SINGULARITY THEORY, HOMEWORK SHEET NO. 3

MARKO BERGHOFF, HU BERLIN, SUMMER 2020

Problem 1

Let $U \subset \mathbb{R}^m$ open. Show that two maps $f, g : U \to \mathbb{R}^n$ have k-th order contact at $x \in U$ if and only if their Taylorexpansions at x up to order k agree.

Problem 2

Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open. Let $f_1, f_2 : U \to V$ and $g_1, g_2 : V \to \mathbb{R}^l$ be smooth mappings so that $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined. Let $x \in U$ and suppose that $f_1 \sim_k f_2$ at x and $g_1 \sim_k g_2$ at $y = f_1(x) = f_2(x)$. Show that $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at x.

Problem 3

Recall the definition of push-forwards (1) and pull-backs (2) of jets:

(1) A smooth map $h : N_1 \to N_2$ induces a mapping $h_* : J^k(M, N_1) \to J^k(M, N_2)$ defined as follows. Let $\sigma \in J^k(M, N_1)_{x,y}$ and let $f : M \to N_1$ represent σ. Then $h_*(\sigma) :=$ the equivalence class of $h \circ f$ in $J^k(M, N_2)_{x,h(y)}$.

(2) A diffeomorphism $g : M_1 \to M_2$ induces a mapping $g^* : J^k(M_2, N) \to J^k(M_1, N)$ defined as follows. Let τ be in $J^k(M_2, N)_{x,y}$ and let $f : M_2 \to N$ represent τ. Then $g^*(\tau) :=$ equivalence class of $f \circ g$ in $J^k(M_1, N)_{g^{-1}(x),y}$.

Show that both maps are well-defined. Show that g^* is a bijection, and that the same holds for h_* if h is a diffeomorphism.

Problem 4

Recall the definition of charts for $J^k(M, N)$:

For $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^m$ open let $T_{U,V} : J^k(U, V) \longrightarrow U \times V \times P^k_{m,n}$ be defined by

$$\sigma \longmapsto (x, y, T_k f_1(x), \ldots, T_k f_n(x))$$

where $x = s(\sigma)$, $y = t(\sigma)$ and $f = (f_1, \ldots, f_n) : U \to V$ is a representative of σ.

Now let M, N be smooth manifolds and $U \subset M$, $V \subset N$ chart domains with coordinate charts $\phi : U \to U' \subset \mathbb{R}^m$ and $\psi : V \to V' \subset \mathbb{R}^n$. We defined a manifold structure on $J^k(M, N)$ by declaring the maps

$$\tau_{U,V} := T_{U',V'}(\phi^{-1})^* \psi_* : J^k(U, V) \longrightarrow U' \times V' \times P^k_{m,n}$$

to be charts.

1. Show that the map $T_{U,V}$ is well-defined and bijective.
2. For $U, U' \subset \mathbb{R}^m$ and $V, V' \subset \mathbb{R}^n$ open, $h : V \to V'$ smooth and $g : U \to U'$ a diffeomorphism show that

$$T_{U',V'}(g^{-1})^* h_* T^{-1}_{U,V} : U \times V \times P^k_{m,n} \longrightarrow U' \times V' \times P^k_{m,n}$$

is smooth.

3. Use the above to show that the charts $\tau_{U,V}$ define a smooth atlas on $J^k(M, N)$.

Problem 5

For $k > l$ consider the canonical projection $\pi^k_l : J^k(M, N) \to J^l(M, N)$. Show that this defines a fiber bundle with base $J^l(M, N)$. What is the fiber over a point $\sigma \in J^l(M, N)$?

Problem 6

Give examples of convergent sequences $(f_n)_{n \in \mathbb{N}}$ in $C^\infty(S^1, \mathbb{R})$ and in $C^\infty(\mathbb{R}, \mathbb{R})$ (in the Whitney C^k-topologies). What happens with convergence if we identify functions in $C^\infty(S^1, \mathbb{R})$ with their “lifts” in $C^\infty(\mathbb{R}, \mathbb{R})$? Compare this with convergence in the *compact-open topology.*