$$

$$\text{for } A^0 = \{(0,0,0)\} \quad A_1^1 = \{x=y=0, z > 0\}$$

$$A_2^1 = \{x=y=0, z < 0\}$$

$$A_1^2 = S \cap \{z > 0\}, \quad A_2^2 = S \cap \{z < 0\}$$



7. Def

Let  $Y$  be a smooth manifold and  $S \subset Y$  closed.

A (topological/  
primary) stratification of  $S$  is a family

of nested closed sets

$$\emptyset = S_{-1} \subset S_{d_1} \subset S_{d_2} \subset \dots \subset S_{d_n} = S \quad \text{such that}$$

1.  $\forall d_i: S^{d_i} \setminus S^{d_i-1}$  is a smooth submanifold of  $Y^d$ ; its (finitely many) connected components  $A^{d_i}$  are called the strata. of dimension  $d_i$
2. the boundary of each stratum  $A$ ,  $\partial A = \bar{A} - A$ , is the union of strata of dimension  $< \dim A$ .

Ex

\* above is a primary stratification of  $S$

$$S_0 = \{(0,0,0)\}, \quad S_1 = \{x=y=0\}, \quad S_2 = S.$$

What about

$$S_1 = \{x=y=0\}, \quad S_2 = S \quad ? \quad \rightarrow \begin{array}{l} 1. \checkmark \\ 2. \sphericalangle \end{array}$$

Def

A (Whitney) stratification is a primary stratification with the following property:

If a stratum  $A$  is incident to a stratum  $A'$  ( $A \prec A'$ ), i.e.  $A \subset \partial A'$ , then for all  $a \in A$

(Whitney condition A) the angle between  $T_a(A')$  and  $\bar{T}_a(A)$  goes to zero as  $|a'-a| \rightarrow 0$ .

(Whitney condition B) if  $r: A' \rightarrow A$  is a local retraction, the angle between  $T_a(A')$  and the vector  $\overrightarrow{a', r(a')}$  tends to zero with  $|a' - a|$ .

9. Theorem Every algebraic variety admits a Whitney stratification

For the proof and examples see the literature, e.g. Pham's book...

### Thom's isotopy theorem

In the following let  $\pi: Y \rightarrow T$  a smooth surjection between the smooth manifolds  $Y$  and  $T$  with  $T$  connected. Further, let  $S \subset Y$  be a stratified set (i.e. Whitney stratified!) ( $\Rightarrow Y$  is stratified by adding the stratum  $Y \setminus S$ )

10. Def  $\pi$  is a locally trivial stratified fiber bundle

if there exists a stratified set  $X$  and locally a homeomorphism  $f: \pi^{-1}(U_\epsilon) \rightarrow X \times U_\epsilon$  such that

$$\begin{array}{ccc} \pi^{-1}(U_\epsilon) & \xrightarrow{f} & X \times U_\epsilon \\ \pi \downarrow & \searrow & \swarrow \text{pr}_2 \\ U_\epsilon & & \end{array}$$

and

$\forall$  A stratum of  $Y$  there exists  $\tilde{A}$  stratum of  $X$

$$f(\pi^{-1}(U_\epsilon) \cap A) = \tilde{A} \times U_\epsilon$$

such that

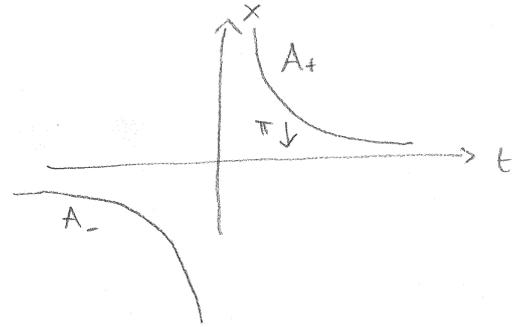
Ex

•  $Y = X \times T$ ,  $X$  stratified &  $T$  connected  
( $\Rightarrow Y$  stratified, low?). Then  $\pi(x,t) = t$  is a trivial stratified fiber bundle

•  $Y = \mathbb{R}^2$ ,  $T = \mathbb{R}$ ,  $S = \{xt - 1 = 0\} = A_+ \cup A_-$

$$\pi(x,t) = t$$

is not locally trivial at the origin.



The main point is that if  $S \subset Y$  is stratified and  $\pi: Y \rightarrow T$  is a locally trivial stratified bundle, then it is also a locally trivial fiber bundle of the pair  $(Y, S)$ .

(EX: Show this!)

In the smooth setting, Thom's isotopy theorem gives a simple characterization of locally trivial stratified bundles:

### II. Theorem

Let  $Y$  be stratified and  $\pi: Y \rightarrow T$  be a proper\* smooth map with  $T$  connected and smooth. If the restriction of  $\pi$  to each stratum of  $Y$  is of rank =  $\dim T$ , then

\* KcT compact  
 $\downarrow$   
 $\pi^{-1}(K)$  compact

$Y$  is a locally trivial stratified bundle. (and some)

Proof: See literature

Important remark! If  $Y = X \times T$ , then:  $\pi$  proper  $\iff X$  compact!

## Landau varieties

12. Def Let  $\pi: Y \rightarrow T$ ,  $Y$  stratified and  $T$  smooth. For every stratum  $A$  of  $Y$  the critical set of  $A$  is

$$cA := \{ a \in A \mid \text{rank } \pi|_A < \dim T \}.$$

The image  $LA := \pi(c\bar{A})$  is called the Landau variety of the stratum  $A$ ,  $\bigcup_A LA =: L$  the Landau variety of  $Y$ , or  $\pi$  respectively.

Proof: Theorem 11

13. Prop Let  $\pi: Y \rightarrow T$  proper and smooth from  $Y$  stratified to  $T$  smooth. Then on every connected component  $C$  of  $T \setminus L$  the map  $\pi|_{\pi^{-1}(C)}$  is a locally trivial bundle. ("An arbitrary projection is locally trivial on the complement of a special set"!)

Proof: Use Theorem 11.

In summary,  $f(t) = \int_{\sigma_0} \omega_t$  can be continued analytically by deforming  $\sigma_0 \rightsquigarrow \sigma_t$  onto  $T \setminus L$  where  $L$  is the Landau variety of  $\pi: (X \times T, S) \rightarrow T, (x, t) \mapsto t$ .

## Integration over relative cycles

In many cases the integration domain  $\sigma_0$  is not closed, but has its boundary lying in a submanifold  $Z_0 \subset X$ , i.e. it's a relative cycle  $\sigma_0 \in \mathbb{Z}_*(X, Z_0)$ .

If we add the condition  $\omega_t|_{Z_0} = 0$ , then Lemma 1 still holds:

$f(t) = \int_{\sigma_0} \omega_t$  with  $\sigma_0 \in \mathbb{Z}_*(X \setminus S_0, Z_0 \setminus S_0)$  and  $\omega_t \in \Omega^*(X \setminus S_t, Z_0 \setminus S_t)$ ,  $d\omega_t = 0$ , is holomorphic in a neighborhood of  $t_0 \in T$ .

Moreover, the machinery we have developed in this chapter can be applied to the relative case as well:

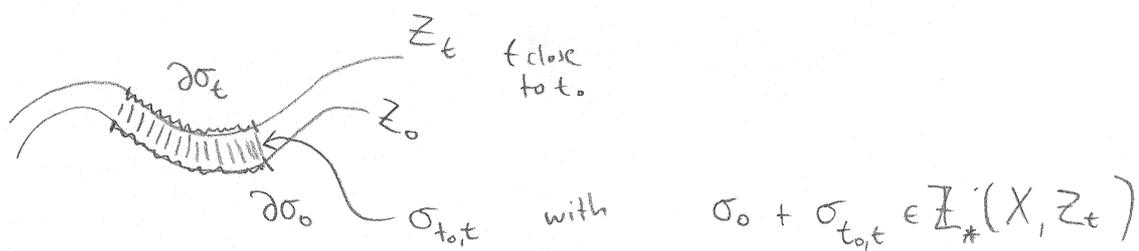
14. Prop.

Let  $\sigma_t \in \mathbb{Z}_*(X, Z_t)$  continuous family of relative cycles,  $\omega_t$  closed &  $\omega_t|_{Z_t} = 0$ , (e.g. a holomorphic form of max. degree)

depending holomorphically on  $(x, t)$  and locally around  $y \in Z_t$  expressible as  $\omega_t(x) = (z_y(x, t))^\alpha \omega_y(x)$  ( $\alpha \in \mathbb{N}_{>0}$ ).

Then  $f(t) = \int_{\sigma_t} \omega_t$  is holomorphic.

Proof: (sketching the idea of Pham's book)



" $\Rightarrow$ "  $f(t) = \int_{\sigma_t} \omega_t = \int_{\sigma_0} \omega_t + \int_{\sigma_{t_0, t}} \omega_t$

$\uparrow$  holom. by Lemma 1

$\uparrow$  tricky... but also holomorphic.

15. Corollary

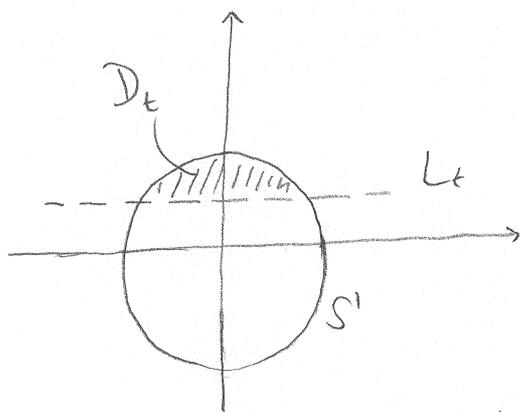
$f(t) = \int_{\sigma_0} \omega_t$  can be analytically continued

for all  $t \in T \setminus L$  where  $L$  is the Landau variety

of  $\pi: (X \times T, Z) \rightarrow T$ ,  $(x, t) \mapsto t$ ,  $Z = \{Z_t \times \{t\} \mid t \in T\} \subset X \times T$

! In fact, we can combine everything above





and consider

$$f(t) = \int_{D_t} dx \wedge dy$$

$$\rightarrow X = \mathbb{R}^2, T = [-1, 1] \text{ (or } \{z \in \mathbb{C} \mid |z| \leq 1\}),$$

(or  $\mathbb{C}^2$ )

$$D_t \in \mathcal{Z}_*(\mathbb{C}^2, \Sigma \cup L_t)$$

$$\text{for } \Sigma = \{x^2 + y^2 = 1\} \subset \mathbb{C}^2$$

Let's look at the fibers over  $t \in \mathbb{C}$  of

$$\pi: (\mathbb{C}^2 \times \mathbb{C}, \Sigma \times \mathbb{C} \cup \{L_t \times \{t\} \mid t \in \mathbb{C}\}) \rightarrow \mathbb{C}$$

$\pi^{-1}(t) = (\mathbb{C}^2, \Sigma \cup L_t)$ . Given  $t \in \mathbb{C}$  the two subspaces  $\Sigma, L_t \subset \mathbb{C}^2$  intersect at  $(x, y) \in \mathbb{C}^2$  with  $y = t$  and  $x^2 + y^2 = 1$ , i.e. at the points  $(\sqrt{1-t^2}, t)$  and  $(-\sqrt{1-t^2}, t)$  except for  $t = \pm 1$  where  $\Sigma \cap L_t = \{(0, \pm 1)\}$ .

Hence,  $\pi$  can not be locally trivial around  $t = \pm 1$ .

ⓘ The first example shows that the points of  $L$  are only "superficially" critical; we have to do more to understand these singularities...

## VII The Picard-Lefschetz theorem

### The setup

We consider the following problem.

Let •  $X$  compact,  $T$  simply connected complex manifolds of dimension  $n$  and  $q$ ,

•  $\pi: Y = X \times T \rightarrow T \quad (x, t) \mapsto t$

•  $S \subset Y$  (Whitney-) stratified set,

$S = S_1 \cup \dots \cup S_k$  union of closed submanifolds of codimension one and in general position

Then (Prop. VI.13) for  $L = \text{Laudan variety of } \pi$

$(Y, S)_{T \setminus L} \rightarrow T \setminus L$  is a locally trivial bundle of pairs.

$\Rightarrow$  every  $[\gamma] \in \pi_1(T \setminus L, t_0)$  defines an ambient isotopy (class) in  $Y_{t_0} = \pi^{-1}(t_0)$  and thus a map

$$\gamma_*: H_*(Y_{t_0}, S_{t_0}) \cong \text{or } H_*(Y_{t_0} \setminus S_{t_0}) \cong$$

Goal: Understand this map!

## The local picture

Let  $A$  be a stratum (of the canonical stratification of  $S$ ),  $A = S_1 \cap \dots \cap S_m \setminus \bigcup_{k>m}^{\ell} S_k$ , → cf. Exerc.  
and  $a \in A$ .

Then we can find a neighborhood  $V$  of  $a$  with

$$S_k \cap V = \emptyset \quad \forall k > m$$

and local coordinates  $y_1, \dots, y_p$  ( $p = n + q$ ) such that

$$a = 0 \in \mathbb{C}^p \quad \text{and} \quad \forall i \in \{1, \dots, m\} : S_i \cap V = \{y_i = 0\}$$

On  $W = \pi(V) \subset T$  we choose local coordinates

$$t_1, \dots, t_q \quad \text{with} \quad \pi(a) = 0 \in \mathbb{C}^q$$

$\Rightarrow$  In  $V$   $\pi$  is given by a system of holomorphic functions  $t_1(y), \dots, t_q(y)$  with  $t_i(0) = 0$ .

**1. Def** A simple loop  $\gamma$  based at  $u_0 \in T \setminus L$  is given by the following construction

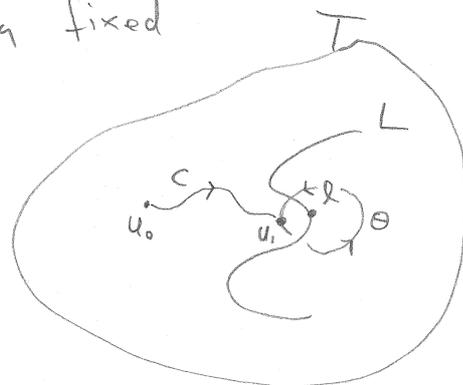
1. a point  $l \in L$
2. a path  $c: [0, 1] \rightarrow T \setminus L$ ,  $c(0) = u_0$ ,  $c(1) = u_1$  close to  $l \in L$

3. in local coordinates  $x$  around  $l \in L$  such that

$L = \{t_1 = 0\}$  a small loop based at  $u_1$

$$\Theta: \tau \mapsto t_1 = \varepsilon e^{2\pi i \tau}, \quad t_2, \dots, t_n \text{ fixed}$$

Then  $\gamma := c' * \Theta * c$



2. Prop  $T$  simply connected  $\Leftrightarrow \pi_1(T \setminus L)$  spanned by simple loops.

Proof: Exercise  $\blacksquare$

### Pinching

Supp.  $a \in cA$  is a critical point of  $A$ , not in the closure of the critical sets of other strata, and of corank one at the target (i.e.  $\text{rank } \pi|_A(a) = q-1$ ).

This implies that locally around  $a$  we have

$$cA \cong LA \quad \text{and} \quad LA \cap W = \{t_1 = 0\},$$

$$cA \cap V = \{y_1 = \dots = y_{n+1} = 0\} \subset cA \cap V = \{y_1 = \dots = y_m = 0\}$$

and

$$t_2(y) = y_{n+2}, \dots, t_q(y) = y_p \quad (\text{recall } p = n+q)$$

and "generically", when restricted to  $A$ ,

$t_1$  will have "quadratic singularity"\*:

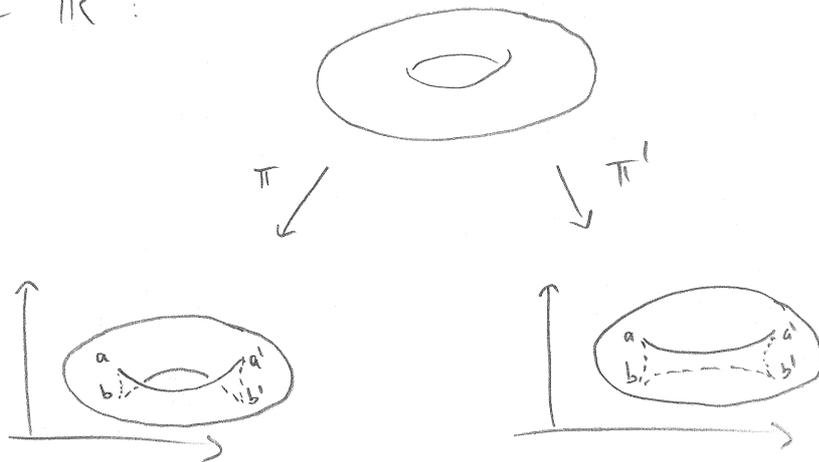
$$\Rightarrow \text{on } A \cap V = \{y_1 = \dots = y_m = 0\} \quad t_1(0, \dots, 0, y_{m+1}, \dots, y_p) = y_{m+1}^2 + \dots + y_p^2$$

Moreover, since  $a$  is only critical for  $A$  the remaining variables  $y_1, \dots, y_m$  must appear linear in  $t_1$ , i.e.

$$t_1(y) = y_1 + \dots + y_m + y_{m+1}^2 + \dots + y_p^2$$

(Ex)

Consider  $A = T^2 \subset \mathbb{R}^3$  and  $\pi, \pi' =$  projections onto the plane  $\mathbb{R}^2$ :



$cA =$  pts in  $T^2$  where one tangent direction is orthogonal to  $\mathbb{R}^2 \rightsquigarrow$  smooth curve

$L = \pi(c\bar{A})$  generically free of singularities, except the four cusps  $a, b, a', b'$ .

\* this is out of the scope of this course; we have to believe...

Finally, setting

$$X_1 = Y_2, X_2 = Y_3, \dots, X_n = Y_{n+1}$$

we see that  $\pi|_V$  takes the simple form

$$\pi(x, t) = t$$

and the  $S_i$  are given by

$$S_2(x, t) = X_1 = 0$$

⋮

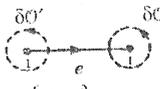
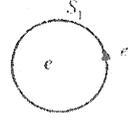
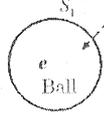
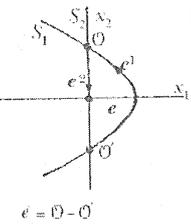
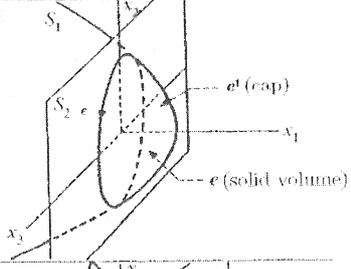
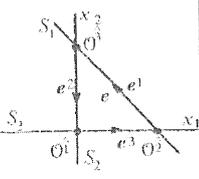
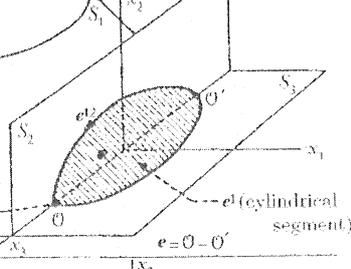
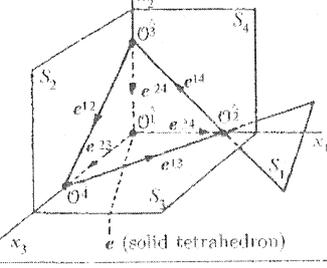
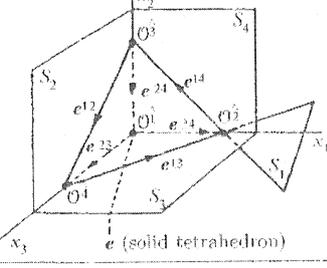
$$S_m(x, t) = X_{m-1} = 0$$

$$S_1(x, t) = (y_1 =) t_1 - (X_1 + \dots + X_{m-1} + X_m^2 + \dots + X_n^2) = 0$$

In the fiber  $Y_t \approx X$  all  $S_{it} = Y_t \cap S_i$  ( $i = \{1, \dots, m\}$ ) are in general position except if  $t_1 = 0$ . In that special case we have

- quadratic pinching  $\stackrel{\text{Def.}}{\Leftrightarrow} m \leq n$  ( $S_i$  quadratic in  $x$ )
- linear pinching  $\stackrel{\text{Def.}}{\Leftrightarrow} m = n+1$  ( $S_i = t_1 - (X_1 + \dots + X_n)$ )  
(in that case  $L = \pi(A)$  for  $A$  with  $\dim A = q-1$ )

! Note that  $m \leq n+1$  is implied by our assumption  $\text{codim } LA = 1$ !

$n \backslash m$	1	2	3
1	 <p> <math>S_1 = \{O, O'\}</math>  <math>e = O - O'</math>  <math>\tilde{e} = \delta O - \delta O'</math> </p>	 <p> <math>S_1</math>  <math>e</math> </p> <p>When <math>t_1 \rightarrow 0</math>, the circle (resp. the sphere) <math>S_1</math> vanishes, and turns into the two 'isotropic lines' (resp. the isotropic cone)</p>	 <p> <math>S_1</math>  <math>e</math> (sphere)              Ball         </p>
2	 <p> <math>S_1 = \{O\}; S_2 = \{O\}</math> </p>	 <p> <math>S_1</math>  <math>S_2</math>  <math>e</math>  <math>e = O - O'</math> </p>	 <p> <math>S_1</math>  <math>S_2</math>  <math>e</math>  <math>e^1</math> (cap)  <math>e</math> (solid volume)         </p>
3	 <p> <math>S_1</math>  <math>S_2</math>  <math>S_3</math>  <math>e</math>  <math>e^1</math> (cylindrical segment)  <math>e = O - O'</math> </p>	 <p> <math>S_1</math>  <math>S_2</math>  <math>S_3</math>  <math>e</math>  <math>e^1</math> (cylindrical segment)  <math>e = O - O'</math> </p>	
4	 <p> <math>S_1</math>  <math>S_2</math>  <math>S_3</math>  <math>S_4</math>  <math>e</math> (solid tetrahedron)         </p>	 <p> <math>S_1</math>  <math>S_2</math>  <math>S_3</math>  <math>S_4</math>  <math>e</math> (solid tetrahedron)         </p>	

from "Singularities of integrals"

F. Pham, Springer 2011

in the following we restrict to the case of quadratic pinching. In order to work coordinate-free, we need some special subspaces of  $U_t := Y_t \cap V$  (from now on we drop the index  $t$ )

**3. Def**

- The vanishing cell  $v$  is the real cell bounded by  $S_1, \dots, S_m$ :  
(oriented)

$$v \stackrel{\text{loc.}}{=} \begin{cases} x_1, \dots, x_n \in \mathbb{R} \\ S_1, \dots, S_m \geq 0 \end{cases}$$

- The vanishing sphere  $e$  is

$$e := \begin{matrix} \partial_m \circ \dots \circ \partial_1 v \\ \uparrow \\ (\partial_i x := \partial x \cap S_i) \end{matrix} \stackrel{\text{loc.}}{=} \begin{cases} x_1, \dots, x_n \in \mathbb{R} \\ S_1, \dots, S_m = 0 \end{cases}$$

- The vanishing cycle  $\tilde{e}$  is

$$\tilde{e} = \begin{matrix} \delta_1 \circ \dots \circ \delta_m e \\ \uparrow \\ (S_i = \text{Leray coboundary wrt } S_i) \end{matrix}$$

Remark: For linear pinching,  $m=n+1$ , the vanishing sphere and cycles do not exist.

**Ex**

Picture on p. 6. Here  $v = \mathbf{e}$  ("bold") and  $e^{i_1, \dots, i_p} :=$

$$\partial_{i_p} \circ \dots \circ \partial_{i_1} \mathbf{e} \quad \text{for } \{i_1 < \dots < i_p\} \subset \{1, \dots, m\} \quad \text{and}$$

$$O^{\hat{i}} := \bigcap_{j \neq i} S_j.$$

For  $t_i \rightarrow 0$  these subspaces of the fiber all reduce to a point. This explains the first half of their name; the second name comes from the coordinate-free description:

### Homological characterisation of $v, e, \tilde{e}$

4. Prop.

$$1. H_{n-m}(U \cap S_1 \cap \dots \cap S_m) \cong \mathbb{Z} \cong \langle e \rangle$$

$$2. H_n(U, S_1 \cup \dots \cup S_m) \cong \mathbb{Z} \cong \langle v \rangle$$

$$3. H_n(U \setminus (S_1 \cup \dots \cup S_m)) \cong \mathbb{Z} \cong \langle \tilde{e} \rangle$$

"Proof":

1.  $U \cap S_1 \cap \dots \cap S_m$  is a complex  $(n-m)$ -sphere which deformation retracts onto the real sphere  $S_{\epsilon}^{n-m} \subset U$ .  
 Moreover, in  $U$  all the maps  $\partial_i$  and  $\delta_i$  are isomorphisms, hence the corresponding homology groups are generated\* by the vanishing cell and cycle (2. & 3.).

Exercise: For  $U = X = \mathbb{C}^n$  and  $m=1$  show that

$$H_{n-1}(S_1(t)) \cong \mathbb{Z} \quad (\text{i.e. } 1.) \quad \text{using } S_1(1) \cong TS^{n-1}$$

\* for  $m=n$ :  $H_0(U \cap S_1 \cap \dots \cap S_n) = \langle \text{two points} \rangle$  (0-sphere!)  
 $\Rightarrow H_n(U \setminus (S_1 \cup \dots \cup S_n)) \cong \mathbb{Z}^2$ . (see picture on page 6)

# The formulae of Picard and Lefschetz

To study the action of  $\pi_1(TL)$  on the various homology groups of  $Y_t$  (resp.  $U_t$ ) it suffices to consider small loops  $\Theta$  around  $u \in L$ ,  $u = \pi(a)$  for  $a \in S_1 \cap \dots \cap S_m$  a pinch point.

## 5. Def

• The transformation  $m_\Theta: Y_{\Theta(0)} \rightarrow Y_{\Theta(1)}$  induced by  $\Theta$ , more precisely by the lift of  $\Theta$ , is called the monodromy of  $\Theta$ .

The induced map  $m_{\Theta*}: H_*(X) \rightarrow H_*(X)$  is called the monodromy operator of  $\Theta$ .

• Let  $h$  be a homology class in  $X$ , represented by a cycle  $\sigma$ . Then  $m_\Theta \sigma$  may be represented by a cycle  $\sigma'$  which differs from  $\sigma$  only inside  $U \subset X$ .\*

Hence, we obtain maps

Var:  $H_n(X \setminus (S_{i_1} \cup \dots \cup S_{i_p}), S_{j_1} \cup \dots \cup S_{j_q})$ ,  $\left( \begin{matrix} \{i_1, \dots, i_p\} \\ \{j_1, \dots, j_q\} \end{matrix} \right) = \left( \begin{matrix} i_1, \dots, i_p \\ j_1, \dots, j_q \end{matrix} \right)$   
 called variation operators of  $\Theta$ .  $\downarrow$

$H_n(U \setminus (S_{i_1} \cup \dots \cup S_{i_p}), S_{j_1} \cup \dots \cup S_{j_q})$   
 def. by  $\sigma \mapsto \sigma' - \sigma$ ,



Some details in

1. This definition vary a bit in the literatur ...

2. \* should be part of the statement of Thm 6.

6. Theorem

For  $\tilde{h} \in H_n(X \setminus (S_1 \cup \dots \cup S_m))$  and  $\theta$  as

above:  $\text{Var}_\theta(\tilde{h}) = N \cdot \tilde{z}$  (Picard's formula)

with  $N = (-1)^{\frac{(n+1)(n+2)}{2}} \langle \nu | \tilde{h} \rangle$  (Lefschetz' formula)

(and similar for the other homologies.) intersection index, see below.

Proof: Picard's formula follows simply from  $H_n(X \setminus (S_1 \cup \dots \cup S_m)) \cong \mathbb{Z}$  and  $\text{Var}_\theta$  is a group homomorphism, the Lefschetz formula can be computed (see Pham)  $\square$

Intersection index  $\langle \nu | \tilde{h} \rangle$  & branch type

Recall Def. III.11. If  $\sigma$  and  $\tau$  are represented by two submanifolds  $S, T$  that intersect transversally (= in general position!), and are closed and oriented, then

$S \cap T = \{x_1, \dots, x_k\}$ . In this case the intersection index  $\langle S | T \rangle$  is given by  $N_+ - N_-$  where

$$N_+ := |\{x_i \mid T_{x_i}T \oplus T_{x_i}S = T_{x_i}X \text{ as oriented tangent spaces}\}|$$

$$N_- := |\{x_i \mid \text{not } \uparrow\}|.$$

(Ex.)  $X$  complex  $n$ -sphere,  $S \subset X$  real  $n$ -sphere. Then  $\langle S | S \rangle = \begin{cases} 0 & n \text{ odd,} \\ 2(-1)^{\frac{n}{2}} & \text{else.} \end{cases}$

In fact, this example is precisely the <sup>case</sup> of the vanishing sphere. Therefore,

$$\langle e | e \rangle = \begin{cases} 2 \cdot (-1)^{\frac{(n-m)(n-m+1)}{2}} & \text{if } n-m \text{ even} \\ 0 & \text{else.} \end{cases}$$

With  $\langle \delta^* \sigma | \tau \rangle = \langle \sigma | \partial \tau \rangle$  (use Poincaré's isomorphism & Stoke's formula) we obtain

$$\langle \nu | \tilde{e} \rangle = \begin{cases} 2 \cdot (-1)^{\frac{(n-m)(n-m+1)}{2}} & \text{if } n-m \text{ even} \\ 0 & \text{else} \end{cases}$$

From this we deduce the variation rules:

- if  $\langle \nu | \tilde{h} \rangle = 0$ , then there is no branching:

- else  $m_{\Theta^*}(\tilde{e}) = \begin{cases} -\tilde{e} & \text{if } n-m \text{ even} \\ \tilde{e} & \text{else} \end{cases}$

and therefore logarithmic branching if

$n-m$  is odd ( $m_{\Theta^*}^k(\tilde{h}) = \tilde{h} + k \cdot N \cdot \tilde{e}$ ) and

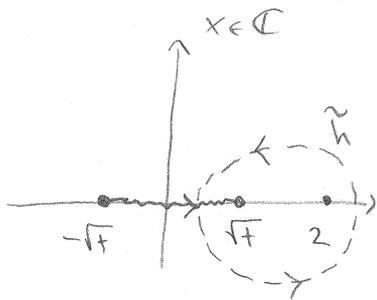
square root branching if  $n-m$  is even ( $m_{\Theta^*}^2(\tilde{h}) = \tilde{h}$ ).

Ex

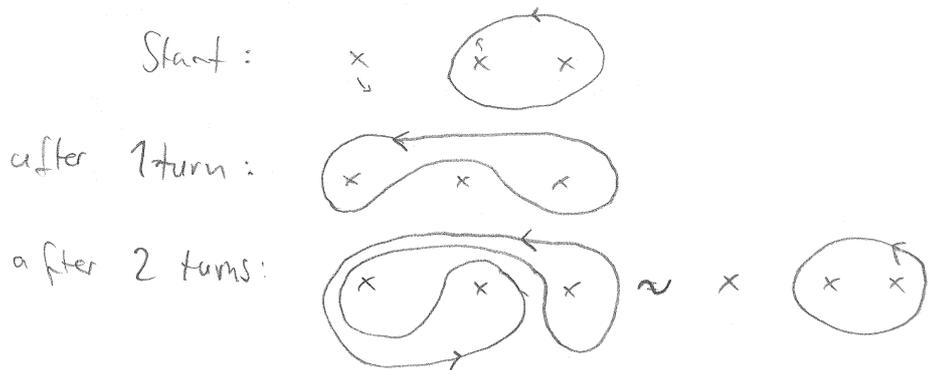
1.

$$X = \mathbb{C}, S_t = \{\pm\sqrt{t}\} \cup \{x=2\} \Rightarrow L = \{t=0\} \cup \{t=4\}$$

Locally around  $t=0 \in L$  the vanishing cell  $v$  is the interval  $[-\sqrt{t}, \sqrt{t}]$ . Let  $\tilde{h}$  be a circle around  $x=\sqrt{t}$  and  $x=2$ . Then with orientations as depicted  $\langle v | \tilde{h} \rangle = -1$  and thus  $\text{Var } \tilde{h} = -\tilde{e}$ .



"Check by hand": Let  $t$  encircle 0



Similarly one shows that there is no branching around  $t=4$  (here  $v = [-\sqrt{t}, 2]$ ).

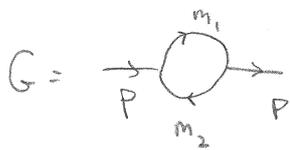
2. Repeat the motivating example, then try to

apply this to  $f(s,t) = \int_a^b dx \int_c^d dy \frac{1}{(x^2-s)(y^2-t)}$

and  $g(s,t) = \oint dx \frac{1}{(x^2-s)(x^2-t)}$  and compare

with the result found by explicit integration.

### 3. A Feynman integral



$$I_G(p) = \int_{M^d} d^d k \frac{1}{(k^2 - m_1^2 + i\epsilon) ((k-p)^2 - m_2^2 + i\epsilon)}$$

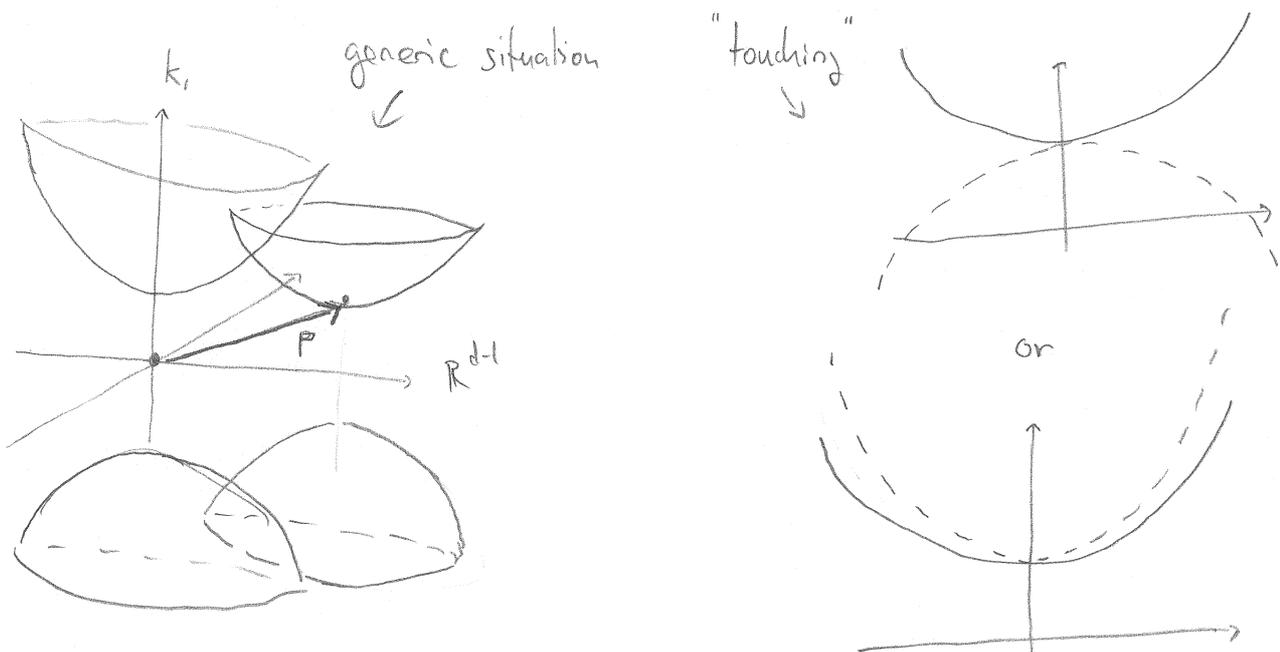
↑ ignore this... ↑

for  $M^d = \mathbb{R}^d$  with  $v^2 = v_1^2 - v_2^2 - v_3^2 - \dots - v_d^2$ .

Singularities of the integrand:  $S_p = \{k^2 = m_1^2\} \cup \{(k-p)^2 = m_2^2\}$

Keeping it real, we argue geometrically to find  $L$ :

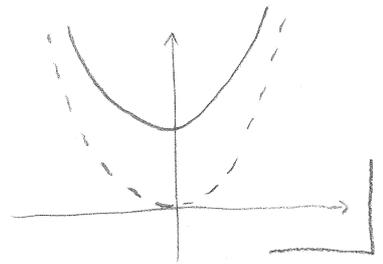
$S_p$  describes two hyperboloids displaced by  $p$ ; the situation is "non-generic" / "topology changes" / "in non-general position" if the two "touch" each other (see picture)



In the upper picture  $p_u = (m_1 + m_2, 0, \dots)$ , while  
 in the other  $p_l = (m_1 - m_2, 0, \dots)$ . So  $L = \{p_u, p_l\} \subset \mathbb{R}^{1, d-1}$ .

Note that there is a third possibility, namely if  $p^2 = 0$   
 but  $p \neq 0$ . Then the two hyperboloids meet at infinity

$\leftrightarrow$  non-compactness of  $X = M^d$ !



If you are brave you can try to compute the variation  
 of  $I_G$ ; it is given (for  $p_u$ ) by

$$\text{Var}(I_G, p_u) = (-2\pi i)^2 \int dk \delta^+(k^2 - m_1^2) \delta^+((k-p)^2 - m_2^2)$$

where  $\delta^+(x^2 - c) := \Theta(x_1) \cdot \delta(x^2 - c)$ .

ⓘ This is actually Theorem V.10, in the sense that  
 the variation of  $I_G$  is given by integration  
 over the vanishing cycle around  $S_{p_u}$ , evaluated  
 as a Leray residue.

$$4. X = \{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum z_i^2 = 1 \}, \quad S = X \cap \mathbb{R}^{n+1} \cong S^n$$

Want to calculate  $\langle S | S \rangle$

① Replace  $S$  by  $\tilde{S}$ , homologous to  $S$ , such that intersection is transversal. Then  $\langle \tilde{S} | S \rangle = \langle S | S \rangle$ .

②  $X \approx TS$

For  $z_j = x_j + iy_j$  the eq for  $X$  is:  $\langle z, z \rangle = 1 \Leftrightarrow \begin{cases} \langle x, x \rangle - \langle y, y \rangle = 1 \\ \langle x, y \rangle = 0 \end{cases}$

Define  $\varphi: X \rightarrow TS$  by  $z_j \mapsto \left( \frac{x_j}{\sqrt{1 + \langle y, y \rangle}}, y_j \right)$

③ Using  $\varphi^{-1}$  we identify each vector field  $V$  on  $S$  with a submanifold  $\tilde{S} \subset X$  that is homologous (in fact, homotopic) to  $S$  which is represented by the zero vectorfield.

④ Take  $V$  as  $y_k(x) = \lambda x_k x_0$  ( $k=1, \dots, n$ ),  $y_0(x) = \lambda(x_0^2 - 1)$  with  $\lambda > 0$ . Its tangent to  $S$  because

$$\langle x, y \rangle = \sum_{k=1}^n \lambda x_k^2 x_0 + \lambda(x_0^2 - 1)x_0 = \lambda x_0 (\langle x, x \rangle - 1) = 0$$

$V$  vanishes at the north and south poles  $P_+, P_-$  where

$$x_1 = \dots = x_n = 0 \quad \text{and} \quad x_0 = \pm 1$$

$$\Rightarrow S \cap \tilde{S} = \{P_+, P_-\}$$

and the tangent spaces at these two points are

spanned by  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  and (for  $P_+$ )

$\frac{\partial}{\partial x_1} + \lambda \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n} + \lambda \frac{\partial}{\partial y_n}$  (and  $\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j}$  at  $P_-$ )

Comparing the orientations induced by  $T_{P_+} S \oplus T_{P_+} \tilde{S}$

and  $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$  (the canonical orientation

of the complex manifold  $X$ ) we find

$$\text{- at } P_+ : \quad \langle \tilde{S} | S \rangle = (-1)^{\frac{(n-1)n}{2}} \quad \left( \text{as the sign of permuting the} \right. \\ \left. \text{basis vectors} \right)$$

$$\text{- at } P_- : \quad \langle \tilde{S} | S \rangle = (-1)^{\frac{(n-1)n}{2}} \cdot (-1)^n$$

$$\Rightarrow \quad \langle \tilde{S} | S \rangle = (-1)^{\frac{(n-1)n}{2}} \cdot (1 + (-1)^n) = \begin{cases} 0 & n \text{ odd} \\ 2 \cdot (-1)^n & n \text{ even} \end{cases}$$

Corollary:  $n$  even  $\Rightarrow$  any vector field on  $S^n$  vanishes somewhere.