# SINGULARITY THEORY, HOMEWORK SHEET NO. 2

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## Problem 1

In Def. 10 we could have added an intermediate definition of transversality of submanifolds:

*Definition:* Two submanifolds $X, Y \subset M$ of a manifold $M$ *intersect transversally*, $X \pitchfork Y$, if

$$\forall x \in X \cap Y : T_xX + T_xY = T_xM.$$

Find a tupel $(f, M, N, S)$ such that $f(M) \pitchfork S$ without $f \pitchfork S$.

## Problem 2

Find the critical points and critical values of the map

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (x, y) \longmapsto (x^3 + xy, y).$$

What is the corank of $f$ at the critical points?

## Problem 3

*Theorem (Whitney '55):* A map of a two-dimensional manifold to a two-dimensional manifold is stable at a point if and only if the map can be described with respect to local coordinates $(x_1, x_2)$ in the source and $(y_1, y_2)$ in the target – with the point under consideration the origin $(x_1, x_2) = (0, 0)$ – in one of the three forms

(1) $y_1 = x_1$, $y_2 = x_2$ - a regular point,
(2) $y_1 = x_1^2$, $y_2 = x_2$ - a *fold*,
(3) $y_1 = x_1^3 + x_1x_2$, $y_2 = x_2$ - a *pleat*.

Use this to classify all singularities of the maps $f$ from Problem 2 above and $p$ from Problem 3, sheet 1.

## Problem 4

Consider the map $f(z) = z^2$, viewed as smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. In the lecture we have seen that $\mathrm{Crit}(f) = \{0\} = \Sigma^2(f)$, but Whitney's theorem or Theorem 8 imply that a generic map from $\mathbb{R}^2$ to $\mathbb{R}^2$ does not have singularities of corank 2. Find a perturbation $f_\epsilon$ of $f$ that has only generic singularities. Assuming you have found a stable perturbation, can you classify its singularities?

## Problem 5

We may upgrade the notion of germs of maps to what is called a *(pre-)sheaf*. An example is given by the following construction. To each open set $U$ in a manifold $M$ associate the algebra

$$\mathcal{C}^\infty(U) := \{f : U \to \mathbb{R} \mid f \text{ is smooth}\},$$

together with restriction maps $\rho_{U,V} : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(U)$ for any pair $U \subset V$ of open sets.

i) Given such data, i.e. a collection of algebras $\mathcal{A}(U)$, one for each $U \subset M$, and maps $\rho_{U,V}$ as above, how can one recover the notion of germs at $x$? (here you should think in terms of functions, but argue abstractly)
ii) Let $\mathcal{C}^\infty_x$ denote the algebra of germs of functions at $x \in M$. The map $\rho_{x,U} : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty_x$ defined by sending $f \in \mathcal{C}^\infty(U)$ to its germ at $x$ is an algebra morphism satisfying $\rho_{x,V} = \rho_{x,U} \circ \rho_{U,V}$ for all $U \subset V$. Is this map surjective, is it injective? What if we replace $\mathbb{R}$ by $\mathbb{C}$ and ask for holomorphicity instead of smoothness?